Fact: Interest rates are stochastic
March 8-9: Björk’s Chapters 20-23 with some detours.

- A hedge experiment.

- I’ll drop Chapter 15 on incomplete models (for now).


- Concrete 1-dimensional models; Ch. 22. Tons of stuff, we can calculate.
  - The (mean-reverting) affine short rate models: Vasicek, Cox-Ingersoll-Ross.
  - Calibration and/or estimation.
March 22-23: Björk’s Chapters 24-25 and beyond.

- Change of numeraire. Needed for option-pricing. Ch. 24, but I won’t do it like that.

- Options on bonds
  - Zero coupon bonds; Ch. 24.
  - Coupon bonds and swaptions; trick & Ch. 25.

- Multi-factor models
  - Affine formalism (Duffie & Kan, Dai & Singleton)
  - Concrete examples (Brigo & Mercurio Ch. 4)
  - Swaption pricing: Unsolved problem (sort of; surprisingly).
After that I’ll leave you in the very capable hands of Fabio Mercurio. (Århus April 19-20.)

And in June Rama Cont will give a 3-day “crash course” on modeling (and option pricing) in with jumps. (Dates TBA)
What determines interest rates? And what moves them around?

- Agents' preferences for consuming now vs. saving for later.
- Supply and demand. Not to be forgotten.
- “Usual macroeconomic suspects”:
  - Growth rates.
  - Expected inflation.
  - Fiscal policy (in the hand of politicians).
  - Monetary policy (largely central banks nowadays).
  - International effects, exchange rates.
- Institutional structure (labor and housing market, legal and political system, …)
**Overload: Too much to model!**

Randomness is a very large component. Let’s just accept that and build empirically plausible stochastic models. Worked well for stocks.

Complication: Many assets (bonds) are different (because they pay at different times) but not *too* different.

That is what interest rate (or *term structure* or *fixed income*) modelling is about.
Fixed income markets are huge. DK government bonds at CSX: Around 1,200 billion ($1.2 \times 10^{12}$) DKK. About the same in mortgage-backed bonds. And that's only the securitized stuff. Add bank loans, credit cards, derivatives.

Nice: The martingale formalism, our fundamental theorems of asset pricing (absence of arbitrage, completeness, . . .) carries over.

We will be looking only at non-trivial special cases. (And the more special, the more non-trivial.)
Björk Chapters 20 and 23: Abstract/general theory.

$T$-maturity ZCB; time-$t$ price denoted $P(t; T)$. As a fct of $T$: Smooth. As a fct of $t$: Erratic (Ito-process).

Continuously compounded ZC yield $y(t, \tau)$ is defined by

$$P(t; t + \tau) = \exp(-\tau y(t; \tau)) \Leftrightarrow y(t; \tau) = -\frac{\ln P(t; t + \tau)}{\tau}.$$  

Note the shift from time of maturity to time to maturity.

Instantaneous forward rates (mathematically convenient)

$$f(t, T) = -\frac{\partial \ln P(t; T)}{\partial T}.$$
Interpretation. Why does this make sense? BLACKBOARD

The term structure of interest rates at date $t$ is the mapping

$$\tau \mapsto y(t; \tau)$$

or some translation of it (eg. into ZCB prices or forward rates). In theory this curve is observable in practice. In practice, well . . .

Short rate:

$$r(t) = f(t; t)$$

Bank account:

$$\beta(t) = \exp \left( \int_0^t r(s) ds \right),$$

so $d\beta(t) = r(t)\beta(t)dt$, and we say/note that this is a locally risk-free asset.
Beware of interest rate quotations: “What does 5% mean?”

(%, quotations. Per year?)
Discrete vs. continuous compounding.
Instantaneous vs. “simple” rates.
Yields on what?

Advice: Show me the money! (& a formula, if you don’t mind)
Dynamics equations (Björk equations 20.1-3)

Short rate ($\top$ means transposition)

\[ dr(t) = a(t)dt + b^\top(t)dW(t) \] (1)

ZCB prices (one eqn' for each $T$; note shift to prop. coefficients)

\[ dP(t; T) = m(t; T)P(t; T)dt + v^\top(t; T)P(t; T)dW(t) \] (2)

Forward rates

\[ df(t; T) = \alpha(t; T)dt + \sigma^\top(t; T)dW(t) \] (3)
No financial assumptions yet. $W$ is just BM under some measure.

Coefficients are adapted (vector-valued) process, but smooth in $T$; subscript-$T$ denotes $T$-differentiation.

We have

$$f(t; T) = -\frac{\partial \ln P(t; T)}{\partial T} \iff P(t; T) = \exp \left( - \int_t^T f(t; s) ds \right)$$

and

$$r(t) = f(t, t).$$

So what’s the connection?
Proposition 20.5 (Quite important result.)

