

Eight Valuation Methods in Financial Mathematics: The Black-Scholes Formula as an Example

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Abstract

This paper describes a large number of valuation techniques used in modern financial mathematics. Though the approaches differ in generality and rigour, they are consistent in a very noteworthy sense: each model has the celebrated Black-Scholes formula for the price of a call-option as a special case.

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1 Introduction

This paper serves as an account of the sophisticated and varied techniques that have been developed in financial mathematics during the last 20 years. We illustrate how these can be used to derive a result that most people with interest in finance are familiar with, the Black-Scholes formula. The real power of the techniques naturally lies in their ability to cope with different generalisations of the basic set-up of Black and Scholes. Some methods are easily described in a more general setting and the Black-Scholes formula then appears as a special case. Other methods are not possible to describe in generality within limited space. When this is the case, we consider the basic Black-Scholes set-up, derive the formula using the particular method, and then indicate the range and applicability of the method. In either case this should allow the reader not familiar with the subject to gain insight into the techniques used in financial mathematics. For people who are familiar with the field, 'we aim to please.' The hope is that any person in this group will see at least one proof and say 'Ah, yes. I hadn't thought of that one.'

The outline of the paper is as follows: In Section 2 we formulate the problem, introduce some central concepts, and present the Black-Scholes formula. In other words we ask the question *and* give the answer up front. The next 8 sections describe ways of getting from point A to point B(.S.) Ways that lead us past answers to more general questions than the one originally posed. By a hedge argument (which was the ingenious insight of Black, Scholes and Merton) the fundamental partial differential equation (PDE) for arbitrage free asset prices is derived in Section 3. Section 4 shows how martingale techniques can be used to solve the pricing problem and stresses the relationship between means of solutions of stochastic differential equations (SDEs) and PDEs. Section 5 shows how the seemingly neutral concept of using different numeraires can turn out to be a very powerful tool, in fact we derive the Black-Scholes formula without calculating a single integral. In Section 6 we initially try to 'mess with your head' by proposing a strategy that seemingly contradicts the previous results. But we show that a careful inspection and some advanced stochastic calculus not only resolves the paradox, but also provides an extra proof. It is shown in Section 7 that the price of the call-option also satisfies a PDE that runs in strike price and maturity date, a forward equation. Not only does this give another proof of the result but also has practical implications for numerical purposes. Section 8 derives the formula as a limiting case of a discrete binomial model. The proposed convergence proof is different from most other convergence proofs in the literature and highlights

an interesting similarity between numeraire/measure changes in discrete and continuous cases. Section 9 shows that we can also derive the formula from the continuous time CAPM model, which links together two of the most celebrated results in financial economics. Utility maximisation of a representative agent with a utility function exhibiting constant relative risk aversion is shown also to do the trick in Section 10. Section 11 sums up the contributions of the paper and discusses the results.

2 The question and the answer

The basic Black-Scholes set-up consists of non-dividend paying stock, the price of which is assumed to be the solution to the SDE

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t^{\mathcal{P}}, \quad (1)$$

where μ and σ are constants and $(W_t^{\mathcal{P}})$ is a Brownian motion on some filtered probability space $(\Omega, (\mathcal{F}_t), \mathcal{P})$, and a bond with price dynamics given by

$$\frac{dB_t}{B_t} = r dt, \quad B_T = 1,$$

where r is the (continuously compounded) interest rate which is assumed to be constant.¹

Our aim is to price a European call-option on the stock with maturity date T and strike price K . This is a security that gives the bearer the right, but not the obligation, to buy one share of stock at time T (and only at that time) for a price of K \$. Hence the contract has a terminal pay-off of

$$\max(S_T - K, 0) \equiv (S_T - K)^+$$

and no intermediate payments.

We will assume that there are no transactions costs, no short-selling constraints and that all assets are perfectly divisible. Furthermore we will allow investors to continuously readjust their portfolios. Specifically, a *trading strategy* (a_t, b_t) is a *predictable* stochastic process satisfying certain technical conditions.² To us, a_t will represent the

¹Notice that this is an ordinary differential equation with solution given by: $B_t = e^{-r(T-t)}$.

²The trading strategy will be stochastic because it depends on the stock and bond whose price evolution is stochastic. However, at any given time t we will know how much to hold. Among other reasons the technical conditions the strategy has to fulfill is to exclude *doubling strategies*. See Duffie (1992) for details.

number of stocks held at time t , while b_t is the bond holdings. $V_t = a_t S_t + b_t B_t$ is the *value process*. A trading strategy is called *self-financing* if

$$dV_t = a_t dS_t + b_t dB_t,$$

which means that we only make an investment today. The gains are reinvested, and we do not use extra funds to cover our losses (this does not necessarily mean constantly calling your stock-broker, 'buy-and-hold'-strategies are evidently self-financing). An *arbitrage opportunity* is a self-financing trading strategy such that either

$$V_0 \leq 0, \mathcal{P}(V_T \geq 0) = 1, \mathcal{P}(V_T > 0) > 0,$$

or³

$$V_0 < 0, \mathcal{P}(V_T \geq 0) = 1.$$

So, an arbitrage opportunity is 'something for nothing' or 'a free lunch'. Reasonably, though human intuition about stochastic phenomena is notoriously poor, we cannot have such strategies in the economic equilibrium. There would be an infinite demand for the 'arbitrage strategy', while no agent (without a serious financial death wish) would be willing to supply it.

