Asset Pricing II: 4th Hand-In

Hand in answers at the lectures on May 5-6, 2009.

Regards,

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Caplets: LIBOR Market Model vs. Vasicek

In this exercise we consider the Vasicek model,

$$dr(t) = \kappa(\theta - r(t))dt + \sigma dW^Q(t),$$

assume (as always) that $\kappa > 0$, and let $A$ and $B$ denote the functions such that $P(t, T) = \exp(A(t, T) - B(t, T)r(t))$.

We now consider a coupon bond that makes deterministic positive payments $\alpha_1, \ldots, \alpha_N$ at dates $T_1, \ldots, T_N$. Clearly the price of this coupon bond is

$$\pi^C(t) = \sum_{i|T_i > t} \alpha_i P(t, T_i).$$

(It is strict inequality, “>”, to keep in line with prices always being ex-dividend.)

The last ingredient we need is a (positive) strike-$K$, expiry-$T$ European call-option on the coupon bond.

Show that there exists a unique $r^* \in \mathbb{R}$ such that $\pi^C(T) \geq K$ if and only if $r(T) \leq r^*$.

Hint: Use the Vasicek assumption and ask yourself “when is $\pi^C(T) = K$?”

Define the adjusted strikes via

$$K_i = \exp(A(T, T_i) - B(T, T_i)r^*).$$

Show that the pay-off of the call can be written as

$$(\pi^C(T) - K)^+ = \sum_{i|T_i > T} \alpha_i (P(T, T_i) - K_i)^+. $$
Hint: Two things can happen to the right hand side. Investigate these separately.

Explain how (given results known from for instance Björk) this leads to a closed-form (up to knowledge of \( r^* \)) expression for the price of the call on the coupon bond.

Assume

- \( \theta = 0.06, \kappa = 0.1, \sigma = 0.01, r_0 = 0.04 \)
- \( K = 3.5, T = 1, N = 4, T_i = i + 1 \) and \( \alpha_i = 1 \) for all \( i \).

Calculate the time-0 price of the call. (Numbers, please; this will involve solving an equation numerically.)

**Caplets: LIBOR Market Model vs. Vasicek**

Consider a world where true yield curves are generated by that Vasicek model with parameters

\[
\theta^Q = 0.06, \kappa = 0.1, \sigma = 0.01,
\]

and where \( r_0 = 0.04 \). To avoid writing a lot of \( \delta \)'s, we only look at cash-flows that occur at dates \( \{0, 1, \ldots, 5\} \).

Determine the (1-year) forward LIBORs, \( L(0, i, i + 1) \) for \( i = 0, \ldots, 4 \) in the notation of Björk’s Definition 20.2. (Numbers please.)

Determine the prices of the 1 to 5 year caplets (i.e. \( T_i = 1, \ldots, 5 \) in the notation from Björk’s Section 24.8). For each expiry date do it for cap rates (the strike in the cap) of \( \{0.01, 0.02, \ldots, 0.07\} \). (So all in all you get you a matrix of caplet prices.)

Calculate (numerically) and plot the implied Black volatilities across strikes and expiry dates (see Björk’s Definition 25.3; just do “spot” volatilities; why is \( T_i = 1 \) special?)

Explain why this is conclusive evidence that the Vasicek model and the log-normal LIBOR market model are truly different models.