Exercises for
Asset Pricing II
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Exercise no. 1 – Girsanov's theorem

Consider the Feynman-Kac formula as given in various versions in Bjørk, propositions 5.6-5.9.

Imagine that the PDE in question is

$$\frac{\partial F}{\partial t}(t,x)+\psi(t,x)\frac{\partial F}{\partial x}(t,x)+\frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial t^2}(t,x)-rF(t,x)=0 \tag{1}$$

with terminal condition

$$F(T,x)=\Phi(x)$$

I.e. the drift term $\mu(t,x)$ is changed relative to (5.22)-(5.25) and (5.34)-(5.38).

Assume furthermore that you *know* the solution (5.36) and want to use the distribution function already known from (5.37). Define $\lambda(s,X_s)$ as

$$\lambda(s,X_s)\equiv\frac{\mu(s,X_s)-\psi(s,X_s)}{\sigma(s,X_s)}$$

Assume that $\lambda(s,X_s)$ is sufficiently nicely behaved in order for the integrals below to be well defined in a sufficiently nice manner. Show that the solution to the PDE (1) can be written as

$$F(t,x)=e^{-r(T-t)}E_{\{t,x\}}\left[e^{-\int_t^T\lambda(s,X_s)dW_s-\frac{1}{2}\int_t^T\lambda(s,X_s)^2ds}\Phi(X_T)\right]$$

Exercise no. 2 – Black-Scholes and the Greeks

Consider the Black-Scholes formula:

$$\begin{aligned}C(S,t) &= S_tN(d)-Xe^{-r(T-t)}N(d-\sigma\sqrt{T-t})\\ d &= \frac{\log(S_t/X)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\end{aligned}$$

1. Prove that
$$C(S,t)\to\max\{0,S-X\}\text{ as }T\to t$$

2. Find the partial derivatives C'_S, C''_{SS} and C'_t.
Hint: $\frac{\partial C}{\partial d}$=0 !

3. Show that $C(S,t)$ satisfies the PDE.

4. It is known that the Δ in the hedging portfolio is $\Delta(S,t)=C'_S$. Use the PDE for $C(S,t)$ to find a similar PDE for $\Delta(S,t)$. Solve it!
Hint: Assume that all regularity conditions in order to be able to interchange the order of partial differentiation w.r.t. S and t are satisfied.

5. Is $\Delta(S,t)$ a martingale in any other diffusion process for S, generated by a parallel shift of the drift term? Or stated differently, can $\Delta(S,t)$ be made a martingale by a suitable choice of *numeraire*?

6. Find the PDE for $\Gamma\equiv C''_{SS}$ and solve it.
Hint: The terminal condition is a so-called *Dirac delta function*:
$$\Gamma(S,T)=\delta_X(S),\ \delta_X(S)=0\text{ for }S\neq X,\ \delta_X(X)=\infty$$

interpreted as:

$$\int_0^\infty \delta_X(S)dS = 1$$

This can be approximated by e.g. $\Gamma(S,T) = \frac{1}{2\Delta S} 1_{S\in[X-\Delta S, X+\Delta S]}$, from where the solution in question can be obtained as a limiting case for $\Delta S \to 0$.

<u>Exercise no. 3 – Arithmetic Brownian motion and option prices</u>

In Bachelier's original thesis from 1900, submitted to Université de Paris, the modern option pricing theory, including the Brownian Motion, was developed to a surprising extent. He "only" missed the arbitrage argument and calculated instead just expected values. (Reminder: The origin of probability theory is clearly found in real life casino and lottery problems and developed in France. Bachelier's thesis was entitled *Theorie de la Speculation*.)

The following problem was solved by Bachelier. It reappeared in a modern context in Brennan (1979) and in Stapleton (1980).

