Hedging with a (Possibly) Wrong Volatility

In this exercise we consider a general arbitrage-free stock price model in a world where there exists a bank-account on which the interest rate is 0, i.e.

\[ dS(t) = \sigma(t)S(t)dW^Q(t), \]

where \( \sigma \) is an arbitrary (up to regularity conditions, that you needn’t worry about) stochastic process.

Consider a simple contingent claim on \( S \) with pay-off function \( g \), i.e. it pays \( g(S(T)) \) to the holder at time \( T \).

Let \( \tilde{\sigma} \in \mathbb{R} \) be given and define the function \( F : [0; T] \times \mathbb{R} \to \mathbb{R} \) as the solution to the (familiar) partial differential equation (subscripts denote differentiation)

\[ F_t(t, x) + \frac{1}{2} \tilde{\sigma}^2 x^2 F_{xx}(t, x) = 0 \quad \text{for } t < T, \quad F(T, x) = g(x). \]

Consider, finally, a trading strategy \( h \) that holds \( h_1(t) = F_x(t, S(t)) \) units of the stock and \( h_2(t) = F(t, S(t)) - S(t)F_x(t, S(t)) \) units of the bank account at time \( t \).

Show that this trading strategy replicates the pay-off of the \( g \)-claim, i.e. that its value process, say \( V^h \), satisfies \( V^h(T) = g(S(T)) \).

Show that the self-financing condition for this trading strategy boils down to the equation

\[ \frac{1}{2}(\sigma^2(t) - \tilde{\sigma}^2)S^2(t)F_{xx}(t, S(t)) = 0 \quad (1) \]

holding (almost everywhere, in an appropriate sense, that you needn’t worry about).

Argue that “usual results” are obtained when \( \sigma(t) = \tilde{\sigma} \), and that in general the \( h \)-strategy can be interpreted as “trying to replicate as if it were the Black/Scholes
model”.

Remark: The result in (1) may alternatively be formulated by saying that \( h \) has an extra financing need of

\[
\frac{1}{2} \int_0^T \left( \sigma^2(t) - \tilde{\sigma}^2(t) \right) S_x(t) F_{xx}(t, S(t)) dt,
\]

and is sometimes called the “1st fundamental theorem of derivative trading”. It has consequences for hedging in misspecified models: If we consider a convex claim (in the sense that \( F_{xx} > 0 \)) and if there is an upper bound on the \( \sigma \)-process, then a super-replicating strategy is achieved by “Black/Scholes \( \Delta \)-hedging with the upper bound”.

### Markovian Representation of HJM-models

Consider a Heath-Jarrow-Morton setup as in Björk chapters 20 and 23. For notational clarification we write \( \sigma_f(t, T) \) for forward rate volatilities and \( \sigma_P(t, T) \) for zero-coupon bond price volatilities. Furthermore we assume dynamics to be driven by a 1-dimensional Brownian motion and the we work directly under an equivalent martingale measure \( Q \), meaning that we are implicitly assuming the model to be arbitrage-free.

Show that

\[
\sigma_P(t, T) = -\int_t^T \sigma_f(t, u) du. \tag{2}
\]

Show that

\[
f(t, T) - f(0, T) = -\int_0^t \sigma_f(s, T) \sigma_P(s, T) ds + \int_0^t \sigma_f(s, T) dW(s). \tag{3}
\]

Assume that

\[
\sigma_f(t, T) = \sigma e^{-\kappa(T-t)},
\]

where \( \sigma \) and \( \kappa \) are (positive) constants. Differentiate (3) wrt. \( T \) (indicated by “subscript \( T \)” and use Leibniz’ rule (ie. “differentiate (under) an integral”), equation (2), and equation (3) “read from right to left” to show that

\[
f_T(t, T) - f_T(0, T) = \int_0^t \sigma^2_f(s, T) ds + \kappa(f(0, T) - f(t, T)) \quad \text{for all} \quad T. \tag{4}
\]
Put 
\[ \phi(t) = \int_0^t \sigma_r^2(s,t)ds. \]
(Of course \( \phi \) can then be found more explicitly, but there is little need to.) Evaluate (4) at “\( T = t \)” and use Björk’s Proposition 20.5 to show that the short rate dynamics are
\[ dr(t) = \left( \kappa(f(0,t) - r(t)) + \phi(t) + f_T(0,t) \right) dt + \sigma dW(t). \]  
(5)

Suppose that we generalize the form of the forward rate volatility to
\[ \sigma_f(t, T) = \sigma(r(t)) e^{-\int_t^T \kappa(u) du}, \]  
(6)
where \( \sigma \) and \( \kappa \) are deterministic functions. (And keep the previous definition of \( \phi \).) Show that in this case we have
\[ dr(t) = \left( \kappa(t)(f(0,t) - r(t)) + \phi(t) + f_T(0,t) \right) dt + \sigma(r(t))dW(t). \]  
(7)
(Your old calculations should almost all carry over verbatim.) Use Leibniz’ rule to show that
\[ d\phi(t) = (\sigma^2(r(t)) - 2\kappa(t)\phi(t))dt. \]  
(8)
Equations (7)-(8) show that under an assumption of multiplicative separability of the forward rate volatility (ie. (6)), the 2-dimensional process \((r, \phi)\) is Markovian wrt. its own filtration. (And that the 1-dimensional process \( r \) might not be.)

**Long Rates and Modelling (In)Consistency**

Björk’s exercise 22.7.

Remark: Björk says that “obviously the limit will depend on \( r(t) \) and \( t \)”. To me it seems more obvious that the limit – if it exists – does not depend on \( r(t) \) and \( t \). Neither is true, but it can be shown that if the process \( f^\infty(t) := \lim_{T \to \infty} f(t,T) \) is well-defined, then it is increasing. And that there are models – fairly strange ones, though – where the limit depends non-trivially on \( r(t) \).