About the problems

- You are allowed to discuss the problems with your fellow students in the course but please hand in individual answers.

- In the simulation exercise, make sure that you make the answer self-contained by cutting and pasting in the relevant figures and - if you need it - tables.

- A satisfactory answer to all problem sets in the course is a requirement for passing the course. You should try to answer all problems. The problem set is accepted if 75% is correct. Each problem carries the same weight.

Problem 1

Consider the stochastic differential equation (SDE)

\[ dY(t) = \sigma Y(t)dW(t) \]  \hspace{1cm} (1)

where \( W \) is a standard one-dimensional Brownian motion. Assume that \( Y(0) = 1 \) and for the numerical exercise let \( \sigma = 0.25 \). If we did not know of the strange properties of Brownian sample paths and used the theory of ordinary differential equations, we might think that the solution to (1) is given by

\[ Y(t) = \exp(\sigma W(t)) \]  \hspace{1cm} (2)

1. Show that this solution would indeed hold if \( W \) were differentiable and we had \( dW(t) = W'(t)dt \).

Using your favorite computer software (R is good, so is Mathematica, but Excel is fine too...) do the following simulation exercise. (In all of the following, think of the time unit of \( t \) as years.)

1
1. Simulate a path of a Brownian motion over 10 years with an observation
point at each $\frac{1}{100}$ years.
2. Using this path, simulate the solution $Y(t)$ to the the SDE (1) over
10 years, subdividing each year into 100 equal time length periods, letting
$Y(0) = 1$ and letting

$$Y(t_{i+1}) = Y(t_i) + \sigma Y(t_i)(W(t_{i+1}) - W(t_i)).$$

3. Use the same path as above to simulate the two processes below in the
interval 0 to 10 years (and using the same subdivision)

$$Y_1(t) = \exp \left( \sigma W_t - \frac{1}{2} \sigma^2 t \right)$$
$$Y_2(t) = \exp(\sigma W_t)$$

Which of these two solutions best approximates the answer from 2?
4. Repeat the simulations above 100 times. Compute the expectations of
$Y(T)$ as found in (2) and $Y_1(T), Y_2(T)$ as found in 3 by taking an average
over the 100 values found in each simulation. (This is an example of Monte
Carlo simulation.)

**Problem 2**
Exercise B.11 on page 442 in TB.

**Problem 3**
Exercise 5.1 on page 75 in TB.

**Problem 4**
Exercise 5.7 on page 77 in TB.

**Problem 5**
Exercise 5.8 on page 77 in TB.

**Problem 6**
Exercise 5.12 on page 78 in TB.

Enjoy!
Best regards, David