Consider an asset developing according to

$$dS(t) = \mu dt + \sigma dW_t$$

1. What is the solution to this SDE?
2. Let this derivative be a European call option with expiration date T and exercise price X. Assuming a constant instantaneous interest rate r, write down the PDE and the boundary condition for this option.
3. Assume that $r = 0$. Prove that the price of this call option becomes

$$\begin{aligned} C(S,t) &= (S(t)-X)N(d_+) + \sigma\sqrt{T-t}N'(d_-) \\ d_{+/-} &= \pm\frac{S_0 - X}{\sigma\sqrt{T-t}} \end{aligned}$$

   where N is the cumulative distribution function of the standard normal and $N^{'}$ the corresponding density function.

4. Find in the same manner the price of the option for a positive interest rate by the procedure outlined in the lectures:
   - Write down the PDE
   - Find the end value distribution of $\tilde{S}$
   - Calculate the relevant expected value
   - Check the result by finding the derivatives and make sure that the solution is actually a solution to the PDE.

<u>Exercise no. 4 – The time-invariant PDE</u>

Consider the following specific version of the fundamental partial differential equation that does not have a specific term $C_t^{'}$ relating to calendar time, and for which volatility is constant:

$$\frac{1}{2}C_{SS}^{''}\sigma^2 S^2 + r[C_S^{'}S - C] = 0$$

S is understood to be a traded asset. Prove that the solution to this PDE has the form

$$C(S) = HS + KS^{-\psi}$$

1. Determine the parameter ψ. The parameters H and K must be determined by some boundary condition. For S itself, self-evidently $H=1$ and $K=0$.

2. Consider a certain given lower threshold value $\underline{S}$. Consider an asset that pays $\alpha\underline{S}$ if S hits $\underline{S}$. Show that

$$C(S) = \alpha S^{-\Psi}\underline{S}^{1+\Psi} = \alpha\underline{S}\,(S/\underline{S})^{-\Psi}$$

by using the boundary conditions $C(S) \to \alpha S$ for $S \to \underline{S}$ and $C(S) \to 0$ for $S \to \infty$.

Imagine that the derivative is paying a continuous "dividend stream" at the rate of d_C.

3. What would change in the PDE?

4. Show that the revised PDE has a solution of the form

$$C(S) = G + HS + KS^{-\Psi}$$

with G, H and K determined by boundary conditions.

5. Consider a perpetuity paying d_C. This perpetuity ceases to pay when S hits the threshold value $\underline{S}$, in which case the owner gets $(1-\alpha)\underline{S}$. When $S \to \infty$ the value of the perpetuity behaves like $C(S) = d_C/r$. Show that the value of this perpetuity is

$$C(S) = (d_C/r) \cdot \left[1 - (S/\underline{S})^{-\Psi}\right] + [(1-\alpha)\underline{S}] \cdot (S/\underline{S})^{-\Psi}$$

6. Interpret this as an expected value.

Exercise no. 5 – Deposit insurance and option theory

In this exercise option pricing theory is applied in order to analyze the price setting of deposits subject to default risk.

At time 0 a bank has invested V kr. in a number of financial activities. The value of the entire portfolio of activities is assumed to follow a process compatible with the assumptions behind the Black-Scholes model.

By changing the composition of the portfolio the management of the bank can change the volatility σ. However, the regulatory authority puts limits on the possible changes: $0 \leq \sigma \leq \overline{\sigma}$.

$\sigma = 0$ means that the entire portfolio is invested in risk free assets.

The funding of the bank – total value $V = 100$ – stems from one-year fixed rate deposits in the amount of $I = 90$ and from equity $S = 10$. The balance of the bank is as follows:

Assets	Liabilities
V=100	I=90
	S=10

No dividends or other payments to the owners take place during the coming year, nor will the bank receive further deposits. The risk free one year rate is assumed to be 10%, continuously compounded.[1]

The fixed rate deposits are not risk free, and depositors are aware of this fact. In order to compensate for this, depositors demand a deposit rate $\tilde{R}$ above 10%. Hence, depositors have a claim of $90 \cdot e^{\tilde{R}}$ against the bank after one year.

[1] Continuous compounding is applied in order to fit into the scenario of the Black-Scholes model.

