... interest rates are stochastic:

![Graph showing interest rates over time]

Asset Pricing II

April 23, 2004

David has taught you well, but ...
And that is what the rest of Asset Pricing II is about.

Rolf: Björk’s Chapters 20-25 with some detours.


- Concrete models; the 1-dimensional affine short rate models. Vasicek, Cox-Ingersoll-Ross. Loads of stuff we can calculate. And we will. And calibrate. Ch. 22. Today, and maybe some on May 13.


- Price-formulas for options on bonds. Ch. 24. May 13, but it’s probably not enough.
What determines interest rates? And what moves them around?

- Agents' preferences for consuming now rather than later.
- Supply & demand.
- Economic growth rates.
- Expected inflation
- Fiscal policy (in the hand of politicians)
- Monetary policy (largely central banks nowadays; Greenspan)
- Labor market structure

- LIBOR market models. Ch. 25. May 14.

- Multidimensional affine models. Matrix-calculations, but much the same as 1-dimensional. However, it is a considerably richer class than you'd think. And strange things can happen. May 14 in the afternoon.
  Literature: Hmmm, something like
  - Duffie & Kan, Math. Fin., 1996
  - Dai & Singleton, J. Finance, 2000
  - Ch. 4 in Brigo & Mercurio's 2001-book is a nice treatment of 2-factor models.

Pierre: Fancy stuff.
International effects, exchange rates.

Overload: Too much to model!
Randomness is a very large component. Let’s just accept that and build (empirically plausible) stochastic models. Worked well for stocks.

Complicating factor: Many assets (bonds) that are different (because they pay at different times) but not too different.

That is what interest rate (or term structure or fixed income) modelling is about.

(I don’t think I have to tell you that fixed income markets are huge. DK government bonds at CSX: Around 1,200 billion (1.2 \times 10^{12}) DKK. About the same in mortgage-backed bonds.)

Björk Chapters 20 and 23: Abstract/general theory.

\(T\)-maturity ZCB; time-\(t\) price denoted \(P(t; T)\). As a fct of \(T\): Smooth. As a fct of \(t\): Erratic (Ito-process).

Continuously compounded ZC yield \(y(t, \tau)\) is defined by

\[
P(t; t + \tau) = \exp(-\tau y(t; \tau)) \Leftrightarrow y(t, \tau) = -\frac{\ln P(t; t + \tau)}{\tau}.
\]

Note the shift from time of maturity to time to maturity.

Instantaneous forward rates (mathematically convenient)

\[f(t, T) = -\frac{\partial \ln P(t; T)}{\partial T}.
\]
Dynamics equations (Björk equations 20.1-3)

Short rate ($T$ means transposition)

\[ dr(t) = a(t)dt + b^T(t)dW(t) \] (1)

ZCB prices (one eqn' for each $T$; note shift to prop. coefficients)

\[ dP(t; T) = m(t; T)P(t; T)dt + v^T(t; T)P(t; T)dW(t) \] (2)

Forward rates

\[ df(t; T) = \alpha(t; T)dt + \sigma^T(t; T)dW(t) \] (3)

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Interpretation. Why does this make sense?

The term structure of interest rates at date $t$ is the mapping

\[ \tau \mapsto y(t; \tau) \]

or some translation of it (eg. into ZCB prices or forward rates). In theory this curve is observable in practice. In practice, well . . .

Short rate: $r(t) = f(t; t)$, Bank account: $\beta(t) = \exp(\int_0^t r(s)ds)$

Beware of interest rate quotations: Discrete vs. continuous compounding. Instantaneous vs. "simple" rates.

Yields on what?

**Advice: Show me the money!** (& a formula, if you don’t mind)
Proposition 20.5 (Quite important result.)

1) If ZCB prices satisfy (2) then forward rates satisfy (3) with

\[ \alpha(t; T) = v_T(t; T)v(t; T) - m_T(t; T) \quad \text{and} \quad \sigma(t; T) = -v_T(t; T). \]

2) If forward rates satisfy (3) then the short rate satisfies (1) with

\[ a(t) = f_T(t, t) + \alpha(t, t) \quad \text{and} \quad b(t) = \sigma(t; t). \]

3) If forward rates satisfy (3) then ZCB prices satisfy (2) with

\[ m(t; T) = r(t) + A(t; T) + \frac{1}{2} S^T(t; T)S(t; T) \quad \text{and} \quad v(t; T) = S(t; T), \]

No financial assumptions yet. \( W \) is just BM under some measure.

Coefficients are adapted (vector-valued) process, but smooth in \( T \); subscript-\( T \) denotes \( T \)-differentiation.

We have

\[ f(t; T) = -\frac{\partial \ln P(t; T)}{\partial T} \iff P(t; T) = \exp \left( -\int_t^T f(t; s) ds \right) \]

and

\[ r(t) = f(t, t). \]

So what's the connection?
3): \( P(t; T) = \exp\left(-\int_t^T f(t; s)ds\right) \), so \( t \) enters in two places in an even trickier way. But recall the Leibniz rule:

\[
\frac{d}{dx} \int_0^x h(t, x)dx = h(x, x) + \int_0^x h_x(t, x)dx.
\]

“Believe in that” for stochastic integrals too! So with \( Y(t; T) = -\int_t^T f(t; s)ds \) we get

\[
dY(t; T) = f(t; t)dt - \int_t^T df(t; s)ds.
\]

Now plug in, interchange & use Ito on \( \exp \).