From pure static arbitrage considerations the only bounds that can be put on the call-option price are:⁴

$$S_t \geq C_t \geq (S_t - B_t K)^+.$$

The main contribution of Black & Scholes (1973) is that they close the gap and give an exact pricing formula by dynamic arbitrage arguments.

Result 1 (The Black-Scholes Formula) *If the setting is as described above then to prevent arbitrage opportunities we must have $C_t = C(S_t, t)$ where*

$$\begin{aligned} C(x, t) &= x\Phi(z) - e^{-r(T-t)}K\Phi(z - \sigma\sqrt{T-t}) \\ z &= \frac{\ln(\frac{x}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \end{aligned} \quad (2)$$

and Φ denotes the cumulative density of the standard normal distribution.

In order to replicate the pay-off of the call-option we should hold

$$a_t = \frac{\partial C}{\partial S}(S_t, t) = \Phi(z)$$

³In incomplete markets the two conditions are not equivalent.

⁴This is given that the call-option contract is the only existing derivative security written on the stock. If there exists several option contracts in the economy, say with different strikes, there would be static arbitrage bounds between these contracts.

shares of stock and

$$C(S_t, t) - a_t S_t$$

in bonds.

This is a remarkable result that has been twisted and turned in the literature for more than twenty years. The most noteworthy thing is that the instantaneous expected return of the stock, μ , does not enter the expression. In other words: Two investors need not agree on the expected return of the stock in order to agree about the option price.⁵

Note the, at least mathematical, simplicity of the replicating portfolio, and that the position in the stock is bounded above by 1.

This method of pricing is *relative*. We price the option in terms of the stock and bond, whose prices are taken as given. We do not need any general equilibrium constraints on the economy other than there being 'no free lunches'. We shall later see that we can arrive at the result from a general equilibrium model, but this is in a sense 'overkill': The above conditions are exactly what we need.

3 The hedge argument and the fundamental PDE

The technique presented in this section is the one originally used by Black & Scholes (1973) to derive the formula that now bears their names. The result was simultaneously and independently derived by Merton (1973*b*). Let Y_t denote the price of a call-option with strike K and maturity T . Now *assume* that Y_t can be written as a twice continuously differentiable function of S_t (hence, no dependence on past S_u 's) and t . That is

$$Y_t = C(S_t, t).$$

The Ito formula applied to Y_t yields

$$dY_t = \left(\mu S_t \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \frac{\partial C}{\partial S} \sigma S_t dW_t^P, \quad (3)$$

where some of the dependences have been notationally suppressed.

In the notation of Section 2 *assume* that a self-financing trading strategy (a_t, b_t) exists

⁵It is worth noting that increasingly frequent discrete sampling of the underlying stock gives an improved estimate of the volatility but high frequency sampling does not necessarily improve the estimate of the drift. So if the stock price is only observed at frequent but discrete times it is likely that investors will agree on the volatility but not necessarily on the drift. For a derivation of this see Ingersoll (1987).

such that

$$a_t S_t + b_t B_t = Y_t, \quad \forall t \in [0, T]. \quad (4)$$

By linearity of stochastic integrals and the self-financing condition we have

$$dY_t = (a_t \mu S_t + b_t r B_t) dt + a_t \sigma S_t dW_t^{\mathcal{P}}.$$

We now have two Ito-expressions for dY_t . This means (by the Unique Decomposition Theorem, see e.g. Duffie (1992)) that the drift and diffusion terms in these must be equal.

Matching diffusion terms yields (since $S_t > 0$ \mathcal{P} -a.s.)

$$a_t = \frac{\partial C}{\partial S}(S_t, t),$$

which gives us the number of shares of stock to hold. On the other hand from (4)

$$S_t \frac{\partial C}{\partial S}(S_t, t) + b_t B_t = C(S_t, t),$$

so

$$b_t = \frac{1}{B_t} (C(S_t, t) - S_t \frac{\partial C}{\partial S}(S_t, t)).$$

From the drift terms we get

$$r S_t \frac{\partial C}{\partial S}(S_t, t) + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2}(S_t, t) + \frac{\partial C}{\partial t}(S_t, t) = r C(S_t, t),$$

which holds if the function C satisfies the PDE

$$r x \frac{\partial C}{\partial x}(x, t) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 C}{\partial x^2}(x, t) + \frac{\partial C}{\partial t}(x, t) = r C(x, t). \quad (5)$$

We immediately note that the exact same argument holds for all types of derivative assets whose prices depend only on time and some Markov process S_t . The only things that differ are the boundary conditions. For this reason (5) is referred to as the fundamental PDE for arbitrage free asset pricing.

As we are considering a European call-option, C should further satisfy the boundary condition⁶

$$C(x, T) = (x - K)^+. \quad (6)$$

To find the arbitrage free call-option price we have to solve (5)-(6). Given Section 2, the easiest thing is to verify the result by a direct calculation. Originally, Black and

⁶In fact we should also require $x \geq C(x, t) \geq (x - e^{-r(T-t)}K)^+$, but this turns out to be automatically satisfied in this case.

