Simple forward rates; LIBOR

A simple forward rate $L(t; S, T)$ specifies the cash-flow for a loan agreement where

- The agreement is made at time $t$
- At time $S$ the borrower receives $1$ (or Euro, or DKK, or ...)
- At time $T$ the borrower pays back $1 + (T - S)L(t; S, T)$
Note that this rate is quoted on a discretely compounded basis. If $L(0; 1, 1.25) = 0.04$ then you have to pay back 1.01; if the 0.04 were taken as continuously compounded you’d have to pay back $\exp(0.25 \times 0.04) = 1.010050$.

The usual simple no-arbitrage argument (DIY) shows that

$$1 + (T - S)L(t; S, T) = \frac{P(t, S)}{P(t, T)} \Rightarrow L(t; S, T) = \frac{1}{T - S} \left( \frac{P(t, S)}{P(t, T)} - 1 \right).$$

Such simple rates are called LIBOR. They are widely used.
With $T = S + \delta$ we may write

$$L_\delta(t; S),$$

$L_\delta(t; t)$ is called ($\delta$-) spot LIBOR.

Immediate (Technical) Observation

$L_\delta(t; T)$ is a $Q^T + \delta$ martingale.
Look at a tenor-structure; a set of dates where something interesting happens

\[
\begin{align*}
\delta & \quad \delta \\
T_0 & \quad T_1 = T_0 + \delta \quad T_i = T_{i-1} + \delta \quad T_N = T_0 + N\delta
\end{align*}
\]

A floating rate bullet bond has the cash-flows

\[
\delta L_\delta(T_{i-1}; T_{i-1}) \quad \text{at } T_i \text{ for } i \leq N-1, \text{ and } 1+\delta L_\delta(T_{N-1}; T_{N-1}) \quad \text{at date } T_N.
\]

\[
\delta L_\delta(T_{i-1}; T_{i-1}) \quad := c_i
\]

The cash-flows are stochastic so finding the arbitrage-free price seems to require a dynamic model.
It doesn’t.

We have

\[ c_i = \frac{1}{P(T_{i-1}, T_i)} - 1 \text{ for } i \leq N - 1 \]

and the time-\(t\) value of the “-1” is of course \(-P(t, T_i)\). Now consider the following trading strategy:

- **time \(t\):** Buy 1 \(T_{i-1}\)-ZCB (price: \(P(t, T_{i-1})\))

- **time \(T_{i-1}\):** Invest the $1 received in \(T_i\)-ZCB. You’ll get \(1/P(T_{i-1}, T_i)\) units & a net-cash-flow of 0.
• time $T_i$: Sit back and receive $\frac{1}{P(T_{i-1}, T_i)}$ from the $T_i$-ZCB.

At a cost of $P(t, T_{i-1})$, this perfectly replicates the $\frac{1}{P(T_{i-1}, T_i)}$-cash-flow from the floating rate bullet. Hence the arbitrage-free price of cash-flow $c_i$ is $P(t, T_{i-1}) - P(t, T_i)$. The arbitrage-free price of the floating rate bullet is

$$\text{FlBull}(t) = \sum_{i=1}^{N-1} (P(t, T_{i-1}) - P(t, T_i)) + P(T_{N-1}) = P(t; T_0),$$

as the sum telescopes. In particular, if $t = T_0$ (“vi står på en terminsdato”) then the floating rate bullet has value 1. *The floating rate bond has par value.*
Note that this result is easily extended to any type of floating rate bond (eg. serial or annuity) with deterministic instalment plan.

If $H(T_i)$ denotes remaining principal and $A(T_i)$ is the principal repaid at time $T_i$ then

$$H(T_{i-1}) = \sum_{j=i}^{N} A(T_j).$$

The $T_i$-cash-flow from the bond is

$$c_i = A(T_i) + \delta L(\delta(T_{i-1}; T_{i-1}) H(T_{i-1}) = A(T_i) + \delta L(\delta(T_{i-1}; T_{i-1}) \sum_{j=i}^{N} A(T_j).$$

A portfolio with $A(T_i)$ units of the $T_i$-bullet has exactly the same cash-flows, and its price (assuming $t = T_0$) is $\sum_i A(T_i) = H(0)$. So the new bond has par value too.
A plain vanilla interest rate swap is contract that consists of

- A long position in a floating rate bullet (or however many $M$ you want as notional principal)

- A short position in a fixed rate bullet (say with fixed rate $\kappa$).

You can think of this as a contract that swaps floating rate interest payments for fixed rate payments (or vice versa).

The value of the swap contract is

$$V_{\text{swap}}(t) = P(t, T_0) - P(t; T_N) - \sum_{i=1}^{N} \delta \kappa P(t; T_i)$$
In practice this equation is used backwards (at the time of initiation of the swap) to set the fixed rate such that $V_{\text{swap}}(t) = 0$, ie.

$$
\kappa^*(t) = \frac{P(t, T_0) - P(t, T_N)}{\sum_{i=1}^{N} \delta P(t; T_i)}.
$$

This is called the (par) swap rate. Note that it is specific to the swap considered; you get different swap rates if you move $T_0$, $\delta$ or $N$ around.

The message is then:

- floating rate bonds trade at par

- swaps can be valued without a dynamic model (there’s no volatility dependence)
A couple of disclaimers/warnings:

It is very important for the “volatility independence” that you swap the exact right rate at the exact right time. Swapping the 6M LIBOR every 3rd month induces volatility dependence. So does moving payments to where they are first known. So-called convexity adjustments try to remedy that.

