Mortgage Choice: Who Should Have Their ARMs Capped?*

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1 Introduction

Floating rate loans with a specific cap-rate (a maximal or "ceiling" interest rate that the mortgagor can pay) have recently been introduced in the Danish mortgage market. Jakobsen & Svenstrup (2001) and Svenstrup (2002) give detailed descriptions of both the construction (of the specific "brand" of loan called Bolig-X) and the pricing (theoretically and market practice) of such contracts. They give several arguments why the capped loans should be attractive to mortgagors. While quite compelling, the arguments are all more or less "practically motivated". The contract exploits market imperfections (such as tax-asymmetry) or it reduces them (such as lack of liquidity stemming from model uncertainty).

In this paper I give a utility-based argument why these capped adjustable rate mortgages ("capped ARMs") could be the optimal/preferred financing choice for a large proportion of mortgagors. This is done by considering a toy interest rate model with 3 simplified loan-types representing fixed rate, floating rate, and "floating +cap" financing. I then look at a (possibly) risk averse mortgagor who is unable to adjust his portfolio dynamically and has to choose his financing from the 3 loan types. For reasonable parameter ranges, no loan-type is dominated, but with empirically plausible parameter values (differences of 1-2% between short and long rates and logarithmic utility) the capped floating rate loan is optimal. So to answer the question in title: If both the market and the mortgagor are (appropriately) risk averse, then the ARM should indeed be capped.

2 The Model

We use the Vasicek model, so under the statistical probability measure $\mathbb{P}$ (some people would call $\mathbb{P}$ the real-world probability measure, but any probability measure on a Brownian filtration is very

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much a mathematical abstraction) the short rate dynamics are governed by

$$dr(t) = \kappa(\theta - r(t))dt + \sigma dW^{\mathbb{P}},$$

where $W^{\mathbb{P}}$ is a Brownian motion.

However, a specification of the $\mathbb{P}$-dynamics of the short rate is not enough to determine prices of bonds and options, see the discussion in Björk (1998). We need to specify a risk-premium. A very convenient choice is to assume it to be constant, and this gives $\mathbb{Q}$-dynamics of the form

$$dr(t) = \kappa((\theta + \lambda) - r(t))dt + \sigma dW^{\mathbb{Q}}.$$

Shortly, we'll see that λ has a natural interpretation.

The arbitrage-free price of a contingent claim with pay-off at time T is

$$\pi(t) = \mathbf{E}_t^{\mathbb{Q}} \left(\exp\left(-\int_t^T r(u)du \right) \times \text{Pay-off} \right). \tag{1}$$

This "pricing by an equivalent martingale measure" approach is reasonable for valuing stochastic cash-flows in financial markets. It is a necessary if we are to exclude the possibility of constructing arbitrages trough dynamic trading. So the use of $\mathbb{Q}$ rest on the belief that there are some participants in the market for which (frequent) dynamic trading is a possibility. This does not mean that *everybody* can trade frequently, that "$\mathbb{P}$ doesn't matter" or risk-neutrality in any way.

For zero coupon bonds the expectation in (1) can be calculated:

$$P(t,T) = \exp\left(A(t,T) - B(t,T)r(t)\right),$$

where $B(t,T) = (1 - \exp(\kappa(T-t)))/\kappa$ and

$$A(t,T) = \frac{B(t,T) - (T-t)(\kappa^2(\theta+\lambda) - \sigma^2/2)}{\kappa^2} - \frac{\sigma^2 B(t,T)^2}{4\kappa}.$$

This gives us an interpretation of λ: The long term level (the unconditional $\mathbb{P}$-mean) of r is θ. Letting $y(t,\tau) = -(\ln P(t,t+\tau))/\tau$ denote the zero coupon yield we see that

$$\mathbf{E}^{\mathbb{P}}(y(t;\text{large } \tau) - r(t)) = \lambda + \text{small terms},$$

so λ expresses the typical slope of the term-structure (difference between long and short rates). The exact magnitude of this difference can be debated, but there is little doubt that it is positive. A positive λ value means that long (and hence more risky wrt. short term price changes) bonds have higher expected rates of return than short bonds, which is in harmony with overall/aggregate market risk aversion. In other words a borrower is rewarded for suing short-term financing.