Scholes (probably) used a Fourier transform technique which is mainly of historical interest nowadays.

Is this price unique, one might ask. Yes it is. Given that (5)-(6) has been solved we have found a (hopefully) self-financing trading strategy that replicates the option. If the option had any other price than the initial investment in this replicating portfolio, we would just sell the option and buy the replicating portfolio (or the other way round, depending on which is the cheaper alternative). This would leave us with a risk-free profit. An arbitrage opportunity !

Now, the reader might argue: 'How do we know that the proposed trading strategy is indeed self-financing?' This is a fair point to make, we have just assumed so far, that this was the case, and made explicit use of it. However, using (5) (which in particular means that C is \mathbf{C}^2) and the Ito formula it is easy to show that (a_t, b_t) is indeed self-financing. We make this seemingly 'round-about' comment because later (in Section 6) we will meet a trading strategy that is obviously self-financing. Except: It isn't !

By a hedge argument this section derived the fundamental PDE for asset prices. By specifying the right boundary condition, the Black-Scholes formula emerged. This approach generalises easily to the case of dividend paying assets. Provided that covariance matrix for stock returns is of full rank ('a complete market'), the results also carry over to higher dimensions (see e.g. Duffie (1992)). The technique also works for American type assets (see Ingersoll (1987)), though the boundary conditions become more subtle, in fact determination of these becomes part of the solution. What this approach is NOT suitable for is non-Markovian settings, such as stochastic volatility in the stock or path dependent features in the derivative (e.g. Asian options). The next section develops a more general valuation technique, that coincides with the PDE approach when the latter is applicable.

4 The martingale approach

In this section we will derive the martingale pricing approach that does not depend on the Markov property of the stock and draw the analogy to the PDE obtained in the previous section. Using the martingale valuation technique we calculate the Black-Scholes formula.

The martingale pricing technique was pioneered by Cox & Ross (1976) and later on further developed and refined by Harrison & Kreps (1979), Harrison & Pliska (1981), and others.

We start by defining the quantity $\eta = (\mu - r)/\sigma$ and the Girsanov factor

$$\xi_t = \exp\left(-\frac{1}{2}\eta^2 t - \eta W_t^{\mathcal{P}}\right).$$

The Girsanov factor is a positive \mathcal{P} -martingale with mean 1, so we can define a new probability measure \mathcal{Q} , equivalent to \mathcal{P} , by setting

$$d\mathcal{Q} = \xi_t d\mathcal{P} \tag{7}$$

on \mathcal{F}_t . According to the Girsanov Theorem we have that under \mathcal{Q}

$$W_t^{\mathcal{Q}} = W_t^{\mathcal{P}} + \eta t$$

is a Brownian motion.⁷ Plugging this into (1) yields that under \mathcal{Q} , the stock price evolves according to the SDE

$$\frac{dS_t}{S_t} = r dt + \sigma dW_t^{\mathcal{Q}}. \tag{8}$$

Let us now define

$$G_t = B_t E_t^{\mathcal{Q}} \left[(S_T - K)^+ \right] = B_t E_t^{\mathcal{P}} \left[\frac{\xi_T}{\xi_t} (S_T - K)^+ \right].$$

We observe that from the definition $(e^{-rt} G_t)$ must be a \mathcal{Q} -martingale with respect to the filtration (\mathcal{F}_t) . By the Martingale Representation Theorem we therefore have that

$$d \left[e^{-rt} G_t \right] = \gamma_t dW_t^{\mathcal{Q}}$$

for some process γ .⁸ Introducing $\Gamma = e^{rt} \gamma / G$, using the Ito formula and reintroducing the \mathcal{P} Brownian motion we get

$$dG_t = G_t (r dt + \Gamma_t dW_t^{\mathcal{Q}}) = G_t ((r + \Gamma_t \eta) dt + \Gamma_t dW_t^{\mathcal{P}}).$$

Now consider a self-financing strategy with value V and no consumption flow before T consisting of a stocks and the remaining amount is put in $b = (V - aS)/B$ bonds. Such a strategy evolves according to

$$\begin{aligned} dV_t &= a_t dS_t + \frac{V_t - a_t S_t}{B_t} dB_t \\ &= (a_t S_t \sigma \eta + r V_t) dt + a_t \sigma S_t dW_t^{\mathcal{P}}. \end{aligned} \tag{9}$$

⁷For the Girsanov Theorem in the context of financial economics see for example Duffie (1992).

⁸A possible reference for the Martingale Representation Theorem is Duffie (1992).

Choosing $V_0 = G_0$ and

$$a = \frac{\Gamma V}{\sigma S}$$

we see that G and V have the same evolution and the same starting value, hence $V_t = G_t$ for all t . Since $C_T = G_T$ we conclude that $C_t = G_t$ for all t . Otherwise there exists an arbitrage because G_t can be replicated through the dynamic trading strategy outlined above. We have now obtained the martingale pricing equation

$$C_t = e^{-r(T-t)} \mathbb{E}_t^{\mathcal{Q}} \left[(S_T - K)^+ \right]. \quad (10)$$

Note that this relation does not depend on μ and/or σ being constants. In fact the measure \mathcal{Q} is unique and the replication argument goes through as long as only the diffusion coefficient, but not necessarily the drift, of the stock is adapted to the filtration generated by the stock.⁹ Investors need not have the same beliefs about the drift of the stock for the arbitrage pricing to be valid. All they have to agree about is the diffusion coefficient. In a continuous-time economy this means that all they have to agree about are the zero-sets for the stock price evolution.