Swaps can be made a lot more exotic with all kinds of embedded option features & strange floating rates. Famous disaster: Proctor and Gamble vs. (literally) Bankers Trust. (Arguably, the problem here was not really the complexity, but the fact that P&G took a huge gamble on rates staying low.)
A caplet contract pays off

$$\delta(L_{\delta}(T_{i-1}, T_{i-1}) - \omega)^+$$ at time $T_i$

Owning a caplet can be thought of as having an insurance against paying high interest.

(A small calculation shows that) 1 caplet can be seen as $(1 + \delta \omega)$ expiry-$T_{i-1}$ strike-1/$(1 + \delta \omega)$ put-options with the $T_i$-ZCB as underlying. So we can price them in a Vasicek or multidimensional Gaussian model.

But there’s another way.
The time-\(t\) arbitrage-free price of a caplet is

\[
\pi_{\text{caplet}}(t) = \delta \beta(t) E_Q^t \left( \frac{(L_\delta(T_{i-1}, T_i-1) - \omega)^+}{\beta(T_i)} \right) \\
= \delta P(t, T_i) E_{t_i}^T ((L_\delta(T_{i-1}, T_i-1) - \omega)^+) 
\]

Recall that \(L_\delta(\cdot; T_{i-1})\) is a \(Q^{T_i}\) martingale.
So one way to specify an arbitrage-free model is as

\[ dL_\delta(t; T_{i-1}) = \gamma^\top(t; T_{i-1})L_\delta(t; T_{i-1})dW^{T_i}(t) \]  

for some deterministic (possible vector-valued) function \( \gamma \).

This is called the (lognormal) LIBOR market model.

Put

\[ v^2(t, T) = \int_t^T ||\gamma(u; T)||^2 du. \]

Then a standard B/S-like calculation (DIY) shows that

\[ \pi_{\text{caplet}}(t; T_{i-1}, \delta, \kappa) = \delta P(t, T_i) \left( L_\delta(t; T_{i-1}) \Phi (d_+) - \kappa \Phi (d_-) \right), \]

where \( d_\pm = (\ln(L_\delta(t; T_{i-1})/\kappa) \pm \frac{1}{2}v^2(t, T_{i-1}))/v(t, T_{i-1}). \)
If $\gamma$ is 1D & constant (in 1st argument) then $v^2 = \gamma^2(T_{i-1})T_{i-1}$ and the formula is the so-called Black's formula.

Other assumption: $\gamma(t, T) = \tilde{\gamma}(T - t)$ where $\tilde{\gamma}$ is piecewise constant. More natural.

Not clear what a reasonable volatility specification is.

A cap contract is a series of caplets; its price is simply the sum of caplet prices.

Market practice is & has been for may years to price – or at least quote – caps with the Black formula. Here is a formal, arbitrage-free model that supports this.
Papers with the model by [Miltersen, Sandmann, Sondermann], [Brace, Gatarek, Musiela] and [Jamshidian] appeared virtually simultaneously in 1997.

Immediate hit. Understandably so. Justifies what was being done & takes as input real observables.

Quoting prices in terms of Black-volatility does not actually mean that you believe in the lognormal model. Cap prices are quoted as “flat volatility”, i.e. the same constant $\gamma$ that when plugged into caplets & summed gives the price.
The models are actually more complicated than they look:

- Strange bond price dynamics

\[
\sigma_P(t, T) = -\sum_{k=1}^{\lfloor (T-t)/\delta \rfloor} \frac{\delta L_\delta(t, T-\delta k)}{1 + \delta L_\delta(t, T-\delta k)} \gamma(t, T - \delta k).
\]

Not a Markovian structure. So simulation requires a lot of bookkeeping.

- Lognormality is not preserved on measure changes. And if 3M LIBOR has lognormal volatility structure, then 6M LIBOR hasn’t.
• Hard to price anything that is not a cap.

• Requires considerable concentration to keep track of all necessary time-indices & integrations.

There is an extensive literature on market models. Nice recent articles by Pelsser, Driessen, deJong.
Another option-type contract is the swaption?

The time-$T_l$ value of the swaption (i.e. the swaption price at its expiry date) is

$$\pi^{swopt}(T_l; T_l, T_m, T_n, \delta, \omega) = \delta(\kappa(T_l; T_m, T_n, \delta) - \omega)^+ \sum_{j=m+1}^{n} P(T_l, T_j).$$
So for \( t < T_l \), the swaption price can be written as

\[
\pi^{\text{swopt}}(t; \ldots) = P(t; T_l) \mathbb{E}_t^{Q_{T_l}} \left( \delta(\kappa(T_l; T_m, T_n, \delta) - \omega)^+ \sum_{j=m+1}^{n} P(T_l, T_j) \right).
\]

If \( T_l = T_m \) then that we can rewrite the swaption pay-off as

\[
\left( 1 - \sum_{j=m+1}^{n} \alpha_j P(T_m, T_j) \right)^+,
\]

with \( \alpha_j = \delta \omega \) for \( j \leq n - 1 \) & \( \alpha_n = 1 + \delta \omega \). So the swaption is really a put option on a coupon-bearing bond. The ideas from earlier in the day was used by BGM to derive an approximate swaption-price formula in a lognormal LIBOR market model.
Put $X(t) = \delta \sum_{j=m+1}^{n} P(t; T_j)$. This a perfectly legitimate choice of numeraire, so it induces an equivalent martingale measure $Q^X$. Then

$$
\pi^{\text{swopt}}(t; T_l, T_m, T_n, \delta, \omega) = X(t) \mathbb{E}_t^{Q^X} \left( (\kappa(T_l; T_m, T_n, \delta) - \omega)^+ \right).
$$

and the process $\{\kappa(t; T_m, T_n, \delta)\}_t$ is a $Q^X$-martingale.

This, known as the swap-measure approach, can lead to Black-type formulas for swaptions.

A few calculations show that lognormal volatility of swap-rates is not consistent with lognormal LIBOR volatility.