My choice of parameters can be seen in Table 1. Figure 1 shows the model yield curve, as well the average Danish yield curve between 1998 and 2002. I have no particular empirical ambitions in this paper, but the parameters are reasonable.

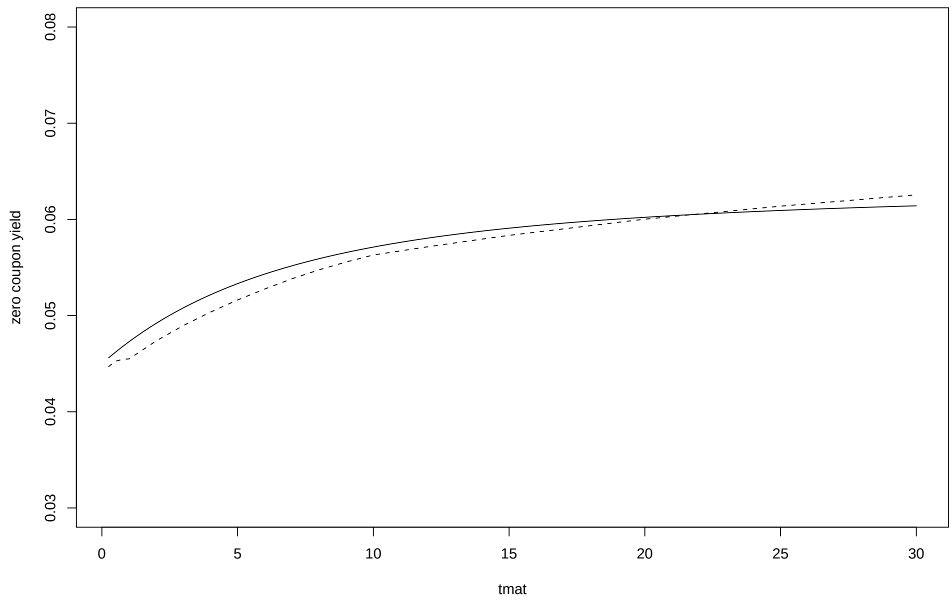

Figure 1: Term-structures. The full curve is the hypothetical yield curve used in our numerical examples. The dotted curve shows the average Danish yield curve over the period 1998 to 2002. (Thanks, Nykredit.)

In the Vasicek model plain vanilla interest rate options can also be priced. Many text-books (Björk (1998) for instance) will tell you that the price of a strike-K, expiry-T_1 call-option on a maturity-T_2 ZCB is given by the Black-Scholes-like expression

$$\pi^{call}(t) = P(t, T_2)\Phi(d_+) - KP(t, T_1)\Phi(d_-),$$

where

$$d_\pm = \frac{\ln(\frac{P(t,T_2)}{KP(t,T_1)}) \pm \frac{1}{2}\Sigma^2}{\sqrt{\Sigma^2}},$$

and

$$\Sigma^2 = \frac{\sigma^2}{2\kappa^3}(1 - \exp(-2\kappa(T_1 - t)))(1 - \exp(-\kappa(T_2 - T_1)))^2.$$

Put-prices follow from from the put/call-parity: $\pi^{call}(t) - \pi^{put}(t) = P(t, T_2) - KP(t, T_1)$.

A caplet contract is insurance against paying high interest rates. It pays

$$\delta(L_\delta(T, T + \delta) - \bar{R})^+ \text{ at time } T + \delta,$$

where the δ-LIBOR rate is $L_\delta(T, T + \delta) = ((1/P(T, T + \delta) - 1)/\delta$. A caplet can may be viewed (see Björk (1998)[Section 19.8]) as $(1 + \delta\bar{R})$ units of strike-$1/(1 + \delta R)$, expiry-T put-options on a maturity-$T + \delta$ ZCB. So we also a closed-from expression for the caplet-price, say $\pi^{caplet}(t, T, \delta; \bar{R})$.