If (S_t) happens to be a Markov process under \mathcal{Q} , then (virtually by definition) we deduce from (10) that the option price is a smooth function of current time and stock price only, i.e. $C_t = C(S_t, t)$. But then the Ito formula and 'coefficient matching' recovers the fundamental PDE of Section 3. In other words, there is consistency between the methods. Note that it is sufficient that (S_t) is Markov under \mathcal{Q} , so nasty drifts do not prohibit us from using a PDE approach. This is of course what one would conjecture since the drift does not enter the fundamental PDE.

To compute the Black-Scholes formula we use that sitting at time 0, $\ln S_T$ is normal under \mathcal{Q} with mean and variance

$$m = \mathbb{E}^{\mathcal{Q}} [\ln S_T] = \ln S_0 + \left(r - \frac{1}{2}\sigma^2\right)T \quad (11)$$

$$v^2 = \text{Var}^{\mathcal{Q}} [\ln S_T] = \sigma^2 T, \quad (12)$$

so

$$\begin{aligned} C_0 &= e^{-rT} \int_{\frac{\ln K - m}{v}}^{\infty} (e^{m+vx} - K) \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx \\ &= S_0 \Phi \left(\frac{\ln(S_0/K) + rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T} \right) - e^{-rT} K \Phi \left(\frac{\ln(S_0/K) + rT}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T} \right). \end{aligned}$$

⁹For obscure stock price processes ξ might not be a \mathcal{P} -martingale, which implies that \mathcal{Q} as defined by (7) is not an equivalent probability measure. But if \mathcal{Q} is well-defined it is unique.

5 Change of numeraire

This section reviews a popular technique for solving the valuation equation

$$C_0 = e^{-rT} \mathbb{E}^{\mathcal{Q}} \left[(S_T - K)^+ \right].$$

The technique is often referred to as the change of numeraire technique because it involves a change of discounting factor from the bank-account to the underlying asset. The technique will prove to be extremely powerful: in this section we will derive the Black-Scholes formula without evaluating a single integral. But more than being a technical tool the change of numeraire approach also exposes an intriguing interpretation of the probabilities in the Black-Scholes formula as will be demonstrated in this section.

The change of numeraire technique showed up in several papers in the late eighties but it was probably known in the financial research community long before.

The idea is the following. We note that the martingale approach in the last section does not depend on the bank-account being used as numeraire for the pay-offs. In fact we could choose S as the numeraire of another martingale measure, \mathcal{Q}' , and under this measure C_t/S_t would be a martingale. To see this observe that

$$e^{-rt} \frac{S_t}{S_0} = \exp\left(-\frac{1}{2}\sigma^2 t + \sigma W_t^{\mathcal{Q}}\right)$$

is a positive \mathcal{Q} -martingale with mean 1. Hence, we can define a new equivalent probability measure related to \mathcal{Q} and \mathcal{P} by:

$$d\mathcal{Q}' = e^{-rt} \frac{S_t}{S_0} d\mathcal{Q} = e^{-rt} \frac{S_t}{S_0} \xi_t d\mathcal{P}$$

on \mathcal{F}_t . The Brownian motion under \mathcal{Q}' is then given by

$$W_t^{\mathcal{Q}'} = W_t^{\mathcal{Q}} - \sigma t = W_t^{\mathcal{P}} + \eta t - \sigma t.$$

Straightforward application of this yields the valuation equation

$$C_0 = S_0 \mathbb{E}^{\mathcal{Q}'} \left[\frac{(S_T - K)^+}{S_T} \right], \quad (13)$$

where

$$\frac{dS_t}{S_t} = (r + \sigma^2) dt + \sigma dW_t^{\mathcal{Q}'}$$

But this does not reduce the complexity of derivation of the Black-Scholes formula. We will still have to evaluate an integral like the one in the previous section. So

instead we reconsider our initial valuation equation. We split up the pay-off to get:

$$\begin{aligned} C_0 &= \mathbb{E}^{\mathcal{Q}} \left[e^{-rT} S_T \mathbf{1}_{\{S_T > K\}} \right] - e^{-rT} K \mathbb{E}^{\mathcal{Q}} \left[\mathbf{1}_{\{S_T > K\}} \right] \\ &= S_0 \mathcal{Q}'(S_T > K) - e^{-rT} K \mathcal{Q}(S_T > K). \end{aligned} \quad (14)$$

We feel that this equation has a very nice interpretation: Given that the European option finishes in-the-money, the option pay-off can be decomposed into two components, the first component is the uncertain amount S_T , and the second component is the fixed amount $-K$. The present value of receiving S_T at time T is of course S_0 . But this has to be weighted with some risk-adjusted probability of finishing in-the-money. \mathcal{Q}' is the right measure to use because it exactly off-sets the ‘ S ’-risk. In other words under \mathcal{Q}' pay-offs are valued as if one were ‘risk-neutral’ with respect to the risk of the underlying stock. The second component $-K$ is a fixed dollar amount. The proper probability measure to apply is therefore the measure \mathcal{Q} under which pay-offs are measured relative to the risk-less bond.

