where $\mu$ and $\sigma$ are functions and $W^P$ is a 1-dimensional BM under the “real world” probability measure $P$. So $r$ is Markov wrt. its own filtration.

Suppose all kinds of ZCB exist. The “formal” equation $P(t; T) = E^Q(\exp(- \int_t^T r(u)du))$ gives the conjecture that

$$P(t; T) = F(t, r(t); T)$$

for some smooth function $F$ (of 3 variables)

Hide $T$-dependence in superscript and use Ito to get

$$\frac{dF^T}{F^T} = \frac{F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T}{F^T} dt + \frac{\sigma F_r^T}{F^T} dW^P(t)$$

$$T$$-dependence in superscript and use Ito to get

$$dF^T = \frac{F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T}{F^T} dt + \frac{\sigma F_r^T}{F^T} dW^P(t)$$

Why is there no “$(dr$-dynamics) $\Rightarrow$ $(df$-dynamics)” in Proposition 15.5? (There isn’t one!)

Surprising? Not really. Think in terms of # traded assets & # sources of risk.

Empirically: Variations in “the” short rate “explains” “a large percentage” of the “variation” of the whole term structure. (“The tail wagging the dog”)

Is all lost? Certainly not. We get consistency relation between ZCBs of different maturities. (So nice we may forget we have a “problem” at all.)

Look (first) at the case where

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW^P(t)$$
Now make a self-financing portfolio with a $T$-ZCB and an $S$-ZCB. $V$ is the value process & $(u_T, u_S)$ relative portfolio weights. From Chapter 6 we have

$$\frac{dV}{V} = \frac{\alpha^S \sigma^T - \alpha^T \sigma^S}{\sigma^T - \sigma^S} dt.$$  

must $= r(t)$ otherwise arbitrage

Rewriting

$$\frac{\alpha^S - r(t)}{\sigma^S} = \frac{\alpha^T - r(t)}{\sigma^T}$$

LHS doesn't depend on $T$, RHS doesn't depend on $S \Rightarrow$ the ratio is independent of maturity:

$$\frac{\alpha^S - r(t)}{\sigma^S} := \lambda(r(t); t).$$

Now make a self-financing portfolio with a $T$-ZCB and an $S$-ZCB. $V$ is the value process & $(u_T, u_S)$ relative portfolio weights. From Chapter 6 we have

$$\frac{dV}{V} = u_T \frac{dF^T}{F^T} + u_S \frac{dF^S}{F^S}$$

$$= (u_T \alpha^T + u_S \alpha^S) dt + (u_T \sigma^T + u_S \sigma^S) dW^P(t)$$

By construction we must have $u_T + u_S = 1$, but still 1 degree of freedom. A clever choice is $u_T \sigma^T + u_S \sigma^S = 0$.

The $dW^P$-term vanishes, we get

$$u_T = \frac{-\sigma^S}{\sigma^T - \sigma^S}$$
where
\[ dr(s) = (\mu - \lambda \sigma) ds + \sigma dW^Q(s) \]

Note: Clearly \( P(t; T)/\beta(t) \) is a \( Q \)-martingale.

Writing
\[
\frac{dP(t; T)}{P(t; T)} = \alpha^T(t, T) dt + \sigma^T(t, T) dW^P = \left( r(t) dt + \sigma^T(t, T) \left( dW^P + \frac{\alpha^T - r(t)}{\sigma^T} dt \right) \right)
\]

shows that pieces fit.

Remark on multi-dimensional model.

We’re still not very concrete.

\[ \lambda: "market \ price \ of \ risk"; \ interpretation \ as \ excess \ expected \ return. \ Has \ to \ be \ exogenously \ specified. \ Usually: \ Postulate \ form \ that \ gives \ same \ structure \ under \ P \ & \ Q. \]

Substitute back and get the term structure PDE:
\[ F_t^T + (\mu - \lambda \sigma) F_r^T \frac{1}{2} \sigma^2 F_{rr}^T = r F^T \quad \text{and} \quad F^T(T; r) = 1 \]

This may be Feynman-Kac represented (by \[ \text{BE 5.12} \]) & we may change measure:
\[ F(t, r(t); T) = \mathbb{E}^Q(\exp(-\int_t^T r(s) ds)) \]