Quantity	Symbol	Numerical value
Initial short rate	$r(0)$	0.045
Long term mean of short rate	θ	0.055
Speed of mean reversion	κ	0.25
Short rate volatility	σ	0.012
Risk premium	λ	0.015, or varying
Maximal rate on capped loan	$\bar{r}$	0.077
Spread on 30Y capped loan*	s	$24.5 * 10^{-4}$, but varying with λ
Mortgagor risk aversion	γ	varying
Maximal possible monthly mortgage payment	$\bar{x}$	2,000
Initial principal on mortgage	$H(0)$	239,603
Monthly mortgage payment on fixed loan*	y_{fix}	1,500

Table 1: Parameter symbols and values. Quantities marked with * are endogenous. The monetary unit can be thought of as EUR.

3 The Products

In this section we introduce 3 different types loans in the hypothetical term-structure model of Section 2. The loans are stylized versions of fixed rate, floating rate, and capped floating rate loans. They are constructed such that the no-arbitrage price (the $\mathbb{Q}$-expected discounted value of the future cash-flows) is to the initial principal. They are trading at par, in other words.

The fixed rate loan

A 30-year annuity loan (with 4 payments per year) with a coupon-rate determined in such a way that it is trading at par (so it's like a cash loan, "kontantlån", really). There are no option features (such as callablility) since this would seriously complicate matters (wrt. both pricing and (sub)-optimal exercise behaviour).

The floating rate loan (ARM for "adjustable rate mortgage" & for the effect)

A floating rate loan where the mortgagor is paying 3-month LIBOR. The instalment plan is fixed at time 0, as if the loan were a 30-year annuity with a coupon rate of initial 3-month LIBOR. Such a loan is always trading at par. In real-life ARMs the instalment plan is not fixed initially, but this simplifies calculations considerably, and after all: You have to repay the principal some time.

The floating rate loan with cap (capped ARM)

The instalments are similar to those of the ARM, but the mortgagor pays an interest rate that is

$$\min(\bar{r}, L_\delta(T, T+\delta) + s) \text{ at time } T+\delta, \quad (2)$$

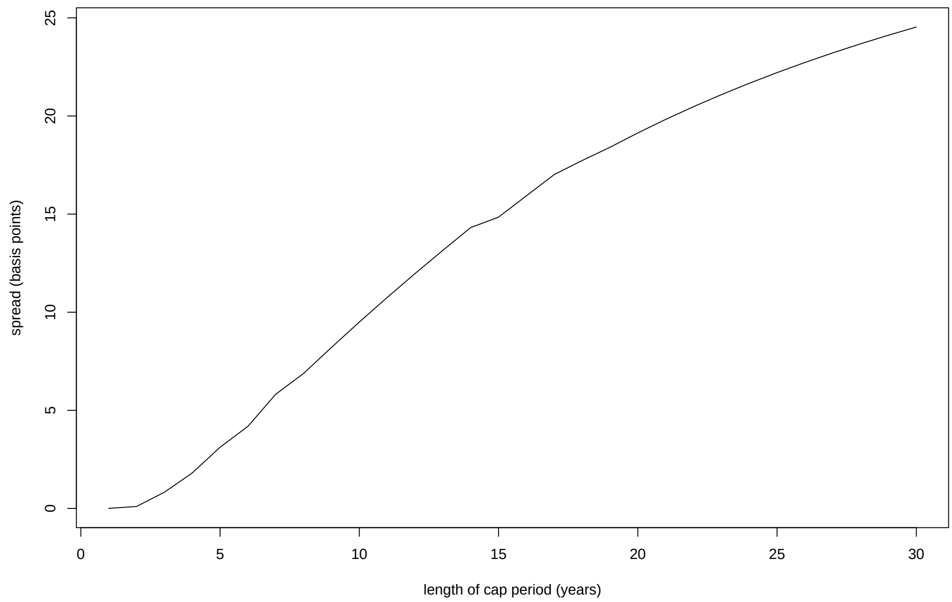

Figure 2: Spreads for capped loans with different lengths. `WHY IS IT JAGGED; CHECK CODE!`