The formula is general, in the sense that it does not depend on the underlying stock following a geometric Brownian motion. In fact, if an equivalent martingale measure with the bank account as numeraire exists (and this measure need not be unique) then one can derive the above formula. In a subsequent section we will show that the European option price of the Cox, Ross & Rubinstein (1979) model has a similar interpretation.

To obtain the Black-Scholes formula we simply have to evaluate the two probabilities in the above equation. We observe that under \mathcal{Q}' , $\ln S_T$ is normal with mean and variance given by

$$\mathbb{E}^{\mathcal{Q}'} [\ln S_T] = \ln S_0 + (r + \frac{1}{2}\sigma^2)T \quad (15)$$

$$\text{Var}^{\mathcal{Q}'} [\ln S_T] = \sigma^2 T. \quad (16)$$

Using this and the distribution of $S(T)$ under \mathcal{Q} given in the previous section we immediately obtain

$$\begin{aligned} \mathcal{Q}'(S_T > K) &= \Phi \left(\frac{\ln(S_0/K) + rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T} \right) \\ \mathcal{Q}(S_T > K) &= \Phi \left(\frac{\ln(S_0/K) + rT}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T} \right), \end{aligned}$$

and thereby the Black-Scholes formula.

The change of numeraire technique is a very powerful tool that can be applied to other types of option contracts and to more general models. In the fixed income literature

this technique has elegantly been applied to option pricing problems under the name of "forward-risk-adjustment", see for example Jamshidian (1989) and El Karoui & Rochet (1989). In the context of exotic options the technique has shown useful in the evaluation of Asian options, lookback options, barrier options, and various other exotica. See for example Ingersoll (1987), Babbs (1992), Dufresne, Kierstad & Ross (1996), and Graversen & Peškir (1995).

6 Shaking your foundation

In this section we will use the so-called *local-time* to derive the Black-Scholes formula. The idea of using local-times in finance is due to Carr & Jarrow (1990). Analysing what is known as the *stop-loss start-gain strategy* they get terms involving local-times. The stop-loss start-gain strategy has also been carefully analysed in the literature by Seidenverg (1988) and Dybvig (1988). The financial insight using this somewhat cumbersome method is the proposal of a trading strategy which is not self-financing and still gives the Black-Scholes formula by taking care of the extra external financing. Moreover one should notice that recently local-times have been used to price American options. See for instance Myneni (1992) and Carr, Jarrow & Myneni (1992). Consider the following trading strategy:

If the present value of the strike price K is below the stock price hold one share of the stock. Finance this by using borrowed funds. If the stock price falls below the present value of the strike price liquidate the position. As we shall see below this strategy will at terminal date T be worth exactly the same as the call-option. Now if the stock price initially is worth less than the present value of the strike price the strategy costs nothing initially. Therefore if the strategy is self-financing this would create arbitrage-opportunities in the Black Scholes-economy. To analyse this strategy we proceed more formally. Let:

$$\begin{aligned} a_t &= \mathbf{1}_{\{S_t > KB_t\}}, \\ b_t &= -\mathbf{1}_{\{S_t > KB_t\}}K, \quad \forall t \in [0, T]. \end{aligned}$$

Then the value of the portfolio at time t , Y_t , is equal to:

$$\begin{aligned} Y_t &= a_t S_t + b_t B_t \\ &= \mathbf{1}_{\{S_t > KB_t\}} S_t - \mathbf{1}_{\{S_t > KB_t\}} K B_t \\ &= (S_t - K B_t)^+. \end{aligned} \tag{17}$$

Now we see that the value of the portfolio is always the lower bound for a call-option and furthermore since $B_T = 1$ we see that we have duplicated the call's payoff.

Therefore if $S_0/B_0 < K$ then the portfolio initially costs nothing. To examine if this portfolio is self-financing we notice that the self-financing condition described with the bond as numeraire is:

$$\frac{Y_t}{B_t} = \frac{Y_0}{B_0} + \int_0^t a_u dF_u, \quad \forall t \in [0, T], \quad (18)$$

where $F_t = S_t/B_t$ is the stock price with the bond as numeraire (the forward price). Inserting a_t and Y_t in (18) we get that the stop-loss and start-gain strategy is self-financing if and only if

$$(F_t - K)^+ = (F_0 - K)^+ + \int_0^t \mathbf{1}_{\{F_u > K\}} dF_u, \quad \forall t \in [0, T]. \quad (19)$$

Fortunately this is not the case - otherwise there could be arbitrage in the economy as described above. To see that the strategy is not self-financing we use the Tanaka-Meyer-formula on Y_t/B_t .¹⁰ It gives us:

$$(F_t - K)^+ = (F_0 - K)^+ + \int_0^t \mathbf{1}_{\{F_u > K\}} dF_u + \Lambda_t(K), \quad \forall t \in [0, T]. \quad (20)$$

We see that the difference between (19) and (20) is the term $\Lambda_t(K)$ which is called the *local time* at K by time t in the stochastic calculus literature. Now we will show that $\Lambda_t(K)$ is positive with positive probability for any t , which shows us that the stop-loss and start-gain strategy is not self-financing.

