Asset Pricing II 2004: Problem Set 3

This set contains 5 (equally-weighted) exercises, of which you must solve 4 (according to the rules outlined below). Solutions to be handed in at the lectures June 7-9.

Exercise 1

Björk’s exercises 22.1 and 22.8. Shouldn’t be too hard.

Exercise 2

Björk’s exercises 22.5 and 22.7. Calibration of the Ho/Lee-model. Like Vasicek, just easier.

Exercise 3

Suppose that ZCB prices/yield curves/forward rates are created by an honest-to-God Vasicek model with parameters

\[ \theta^Q = 0.06, \kappa = 0.25, \sigma = 0.01, \]

and that \( r_0 = 0.03 \). We won’t worry about \( \mathbb{P} \)-parameters in this exercise; we did that ’till the cows came home in Set 2.

Show that in this case Björk’s equation (22.47) does indeed give back a flat \( \Theta \)-function. Here “show” does not does not mean “show algebraically” (which is exactly what is done in the book, so that’s OK) but rather “show numerically”. This means: Write a program that takes as input \( r_t \), \( \kappa \) and \( \sigma \) and some smooth observed yield curve (or forward curve, Björk’s \( f^* \)) and calculates (and plots) the right hand side of (22.47). In this case the observed yield curve just happens to come from the Vasicek-model.

Suppose that 3 months (0.25 years) pass and that the short rate is now 4\%, ie. \( r_{0.25} = 0.04 \). Keeping the \( \Theta \)-function fixed, show that the calibrated model does indeed match the observed yield curve. It’s probably easiest to Björk’s Proposition 22.8 And again: I want numbers.
Arguably, this is a rather boring “back and forth”-exercise. But now let’s make it interesting: The true yield curve still comes from the Vasicek model, but suppose that someone uses linear interpolation to get what is going to play the role of the observed yield curve. More specifically,

\[ f^*(0, \tau) = r_0 + \frac{\tau}{10} (f^{true}(0, 10) - r_0) \]

Show that if you feed this forward rate curve into the machinery above, then you do not get a flat \( \Theta \)-function. Further, illustrate that after 3 months, the calibrated model will not match the term structure (true or linearly-interpolated observed, it doesn’t matter which, and just use \( r_{0.25} = 0.04 \) again).

This is an example of what Björk (in the notes to Chapter 23.5) calls inconsistent modelling: The functional form of the yield curve is different from what the dynamic model can generate, and that leads an unstable situation, where you need to (frequently) re-calibrate.

**Exercise 4**

As in Exercise 3 we look at a world where true yield curves are generated by that Vasicek model with parameters

\[ \theta^Q = 0.06, \kappa = 0.25, \sigma = 0.01, \]

and where \( r_0 = 0.03 \). We won’t worry about \( \mathbb{P} \)-parameters, and there’s no difference between theoretical and observed yield curves. (“Been there, done that.”) To avoid writing a lot of \( \delta \)’s, we only look at cash-flows that occur at dates \( \{0, 1, \ldots, 5\} \).

Determine the (1-year) forward LIBORs, \( L(0, i, i + 1) \) for \( i = 0, \ldots, 4 \) in the notation of Björk’s Definition 20.2. (Numbers please.)

Determine the (par) swap rates for swaps yearly payments and lengths of up to 5 years. With a little extra notation added to Björk’s Proposition 20.7, we could call these \( R(0; i) \).

Determine the prices of the 1 to 5 year caplets (see Björk’s 24.8 if need be). For each expiry date (of which there are 5) do it for cap rates (the strike in the cap) of \( \{0.01, 0.03, \ldots, 0.07\} \) as well as \( R(0; i) \). (So all in all you get you a matrix of caplet prices.)

Calculate (numerically) and plot the implied Black volatilities across strikes and expiry dates. Do it for both flat and forward/spot volatilities in the sense of Björk’s Definition 25.3.
Exercise 5

If numbers really don’t agree with you, then you may skip either Exercise 3 or 4 and do this one instead.

Consider the Two-Additive-Factor Gaussian model from Section 4.2 in Brigo and Mercuio. (We’ve already done enough calibration, so just assume $\varphi$ is constant.) Use the ODE-technique for affine models (that we have from Duffie and Kan) to prove Brigo and Mercuio’s Theorem 4.2.1. (Now look at Brigo and Mercuio’s proof that goes via Lemma 4.2.1 and conclude that this is much easier.)