1. Explain why the depositors' claim on the bank after one year can be perceived of as a combination of a risk free asset and a sold put with the bank's total market value (V) as underlying asset. (Draw a diagram and explain that the depositors' claim after one year is $\min\{V, 90 \cdot e^{\tilde{R}}\}$.)

2. Explain by means of the put-call parity that the shareholders have a call option on the bank's assets.

3. How large should $\tilde{R}$ be in order for the bank to attract deposits? You can assume that the depositors trust that the volatility in question ($\sigma = 10\%$ p.a.) will *not* be changed *after* the deposits have been made.
   Hint: Depositors are paying 90 and demand $\tilde{R}$ so large, that

   $$90 = 100 - C(100, 90 \cdot e^{\tilde{R}})$$

   Use Black-Scholes' formula together with a table of the cumulative normal distribution. There are also a number of polynomial approximations around with high degree of precision (actually, tables are made from such polynomial approximations.)

Assume now that the management of the bank can choose an asset portfolio within the boundaries $0 \leq \sigma \leq 0{,}30$ without being subjected to intervention from the regulatory authority.

4. At what value of σ is the value of the equity maximized? (No calculations required!)

5. What is the value of the equity and the deposits, resp., given the investment policy that maximizes the value of the equity?

6. Assume now that depositors have correctly foreseen this way of acting, where the management of the bank can extract some of the value of the deposits by a change in investment policy.

   What would the required deposit rate be under such circumstances?

Assume now that the government establishes a deposit insurance in order to protect depositors. The guarantee covers the depositors in full, accrued interest included, given that the bank is unable to meet its obligations.

7. What would the required deposit rate be under such circumstances?

8. How does the management maximize the value of the equity under such circumstances, and what is the maximum value of the equity?

9. What is the value of the deposit insurance from the point of view of the shareholders?

Exercise no. 6 – Life-time consumption and saving (Take home problem 2000)

An investor with initial wealth V_0 wants to plan a consumption- and investment policy over the time horizon $[0, T]$. The investment possibilities consist of

- a bank account with an infinitesimal rate of interest r
- a portfolio of shares S_t, whose price process follows a Geometric Brownian Motion:

  $$dS_t = \mu S_t dt + \sigma S_t dZ_t$$

  The inital value of the portfolio is normalized so that $S_0 = V_0$.

The investor wants to be able

- to consume continuously a fixed fraction ρ of his wealth V_t, $t \in [0,T]$, i.e. $c_t = \rho V_t$
- and simultaneously that the end-of-period wealth follows the development in S_T with a so-called “floor constraint”:
$$V_T = \max\{S_0, kS_T\}$$
for a fraction $k \in (0,1)$.

1. State the PDE which for given values of ρ og k determines the arbitrage-free initial value of that portfolio, which exactly finances the desired consumption stream and- simultaneously - fulfills the end-of-period restriction for V_T.
2. Solve this PDE.
3. For *arbitrarily* chosen values of ρ og k this initial value will *not* equal V_0. Determine a relation between ρ and k (and possibly other parameters), which ρ and k must satisfy in order for the initial value to be V_0.
4. Show that for a given value of $\rho < r$ a value k exists so that the required constraints on the consumption- and investment policy can be fulfilled.

Exercise no. 7 – Portfolio insurance (Take home problem 2001)

Philip Dorset is a 45 year old man has just inherited 1 mio. kr. Realizing that his pension saving is inadequate he decides to invest his inheritance in an index linked savings arrangement, where he can choose the underlying portfolio. Provided that his savings is tied up for at least 20 years the return on his saving is tax-exempt.

He is convinced at the outset that a stock portfolio will outperform a bond portfolio for the horizon in question. He also remembers from an undergraduate finance course at the Business School that investors should diversify and buy the market portfolio. Hence, he chooses to invest in an index fund tracking closely the development in the true market portfolio. This index portfolio has an expected return of 10% p.a. (“continuously compounded”) and a volatility of 20% p.a. I.e., the development in the value of the index fund follows the stochastic differential equation

$$dS_t = 0.1S_t dt + 0.2S_t dz_t$$

where one unit of time in the model corresponds to one calendar year. Thr riskless rate of interest is 5% p.a. (“continuously compounded”).