If we take the “risk-neutral” expectation of (20) we get:

$$\mathbb{E}_0^{\mathcal{Q}} [(F_t - K)^+] = (F_0 - K)^+ + \mathbb{E}_0^{\mathcal{Q}} [\Lambda_t(K)], \quad \forall t \in [0, T], \quad (21)$$

where the expectation of the “ dF_u ” integral is zero because F is a \mathcal{Q} -martingale and thereby we have that the integral-term is a \mathcal{Q} -martingale. It is obvious from the results in the previous sections that $\mathcal{Q}(F_t > K) > 0$, $\mathcal{Q}(F_t < K) > 0$.

Furthermore: Because $g(x) = (x - K)^+$ is strictly convex over an interval containing K we get that Jensen's inequality holds strictly for $g(x)$. I.e.:

$$\mathbb{E}_0^{\mathcal{Q}} [(F_t - K)^+] > (\mathbb{E}_0^{\mathcal{Q}} [F_t] - K)^+ = (F_0 - K)^+. \quad (22)$$

Combining (21) with (22) we get $\mathbb{E}_0^{\mathcal{Q}} [\Lambda_t(K)] > 0$. Since $\Lambda_t(K) \geq 0$ it follows that $\mathcal{Q}(\Lambda_t(K) > 0) > 0$, $\forall t \in (0, T]$. Therefore $\mathcal{P}(\Lambda_t(K) > 0) > 0$, $\forall t \in (0, T]$, which

¹⁰The Tanaka-Meyer formula can for instance be found in Karatzas & Shreve (1988) p. 220.

shows us that the strategy is not self-financing.

In our setting $\Lambda_t(K)$ has a very nice interpretation. Suppose that we change our strategy in the following way:

Buy one share of stock each time F rises from K to $K + \epsilon$, $\epsilon > 0$. In this case we should also go short in K bonds. We see that every time the transaction takes place it requires an additional ϵ bonds. Furthermore we liquidate the portfolio every time F goes back to K .

Now let $U_t(\epsilon)$ denote the number of times F has risen from K to $K + \epsilon$ until time t . Then we see that with the above mentioned strategy we would have to invest in $\epsilon U_t(\epsilon)$ bonds at time t to handle the external financing. Now it can be shown that:

$$\lim_{\epsilon \downarrow 0} \epsilon U_t(\epsilon) = \Lambda_t(K).$$

That is: The additional local time term from equation (20) can be interpreted as the external financing required to trade by the stop-loss start-gain strategy.

Now we will show that evaluating (21) for $t = T$ yields the Black-Scholes formula. From the previous section we notice that $C_0/B_0 = E_0^Q [(F_T - K)^+] = E_0^Q [(S_T - K)^+]$. I.e.:

$$C_0 = (S_0 - e^{-rT}K)^+ + e^{-rT} E_0^Q [\Lambda_T(K)]. \quad (23)$$

(23) has a nice interpretation: The first term on the right-hand side is the option's intrinsic value and is according to (17) equal to the initial investment required in the stop-loss start-gain strategy. The residual ($e^{-rT} E_0^Q [\Lambda_T(K)]$) is referred to as the option's time value which in this case is the present value of the expected external financial costs.

From the previous section we know that F is a Q -martingale. By Girsanov's theorem we therefore have that:

$$dF_t = \sigma F_t dW_t^Q.$$

That is:

$$F_t = F_0 \exp\{\sigma W_t^Q - \frac{1}{2}\sigma^2 t\}.$$

Therefore we get the following transition density for F :

$$\psi(F_t, t; F_0, 0) = \frac{1}{F_t \sigma \sqrt{t}} \phi\left(\frac{\ln\left(\frac{F_0}{F_t}\right) - \frac{1}{2}\sigma^2 t}{\sigma \sqrt{t}}\right), \quad (24)$$

where $\phi(z) \equiv \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right)$ is the standard normal density function.

Using a theorem about local times yields:¹¹

$$\mathbb{E}_0^Q \left[\int_0^T k(F_s) d \langle F \rangle_s \right] = \mathbb{E}_0^Q \left[2 \int_{-\infty}^{\infty} k(x) \Lambda_T(x) dx \right], \quad (25)$$

where k is a Borel-measurable function. Using that $d \langle F \rangle_t = \sigma^2 F^2(t) dt$ on the left side and employing Fubini's theorem on the right side of (25) gives us:

$$\int_{-\infty}^{\infty} k(x) \int_0^T \sigma^2 x^2 \psi(x, t; F_0, 0) dt dx = \int_{-\infty}^{\infty} k(x) 2\mathbb{E}_0^Q [\Lambda_T(x)] dx. \quad (26)$$

Now choose $k(x) = \mathbf{1}_{\{x \in A\}}$, where $A \in \mathcal{F}$. (26) then becomes:

$$\int_A \int_0^T \sigma^2 x^2 \psi(x, t; F_0, 0) dt dx = \int_A 2\mathbb{E}_0^Q [\Lambda_T(x)] dx.$$

