In a simplified setup with two assets, a money market account (or bank account) and a risky asset:

\[ dM_t = rM_t dt; \quad M_0 = 1 \]  \hspace{1cm} (1)

The risky asset can be described by the price process \( S_t \) developing in accordance with the lognormal process

\[ dS_t = \alpha S_t dt + \sigma S_t d\zeta_t; \quad S_0 \text{ given} \]  \hspace{1cm} (2)

The investor's (financial) wealth at any given point in time \( t \) is defined by his portfolio position \( (h_0(t), h_1(t)) \):

\[ V_t = h_0(t)M_t + h_1(t)S_t \]  \hspace{1cm} (3)

(3) is a valid snapshot for the financial balance at any point in time, but it is also the result of a sequence of decisions made that are followed by realizations of market price changes. A more elaborate expression is as follows:

\[ V_t = \underbrace{h_0(t - \Delta t)M_t + h_1(t - \Delta t)S_t}_{\text{Entering wealth}} + \underbrace{h_0(t)M_t + h_1(t)S_t + C(t)\Delta t}_{\text{Exiting wealth}} \]  \hspace{1cm} (4)

where \( C(t)\Delta t \) is the consumption expenditure planned for the period \( (t, t + \Delta t) \). This decision must be made at the beginning of the period — a so-called predictable process.
\( C(t) \) could be labour income (for the period just passed) or dividend payments from assets (for the period just passed), in which case

- the sign is obviously reversed (“negative consumption”)
- you enter into the new period with payment known “in advance”

By leading this relation one time step we get

\[
V_{t+\Delta t} = h_0(t)M_{t+\Delta t} + h_1(t)S_{t+\Delta t}
\]  

(5)

Upon subtraction of (4) from (5) we have the relation

\[
V_{t+\Delta t} - V_t = h_0(t)[M_{t+\Delta t} - M_t] + h_1(t)[S_{t+\Delta t} - S_t] \\
- M_t[h_{0}(t - \Delta t) - h_{0}(t)] - S_t[h_{1}(t - \Delta t) - h_{1}(t)]
\]  

(6)

It would be tempting to let \( \Delta t \to 0 \) and just get the limiting result

\[
dV_t = h_0(t)dM_t + h_1(t)dS_t - h_0(t)dM_t - h_1(t)dS_t
\]

\[
= h_0(t)rM_tdt + h_1(t)dS_t - M_tdh_0(t) - S_tdh_1(t)
\]  

(7)

and combine this with (4) to get

\[
C(t)dt = -M_tdh_0(t) - S_tdh_1(t)
\]  

(8)

This limiting form of the budget dynamics is actually true in a deterministic setting, where the result is

\[
dV_t = h_0(t)rM_tdt + h_1(t)dS_t - C(t)dt
\]  

(9)
However, it is **not true** in a continuous time stochastic environment. This is due to the fact that stochastic calculus deals with **forward oriented increments**, and that such limiting operations must be done with great care. The terms $M_t [N_0(t - \Delta t) - N_0(t)]$ and $S_t [h_1(t - \Delta t) - h_1(t)]$ are approximating terms in a limiting operation converging to a stochastic integral, and it is clear that $M_t$ and $S_t$ are weighted by **backward oriented increments**. This must be corrected.

Rewriting the budget dynamics in the following manner corrects this problem:

$$V_{t+\Delta t} - V_t = h_0(t) [M_{t+\Delta t} - M_t] + h_1(t) [S_{t+\Delta t} - S_t]$$
$$+ [M_t - M_{t-\Delta t}] [h_0(t) - h_0(t - \Delta t)] + [S_t - S_{t-\Delta t}] [h_1(t) - h_1(t - \Delta t)]$$
$$+ M_{t-\Delta t} [h_0(t) - h_0(t - \Delta t)] + S_{t-\Delta t} [h_1(t) - h_1(t - \Delta t)]$$

(10)

because the cross product terms are producing $dt$-terms in the limit that catch up covariance properties between positions and price movements and not a stochastic integral term. As a matter of fact, the cross product term related to the bank account vanishes in the limit — the $M$-part is in itself non-stochastic and cannot produce any covariance term.

The definition of a self-financing portfolio is at odds with the literature in general. If $c(t)$ is interpreted as consumption, (6.11) is a budget restriction. The definition is only valid if $c(t)$ refers to dividends (i.e. a negative value of $c(t)$) coming from the assets themselves, does the definition make sense. But this is obviously not what Björk has in mind.