1) If ZCB prices satisfy (2) then forward rates satisfy (3) with

\[ \alpha(t; T) = v^T_T(t; T)v(t; T) - m_T(t; T) \quad \text{and} \quad \sigma(t; T) = -v_T(t; T). \]

2) If forward rates satisfy (3) then the short rate satisfies (1) with

\[ a(t) = f_T(t, t) + \alpha(t, t) \quad \text{and} \quad b(t) = \sigma(t; t). \]

3) If forward rates satisfy (3) then ZCB prices satisfy (2) with

\[ m(t; T) = r(t) + A(t; T) + \frac{1}{2}S^T(t; T)S(t; T) \quad \text{and} \quad v(t; T) = S(t; T), \]
where $A(t; T) = -\int_t^T \alpha(t; s)ds$ and $S(t; T) = -\int_t^T \sigma(t; s)ds$.

You’ll forget terms if you aren’t careful, so let’s look at a proof.

1): Take logs, use Ito & differentiate wrt. $T$

The complication with the last two statements is that we have $t$ appearing both under the integral sign and in the limit.

Recall the Leibniz rule (where $h : \mathbb{R}^2 \mapsto \mathbb{R}$ is a smooth function)

$$\frac{d}{dx} \int_0^x h(t, x)dx = h(x, x) + \int_0^x h_x(t, x)dx.$$
2): \( r(t) = f(t; T) \), but by Leibniz, we’re inspired to write

\[
dr(t) = d_t f(t; T)|_{T=t} + \left( \underbrace{d_T f(t; T)}_{T=t} \right) |_{T=t} = f_T(t; T) dt
\]

and the result follows.

Björk has a (real) proof on integral form — except his “changing the order and identifying” may leave a little too much to the reader. Details on homepage.
3): \( P(t; T) = \exp \left( - \int_t^T f(t; s) ds \right) \), so \( t \) enters in two places in an even trickier way.

Björk gives “a heuristic proof”; even if I wanted to, I can’t repeat the arguments in a way that sounds convincing.

So let me sketch a proof. BLACKBOARD and details on homepage.
An application: The HJM drift condition

Assume that the model of forward rates is given by (3) under some measure $P$. Suppose further that the model is arbitrage-free.

Then there exists an equivalent martingale measure $Q \sim P$ such that

$$\frac{P(t; T)}{\beta(t)}$$

is a $Q$-martingale for all $T$.

So

$$dP(t; T) = r(t)P(t; T)dt + S^\top(t; T)P(t; T)dW^Q(t).$$
Note the subtle application of Girsanov's theorem: Equivalent changes of measure change drift – not volatility.

But from Proposition 20.5 3) we get

\[ r(t) = r(t) + A^Q(t; T) + \frac{1}{2} S^\top(t; T) S(t; T) \Rightarrow -A^Q(t; T) = \frac{1}{2} S^\top(t; T) S(t; T). \]

Differentiate both sides wrt. \( T \) and get the Heath-Jarrow-Morton drift condition

\[ \alpha^Q(t; T) = \sigma^\top(t; T) \int_t^T \sigma(t; s) ds. \]
But what about drifts under $P$?

From Girsanov’s theorem we know that there exists a stochastic process $\lambda$ such that

$$dW^Q = dW^P - \lambda(t)dt$$

defines a $Q$-BM.

Important: $\lambda$ doesn’t depend on $T$.

Not important: Whether I choose to write “+” or “-”.

We have

$$\frac{dP(t; T)}{P(t; T)} = (r(t) + A^P(t; T) + \frac{1}{2}S^\top(t; T)S(t; T))dt + S^\top(t; T)dW^P(t)$$

$$= r(t)dt + S^\top(t; T)dW^Q(t) + \left(A^P(t; T) + \frac{1}{2}S^\top(t; T)S(t; T) + S^\top(t; T)\lambda(t)\right)dt.$$

The term in the curly braces must $= 0$. 
This means that 

\[-A^P(t; T) = \frac{1}{2} S^\top(t; T) S(t; T) + S^\top(t; T) \lambda(t)\].

Differentiating wrt. \( T \) gives

\[
\alpha^P(t; T) = \sigma^\top(t; T) \left( \int_t^T \sigma(t; s) - \lambda(t) \right).
\]

In sloppy matrix notation we may write

\[
\lambda(t) = -\frac{E^P(\text{return on ZCB}) - r(t)}{\text{Vol}(ZCB)}.
\]

If \( \sigma \) (forward rate volatility) is chosen positive then (typically) \( \lambda(t) \) will be positive. Otherwise it’s negative.
Another application (& exercise): HJM and the Markov-property

How much has Bjarne said about the Markov property?

The drift condition makes the dynamics of one forward rates dependent on all other forward rates ⇒ Non-markovian,

But sometimes the models are Markovian.

and here I solve the exercise

There's more literature on this where people do cunning stuff.
A third application: Musiela formulation/parametrization

Change to time to maturity in forward rates: \( r(t, x) = f(t; t + x) \). “How rates are quoted” (practice) and we get an object that lives on a rectangular domain (mathematical).

By Leibniz

\[
\begin{align*}
dr(t, x) &= dt f(t; T)|_{T=t+x} + dT f(t; T)|_{T=t+x} = df(t, t + x) + \frac{\partial}{\partial x} r(t, x),
\end{align*}
\]

and in fancy notation the drift condition gives

\[
\begin{align*}
dr(t, x) &= (\mathcal{F} r(t, x) + D(t, x)) dt + \underbrace{\sigma_0(t, x)}_{=\sigma(t, t+x)} dW^Q(t).
\end{align*}
\]