Realizing that the integrands are nonnegative and that both integrals are equal for any $A \in \mathcal{F}$ we get:

$$\int_0^T \sigma^2 x^2 \psi(x, t; F_0, 0) dt = 2\mathbb{E}_0^Q [\Lambda_T(x)]. \quad (27)$$

Combining (24) with (27) yields:

$$\mathbb{E}_0^Q [\Lambda_T(K)] = \frac{\sigma K}{2} \int_0^T \frac{1}{\sqrt{t}} \phi \left(\frac{\ln \left(\frac{F_0}{K} \right) - \frac{1}{2} \sigma^2 t}{\sigma \sqrt{t}} \right) dt. \quad (28)$$

If we substitute (28) back into (23) we get:

$$C_0 = (S_0 - e^{-rT} K)^+ + e^{-rT} \frac{\sigma K}{2} \int_0^T \frac{1}{\sqrt{t}} \phi \left(\frac{\ln \left(\frac{S_0}{K e^{-rT}} \right) - \frac{1}{2} \sigma^2 t}{\sigma \sqrt{t}} \right) dt.$$

Now changing variable by $\nu \equiv \frac{\sqrt{t}\sigma}{\sqrt{T}}$ gives us:

$$C_0 = (S_0 - e^{-rT} K)^+ + e^{-rT} K \sqrt{T} \int_0^\sigma \phi \left(\frac{\ln \left(\frac{S_0}{K e^{-rT}} \right) - \frac{1}{2} \nu^2 T}{\nu \sqrt{T}} \right) d\nu.$$

Finally we have reached the Black-Scholes formula. This is seen by noticing that (2) differentiated with respect to σ is:

$$e^{-rT} K \sqrt{T} \phi \left(\frac{\ln \left(\frac{S_0}{K e^{-rT}} \right) - \frac{1}{2} \sigma^2 T}{\sigma \sqrt{T}} \right), \quad (29)$$

with the boundary condition that $C_0 = (S_0 - e^{-rT} K)^+$ when $\sigma = 0$.

¹¹The theorem can for instance be found in Karatzas & Shreve (1988) p.218.

7 The forward equation of European option prices

Compared to the technique applied in Section 5 the following derivation of the Black-Scholes formula might seem cumbersome. On the other hand the result that we will derive will shed further light on the European option pricing problem and will expose an interesting duality of the pricing problem that we consider. In the spirit of Dupire (1993) we derive a forward partial differential equation for the European option prices. In this equation the variables are the strike and the maturity whereas the current spot and time are fixed. This is opposed to the standard backward partial differential equation derived in Section 3, where the spot and time are the variables and strike and maturity are fixed. Examining this forward equation reveals that the option pricing problem can be solved in a dual economy where every parameter is turned upside down: the strike price is the underlying, the option is a put with strike equal to the current spot, time is reversed, etc.

Again we start from the valuation equation

$$C_0 = e^{-rT} \mathbb{E}^{\mathcal{Q}} [(S_T - K)^+] = e^{-rT} \int_K^{\infty} (x - K) \psi(x, T) dx, \quad (30)$$

where $\psi(x, T)$ is the \mathcal{Q} -density of S_T in the point x given S_0 at time 0.

Due to the Markov property of the spot price we have that ψ solves the forward Fokker-Planck equation¹²

$$0 = -\frac{\partial \psi}{\partial T} - \frac{\partial}{\partial x} [rx\psi] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [\sigma^2 x^2 \psi]$$

subject to the initial boundary condition $\psi(x, 0) = \delta(x - S_0)$, where $\delta(\cdot)$ is the Dirac Delta function.¹³ We will use this to derive a forward equation for the option prices.

Assuming that

$$rx\psi(x, T) \rightarrow 0, \quad \sigma^2 x^2 \psi(x, T) \rightarrow 0, \quad \frac{\partial}{\partial x} [\sigma^2 x^2 \psi(x, T)] \rightarrow 0,$$

for $x \rightarrow \infty$, which is clearly satisfied in the Black-Scholes model, integration of the forward equation over the interval (y, ∞) yields:

$$0 = -\frac{\partial}{\partial T} \int_y^{\infty} \psi(x, T) dx + ry\psi(y, T) - \frac{1}{2} \frac{\partial}{\partial y} [\sigma^2 y^2 \psi(y, T)].$$

Integrating once more, this time over (K, ∞) , yields

$$0 = -\frac{\partial}{\partial T} \int_K^{\infty} \int_y^{\infty} \psi(x, T) dx dy + r \int_K^{\infty} y\psi(y, T) dy + \frac{1}{2} \sigma^2 K^2 \psi(K, T)$$

¹²See Revuz & Yor (1991) p 269 for the Fokker-Planck equation.

¹³The Dirac Delta function is defined by $\delta(x) = 0$, for all $x \neq 0$, and $\int_{-\epsilon}^{\epsilon} \delta(x) dx = 1$, for all $\epsilon > 0$.