where $\bar{r}$ is a maximal rate and s is a spread that makes the mortgagor pay for the embedded option through extra higher interest payments. Given a $\bar{r}$-value (7.7% in the examples), s is determined is such a way that the capped ARM is also trading at par. It can be calculated in following way: Supposing for a minute that the interest rate is of the form

$$\min(\tilde{r} + s, L_\delta(T, T + \delta) + s), \tag{3}$$

then s goes outside the "min"-operator, and it's easy to show that the spread that makes the capped loan trade at par is

$$\tilde{s}(\tilde{r}) = \frac{\sum H(T_{i-1}) \pi^{caplet}(0; T_{i-1}, \delta; \tilde{r})}{\sum \delta H(T_{i-1}) P(0, T_i)}, \tag{4}$$

where $H(T)$ denotes the remaining outstanding principal at time T. Given a maximal rate $\bar{r}$ in "equation (2)"-sense we simply use equation (4) to find $\tilde{r}$ such that $\tilde{r} + \tilde{s}(\tilde{r}) = \bar{r}$ (this is easy to do numerically).

Figure 2 shows spreads for capped ARMs of different lengths. Naturally, the longer the cap-period the higher the spread, but we see that the curve is flatter in the beginning (where it is quite unlikely that the cap will ever become effective) and towards the end (where most of the principal has been paid off). So: Capping for 30 rather that 20 years shouldn't cost you very much.

Real-life capped ARMs (such as Bolig-X) have some more "bells & whistles", (shorter cap period, some averaging in the fixing rate, non-deterministic instalments) but again this simplifies computations and captures the general idea. Bolig-X loans cap for no more than 5-7 years, but have spreads that are about the size of "our" spread on a loans with a 30 year cap. `CHECK THIS WITH SOMEBODY WHO KNOWS!`

4 The Mortgagor

Consider a specific mortgagor. Consistent with standard economic reasoning, he tries to maximize expected discounted utility,

$$\mathbf{E}\left(\sum_t d_i(t)u(\bar{x} - y_t)\right) \tag{5}$$

where d_i is some subjective, mortgagor specific discount-function. We use the initial curve $P(0;t)$. The utility function is assumed to be of the form

$$u(c) = \frac{c^\gamma - 1}{\gamma},$$

where the "$\gamma = 0$"-case should be interpreted as logarithmic utility. If $\gamma = 1$, the mortgagor is risk-neutral, while $\gamma < 1$ means risk aversion. Notice that utility is derived from consumption, which we assume to be of the form "some fixed amount ($\bar{x}$) less what you pay on your mortgage". The choice of $\bar{x}$ also leaves room for "fudging". If $\bar{x}$ is much larger than the typical value of y_t then the utility function is well approximated by a linear function over a large range, and in essence the mortgagor becomes risk-neutral. In the numerical example 1,500 is a typical value of mortgage payments, and the $\bar{x}$ level is set at 2,000 (if y_t gets above 1,999 we truncate it).

Suppose the mortgagor must choose how to finance his house only at time 0, and that the 3 loans from Section 3 are his only options. Then the portfolio choice problem becomes a quite manageable one: There are only 3 points at which to evaluate the criterion function. The optimal loan is the one producing the highest value. The expectation in (5) can readily be evaluated by simulation: Simulate a short rate path (under $\mathbb{P}$). We can do exact simulation, so one path requires only 120 draws from a normal distribution and can easily be vectorized (for speed). Such a path determines the cash-flows from each of the 3 loans, and therefore specific outcomes of utility. Now repeat, calculate sample average utilities, and stop when they differ at a suitably significant level. (A few thousand paths are usually enough.) Repeat for different parameter values, the most interesting ones to vary being the market risk-premium λ and the mortgagor risk-aversion γ.

