Asset Pricing II: 3rd Hand-In

Hand in answers at the lectures on April 16-17, 2009.

The first two exercises are of “mostly conceptual nature”. The last one is very much the opposite.

Kindly,

Rolf

Self-Financing Strategies

On the homepage I have posted an exercise from a book about continuous-time finance. One that Bjarne came across. You are not asked — directly at least — to solve the exercise. Rather, you must:

1. Explain why the result that parts a.-e. lead to is strictly speaking correct, but completely obvious and somewhat beside the point. (What is the only way that \((1, -\delta(t))\) can be self-financing?)

2. Comment on the last five lines of the exercise. (Is the “it’s OK in discrete time”-statement correct?)

Stochastic Volatility Models

First, read Björk’s Chapter 15 and convince yourself that that looks a lot like Chapter 21, which was covered at the lectures.

Second: A stochastic volatility model for a stock price, \(S\), could look like this

\[
\begin{align*}
    dS(t) &= \beta S(t) + \sqrt{V(t)}S(t)dW_1(t), \\
    dV(t) &= \xi V(t)dt + \gamma V(t)dW_2(t),
\end{align*}
\]
where $W_1$ and $W_2$ are independent Brownian motions, and we think of $\xi$ and $\gamma$ as constants. Argue that this puts us in the realm of Björk’s Chapter 15. Think about the following:

- In the sense of Assumption 15.3.1: What should the $X$-variables be? What are the $\mu$- and $\delta$-functions?

- What is the system of equations that the $\lambda$-vector(process) must solve? In particular and in the sense of equations (15.14-15): What is the first entry of the $\alpha$-vector? What is the first row of the $\sigma$-matrix?

- Can we tell what $\lambda_1$ is? What about $\lambda_2$?

Show that under an equivalent martingale measure we have

$$dS = rS(t)dt + \sqrt{V(t)}S(t)dW_1^Q(t)$$

and

$$dV(t) = (\xi - \gamma\lambda_2(t, S(t), V(t)))V(t)dt + \gamma V(t)dW_2^Q(t),$$

and argue that if we assume that $\lambda_2$ is an honest-to-God constant, then the change of measure can be subsumed (“soaked up”) by a change of parameter. Suppose the dynamics of the volatility is changed to

$$dV(t) = \kappa(\theta - V(t))dt + \sigma\sqrt{V(t)}\left(\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t)\right),$$

How would you interpret the parameter $\rho$? How does this change the analysis of the independent geometric Brownian motion case from above? What are now $dS$ and $dV$ under an equivalent martingale measure? To ensure that the $P$ to $Q$ change can be “soaked up” by a parameter change, what assumptions need to be made about the functional form of $\lambda_2$? What if $\lambda_1 = \lambda_2 = 0$? What if $\lambda_2 = 0$ but $\lambda_1 \neq 0$?

The CIR ZCB Formula

(Arguably, this is not the most creative of exercises, but afterward you can walk around the finance community with a smug expression on your face.) Verify the formula for zero-coupon bonds prices the Cox-Ingersoll-Ross model given in Proposition 22.6 in Björk.
A constructive hint is the following: The ODE for \( B \) reads

\[
\frac{dB}{dt} = ab^2 - bB - 1 = a(B - c_1)(B - c_2)
\]

for some appropriate constants (it’s your job to find them; the are roots of a quadratic equation). This ODE is of a form you can use separation of variables on. Consult the differential equation section of your 1st year math-book. The message is that you have to look at

\[
\int \frac{dB}{(B - c_1)(B - c_2)} + c = a \int dt,
\]

where you “forget” that \( B \) is a function in the left-hand side integral, which then becomes

\[
\frac{\ln |B - c_2|}{c_2 - c_1} + \frac{\ln |B - c_1|}{c_1 - c_2}.
\]

The right-hand side is just \( at \). Determine the constant of integration, \( c \), from the boundary/terminal condition, and solve for \( B \) in terms of \( t \). (You need to take appropriate “sign care” when removing the \( | \cdot | \)’s.)

For the \( A \)-function, a change of variable leads to an integral, that is almost of the same form as above — and can at least be looked up. Note that Björk’s \( A_0 \)-function in Proposition 22.6 is the logarithm of the \( A \)-function in Proposition 22.2.