1. Compare with an alternative investment strategy, where all the money is invested in riskless assets. What is the probability that the value of the index fund after 20 years is larger than the value of this alternative riskless investment strategy? (It is *not* expected that you find an exact value of this probability. It is fully sufficient to state this probability in terms of the value of a fractile in a known probability distribution.)

Philip realizes that by investing 100% in the index fund a “downside risk” exists. Believing in future low inflation rates he is content with knowing that his savings is worth at least 1 mio. kr. in 20 years. His bank suggest to him that he invests the necessary amount in risk-free 20 year zero coupon bonds that the bank itself has issued. The remaining wealth can then be invested in the index fund.

2. How much should be invested in these zero coupon bonds?

3. Under what circumstances is the value of this mixed investment strategy larger than the value of a pure index fund investment strategy? And with what probability? (Again it is fully sufficient to state the result as a fractile in a known probability distribution).

Philip has also seen advertisements from investment banks promoting a financial product called "portfolio insurance". Portfolio insurance involves using part of his money to buy a put option with exercise price 1 mio. kr. and invest the remaining wealth, after paying for the put option, in the index fund. In this way the underlying asset becomes the position the index fund. Also, the downside risk is eliminated, but the upside potential remains.

4. Formulate an equation with one unknown to determine the magnitude of the put option premium. (Hint: Philip's portfolio position is a derivative of the index fund.)

5. Choose one of the following two questions:

   (a) Solve for this put option premium numerically using your favourite spreadsheet or other numerical solution software.

   (b) Show that there is a unique solution to the equation.

Assume now that the index fund is paying dividends continuously at the rate 2% p.a. of actual market value.

6. What would be the necessary changes in the above equation under the assumption that Philip is reinvesting the dividends instantaneously in the index fund? You are not supposed to do any numerical calculations or proofs of uniqueness. However, indicate clearly how you would proceede if you were to do so.

Exercise no. 8 – (Take home problem 2003)

It is an unsettled empirical issue whether share prices and/or share returns follow mean-reverting processes or not. Consider the following three modifications of the Black-Scholes model and its lognormal distribution. We assume that the continuously compounded risk-free rate of interest r is constant:

$$\begin{aligned}
\text{Model I}: \quad dS_t &= (a - \rho S_t)dt + \sigma S_t dB_t \\
\text{Model II}: \quad dy_t &= (a - \rho y_t)dt + \sigma dB_t \\
S_t &\equiv \exp(y_t) \\
\text{Model III}: \quad dS_t &= (r + \lambda_t \sigma) S_t dt + \sigma S_t dB_t^1 \\
d\lambda_t &= a(\bar{\lambda} - \lambda_t)dt + v dB_t^2
\end{aligned}$$

In Model III, λ_t is the risk-premium which follows a mean-reverting process. There are two Brownian motions, B_t^1 and B_t^2, resp., which may be correlated.

In the following questions you are asked for answers on an **operational** level; we simply assume that more profound regularity requirements associated with Girsanov's theorem are fulfilled.

1. Demonstrate that the process in Model I is neither a normal model nor a lognormal model.

2. Find an analytical expression for the solution to S_t in Model II.

3. Demonstrate that the process for S_t in Model III is lognormal.

4. Find, if possible, the price of a standard European call-option in these three models. Explain why or why not you are able to find an analytical price for the standard options in these three models.

5. Suppose you want to formulate the fundamental PDE in terms of y in Model II instead of in terms of S. Write down this PDE.

6. Suppose next the above processes originate from the dynamics of an exchange rate, where the "foreign currency" pays a continuously compounded interest rate r^*. More precisely, S_t is supposed to be the balance in terms of DKK on a foreign bank account, where all interest payments are assumed accumulated (reinvested) in the account. You are asked to derive

   (i) the processes for the exchange rate itself in Models I, II and III

   (ii) the call option price (if possible, cf. question 4) for a European call option with the exchange rate itself as the underlying asset.

References

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