This produces the "map" shown in Figure 3. This is a (risk-aversion, risk-premium)-coordinate system where the letters indicate the optimal loan type ("F" for fixed; "A" for adjustable/floating, and "C" for capped). We see that if the mortgagor is risk-averse and the market offers no particular reward for short-term financing, then fixed rate loans are preferred (the south, south/east part of the map). On the other hand, with low or no mortgagor risk-aversion, and positive market risk-premia (west, north/west), the mortgagor uses plain floating loans; ie. he is unwilling to pay for any sort of insurance. In either case the optimal choice of loan is a "no-brainer", it is when we consider

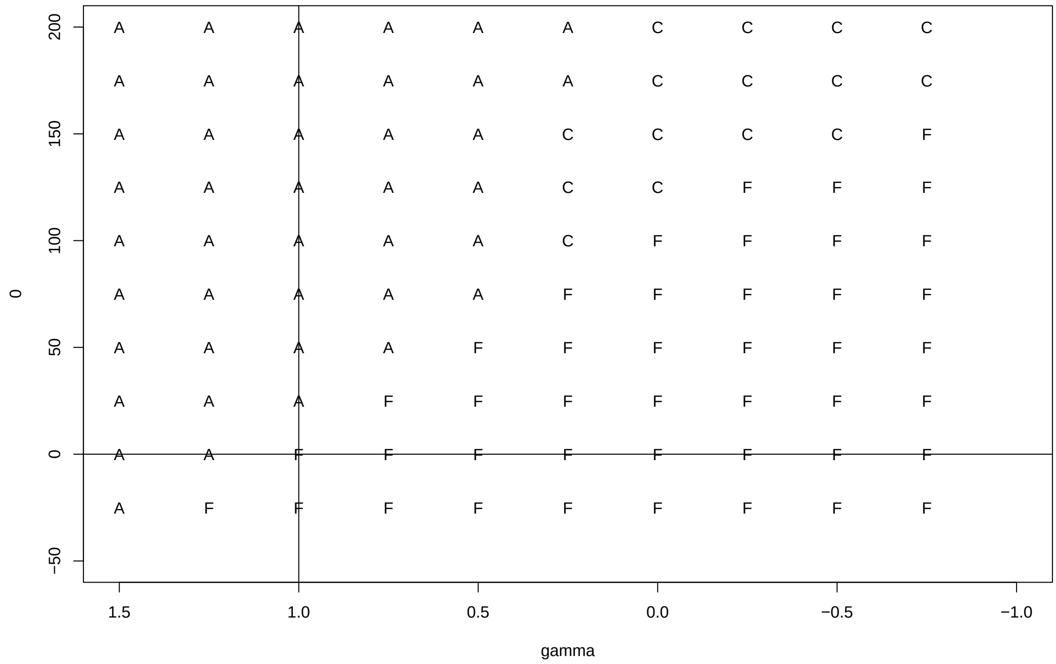

Figure 3: Optimal choice of loan for different levels of (mortgagor) risk-aversion ($\gamma = 1$ is risk-neutrality, and the smaller γ is the more risk-averse) and (market) risk-premium (λ; $\lambda = 0$ means that $\mathbb{P} = \mathbb{Q}$ and in general λ is (approximately) the typical difference between long and short rates). The letters indicate which type of loan is optimal ("F" for fixed, "A" for adjustable/floating and "C" for capped floating)

the north/east-direction of the map that things get interesting. Here there is real trade-off: The mortgagor doesn't like taking risk, but the market rewards him for it. This is by far the empirically most plausible situation. In this case the optimal loan problem is non-trivial; either loan can be optimal. We see that there is a sizeable parameter-area where the capped loan is the preferred choice. Looking at the Danish market over the period 1998 to 2002, the average difference between long and short rates was around 150 basis points. Combining that with a firm belief in logarithmic utility, you may indeed argue that capped floating rate loans are the optimal financing choice.

5 Conclusion

The optimal way to finance your house is a highly complex dynamic portfolio choice problem. In this paper I considered a simplified (and hence solvable) version of the problem, but one that I think still captures important aspects of both market and mortgagor behaviour. Key features are your risk-aversion and the reward offered by the market for taking on risk. The analysis lends support to the claim that capped floating rate loans are a significant improvement of the Danish mortgage market.

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