Of course the only real way to prove it is to write things out in integral form, see the original HJM-article or the note on the course homepage. Takes about two hours of me writing integrals on the blackboard.

\[
\text{where } A(t; T) = -\int_t^T \alpha(t; s)ds \quad \text{and} \quad S(t; T) = -\int_t^T \sigma(t; s)ds.
\]

You’ll forget terms if you aren’t careful, so let’s sketch a proof:

1): Take logs, use Ito & differentiate wrt. \( T \)

2): \( r(t) = f(t; t) \), but “the chain rule” inspires us to write

\[
dr(t) = d_tf(t; T)|_{T=t} + d_Tf(t; T)|_{T=t} = f_T(t; T)dt
\]

and the result follows.
Note the subtle application of Girsanov’s theorem: Equivalent changes of measure change drift – not volatility.

But from Proposition 20.5 3) we get

\[
r(t) = r(t) + A^Q(t; T) + \frac{1}{2} S^\top(t; T) S(t; T) \Rightarrow -A^Q(t; T) = \frac{1}{2} S^\top(t; T) S(t; T).
\]

Differentiate both sides wrt. \( T \) and get the Heath-Jarrow-Morton drift condition

\[
\alpha^Q(t; T) = \sigma^\top(t; T) \int_t^T \sigma(t; s) ds.
\]

An application: The HJM drift condition

Assume that there model of forward rates is given by (3) under some measure \( P \).

Suppose further that the model is arbitrage-free.

Then there exists an equivalent martingale measure \( Q \sim P \) such that

\[
\Delta \frac{P(t; T)}{\beta(t)} \text{ is a } Q\text{-martingale for all } T.
\]

So

\[
dP(t; T) = r(t)P(t; T)dt + S^\top(t; T)dW^Q(t).
\]
This means that $-A^P(t; T) = \frac{1}{2}S^\top(t; T)S(t; T) + S^\top(t; T)\lambda(t)$. Differentiating wrt. $T$ gives
\[ \alpha^P(t; T) = \sigma^\top(t; T)\left(\int_t^T \sigma(t; s) - \lambda(t)\right). \]

In sloppy matrix notation we may write
\[ \lambda(t) = -\frac{E^P(\text{return on ZCB}) - r(t)}{\text{Vol}(ZCB)}. \]

If $\sigma$ (forward rate volatility) is chosen positive then (typically) $\lambda(t)$ will be positive. Otherwise it's negative.

But what about drifts under $P$?

From Girsanov’s theorem we know that there exists a stochastic process $\lambda$ such that
\[ dW^Q = dW^P - \lambda(t)dt \]
defines a $Q$-BM.

Important: $\lambda$ doesn’t depend on $T$.

Not important: Whether I choose to write “+” or “-”.

We have
\[
\frac{dP(t; T)}{P(t; T)} = (r(t) + A^P(t; T)) + \frac{1}{2}S^\top(t; T)S(t; T))dt + S^\top(t; T)dW^P(t)
= r(t)dt + S^\top(t; T)dW^Q(t) + \left(A^P(t; T) + \frac{1}{2}S^\top(t; T)S(t; T) + S^\top(t; T)\lambda(t)\right).
\]

must = 0
A third application: Musiela formulation/parametrization

Change to time to maturity in forward rates: \( r(t, x) = f(t; t + x) \). “How rates are quoted” (practice) and we get an object that lives on a rectangular domain (mathematical).

By the chain rule

\[
\begin{align*}
dr(t, x) &= dt \frac{df(t)}{dt} \bigg|_{T=t+x} + dT \frac{df(t; T)}{dT} \bigg|_{T=t+x} = df(t, t + x) + \frac{\partial}{\partial x} r(t, x), \\
&= \mathcal{F}(t, x) + D(t, x) dt + \sigma(t, t + x) dW^Q(t)
\end{align*}
\]

Another application (& old exam question): HJM and the Markov-property

The drift condition makes the dynamics of one forward rates dependent on all other forward rates ⇒ Non-markovian,

(How much has David talked about the Markov-property? If you don’t know what that is, then the statement is void.)

But sometimes the models are Markovian.

and here I solve the exercise

There’s more literature on this where people do cunning stuff.
Other words you’re bound to hear

(Forward) LIBOR (rates): Definition 20.2; more in Ch. 25

Caplets & caps: (Sums of) options on LIBOR. We’ll price them later.

Floating rate bonds: Easier than you’d think.

Swaps: “Plain vanilla” = (floating - fixed) rate bond.

Swaptions: We’ll price them later too.

The term structure can now be analyzed with tools for infinite dimensional SDEs. Very tricky stuff!