THE EQUATION \( \sigma_p (s, t) = \int_t^T c(s, u) dW(u) \)

Follows from Th 18. in Brock (The Asset Pricing Model).

We know that under friction:

\[ dP(t, T) = r(t) P(t, T) dt + \sigma_p (t, T) P(t, T) dW^q(t) \]

This fits into Prop 15.5 with:

\[ m(t, T)'' = r(t) \]
\[ \nu(t, T)'' = \sigma_p (t, T) \]

This means that \( m'' = 0 \) and \( \nu'' = -\sigma_q \)

Hence from (1) in Prop 16.5

\[ d[f(t, T)] = -\sigma_q (t, T) \sigma_p (t, T) dt + \sigma_q (t, T) dW^q(t) \]

Or on differential form

\( \frac{d}{dt} [f(t, T) - f(0, T)] = -\int_0^t \sigma_q (s, T) \sigma_p (s, T) ds + \int_0^t \sigma_q (s, T) dW^q(s) \)
Now we assume

\[ \sigma(T,t) = \sigma e^{-k(t-T)} \]

\[ \text{Diff (**) w.r.t. } T \text{ a integrating.} \]

\[ \frac{g_T(t,T) - g_T(0,T)}{h_1} = - \int_0^t \left( \sigma_T(s,T) \sigma_T(s,T) + \sigma_T(s,T) \sigma_T(s,T) \right) ds \]

\[ + \int_0^t \left( \frac{\partial}{\partial T} \sigma_T(s,T) \right) dW^9(s) \]  (eq. 2)

We have

\[ \frac{\partial}{\partial T} \sigma_T(s,T) = - \sigma_T(s,T) \]

AND

\[ \frac{\partial}{\partial T} \sigma_T(s,T) = - k \sigma_T(s,T) = - k \sigma_T(s,T) \]

So close the calculus (eq. 2)

(eq. 2) = \[ \int_0^t \sigma_T^2(s,T) ds \]

\[ + k \left( \int_0^s \sigma_T(s,T) \sigma_T(s,T) dW^9(s) - \int_0^t \sigma_T(s,T) dW^9(s) \right) \]

\[ = - (g(t,T) - g(0,T)) \text{ by (eq. 1))} \]
\[ \mathcal{S}_T (l, t) - \mathcal{S}_T (0, t) = \int_0^t \sigma_g (s, t) \, ds + K (\mathcal{S}_T (0, t) - \mathcal{S}_T (l, t)) \]  

**Put** \( \phi (t) = \int_0^t \sigma_g (s, t) \, ds \)

**Setting** \( T = t \) in (43) gives

\[ \mathcal{S}_T (l, t) = \mathcal{S}_T (0, t) + \phi (t) + K (\mathcal{S}_T (0, t) - \mathcal{S}_T (l, t)) \]

**Apply (2)** from Prop. 15.5 and note that

\[ "d (l, t)" = - \sigma_g (l, t) \frac{\partial \mathcal{S}_T (l, t)}{\partial l} = 0 \]

\[ "\sigma (l, t)" = \sigma_g (l, t) \frac{\partial \mathcal{S}_T (l, t)}{\partial l} = 0 \]

\[ \mathcal{S}_T (l, t) \text{ given by (44)} \]

**To get**

\[ d \mathcal{S}_T (l, t) = \left[ K (\mathcal{S}_T (l, t) - r(t)) + \phi (t) - \mathcal{S}_T (l, t) \right] dt + \sigma \, dW \]
In this case

\[ \mathrm{d}X(t) = M(t, r(t)) \, \mathrm{d}t + b(t, r(t)) \, \mathrm{d}W(t) \]

for PCR's \( M = b \) ("only dep on \( r(t) \)
- and not "things" in drift & vol.")

Hence \( r \) is Markovian w.r.t. its own filtration

I.o.w. the dist.

\[ F_{\tau \mid S} \] depends on on \( S \)'s_value.

This means that a one approach can be
used to price derivatives.

When/If

\[ \sigma(t, \tau) = \sigma(r(t)) \, e^{-\int_0^T \mu(u) \, \mathrm{d}u} \]

The exact same calculations show that

\[ \mathrm{d}R(t) = \int k(t) \left( \mathcal{F}_t - r(t) \right) + \phi(t) + \mathcal{F}_t \, \mathrm{d}W(t) \]
\[ \Phi(t) = \int_0^t \sigma^2(r(s)) \Theta^2 \int_s^t k(u) \, du \, ds \]

Note that \( \Phi \) isn't deterministic; it captures some path dependence. For \( \Phi \) is not Markovian with its own filtration.

But from Leibniz's Rule

\[ d \Phi(t) = \left( \sigma^2(r(t)) - 2 \int_0^t k(u) \, du \right) \, dA \]

So the 2-Dim System \((r, \Phi)\) is Markovian.

And a PDB approach with an extra state variable (to which no 2\textsuperscript{nd} derivatives are connected) can be used.