Now we go back to the pricing equation. Integrating by parts we get that

$$C_0 = e^{-rT} \int_K^\infty \int_y^\infty \psi(x, T) dx dy,$$

so

$$\frac{\partial}{\partial T} \int_K^\infty \int_y^\infty \psi(x, T) dx dy = r e^{rT} C_0 + e^{rT} \frac{\partial C_0}{\partial T}.$$

Further we have that

$$\begin{aligned} \int_K^\infty y \psi(y, T) dy &= e^{rT} C_0 - K e^{rT} \frac{\partial C_0}{\partial K} \\ \psi(K, T) &= e^{rT} \frac{\partial^2 C_0}{\partial K^2}. \end{aligned} \tag{31}$$

If we let $C(K, T)$ denote the initial price of a European call-option with strike K expiring at time T , we obtain the following forward partial differential equation for the European call-option prices

$$0 = -\frac{\partial C}{\partial T} - rK \frac{\partial C}{\partial K} + \frac{1}{2} \sigma^2 K^2 \frac{\partial^2 C}{\partial K^2} \tag{32}$$

subject to the initial boundary condition $C(K, 0) = (S_0 - K)^+$.

The forward equation can now be solved to yield the Black-Scholes formula.

The advanced reader might observe that the forward equation could be derived from the valuation equation under the \mathcal{Q}' measure, (13), combined with the time homogeneity of the stock price process in the Black-Scholes model. Under the assumption of a positive dividend yield of the underlying stock, Andreasen & Gruenewald (1996) apply this technique to obtain a forward equation for American call-options in the Black-Scholes model as well as in the jump-diffusion model of Merton (1976).¹⁴ But the forward equation (32) is more general; under assumption of sufficient regularity it holds for all Ito processes of the type

$$\frac{dS_t}{S_t} = \mu(t; \omega) dt + \sigma(S_t, t) dW_t^{\mathcal{P}}$$

if additionally the interest rate is only a function of time and the stock price. For stock option pricing and short maturities it is in most cases reasonable to assume (at most) time-dependent interest rates. Given today's yield curve it is then possible to uniquely determine the function $\sigma(S, t)$ from a full double continuum of option prices, $C(K, T)$, by the forward equation (32). The trick is simply to estimate the

¹⁴The positive dividend yield implies that the American call-option might be exercised prematurely.

derivatives in (32) and isolate the function $\sigma(\cdot, \cdot)$. In other words from a full set of marketed options we can ‘infer the option pricing model of the market’. This was first observed by Dupire (1993), but already Breeden & Litzenberger (1978) noted the relation (31). This relation tells us that we can infer the stock’s risk-adjusted distribution at a given maturity date from a continuum of option prices of different strikes.¹⁵

Another interesting implication of the forward equation is that when the volatility coefficient (now possibly a function of time and spot) is given we can price all options on the market by only solving one partial differential equation numerically. Andreasen (1996) observes that this also goes for the hedge ratios of the European options. To see this define

$$\Delta(K, T) = \frac{\partial C(K, T)}{\partial S} \Big|_{S=S_0}. \quad (33)$$

Differentiation of the forward equation (32) now yields

$$0 = -\frac{\partial \Delta}{\partial T} - rK \frac{\partial \Delta}{\partial K} + \frac{1}{2} \sigma^2 K^2 \frac{\partial^2 \Delta}{\partial K^2}$$

subject to the initial boundary condition $\Delta(K, 0) = \mathbf{1}_{\{S_0 \geq K\}}$.

Similar forward equations might be derived for other ‘Greeks’, i.e partial derivatives of the option price w.r.t. other parameters.

The equation can also be used to show how to hedge exotic options statically. In Carr, Ellis & Gupta (1998) a model with zero interest and a symmetry condition on $\sigma(\cdot, \cdot)$, which is clearly satisfied in the Black-Scholes setting, is developed. It is then shown how to hedge for instance down-and-out calls statically by ‘standard’ options. The last point to be stated is the duality of the option pricing problem implied by the forward equation. Suppose that the time axis is reversed, S_0 is a fixed quantity, and that we are sitting at time T evaluating

$$\mathbb{E}^{\mathcal{R}} \left[(S_0 - K_0)^+ \mid K_T = K \right] \quad (34)$$

for the process

$$\frac{dK_t}{K_t} = (-r)d(-t) + \sigma dW_t^{\mathcal{R}}$$

where $W^{\mathcal{R}}$ is some backward running Brownian motion under some probability measure \mathcal{R} . Then the forward equation (32) is the backward equation resulting for this problem. So we conclude that the option pricing problem might be solved in a dual economy where time is reversed, the strike is the underlying that pays a proportional

¹⁵For further exploration of this see for example Shimko (1991), Derman & Kani (1994), Rubinstein (1994), and Jackwerth & Rubinstein (1996).

dividend of r , the option is a put on the initial stock price, and finally the interest rate is equal to zero. Further we see that in this "space" the hedge ratio of the original economy will be a digital option. The option pricing can therefore be performed in a reversed binomial tree.¹⁶

Note that under \mathcal{R} we have

$$K_0 = K \exp \left(\left(-r - \frac{1}{2} \sigma^2 \right) T + \sigma (W_0^{\mathcal{R}} - W_T^{\mathcal{R}}) \right).$$

Using this in (34) gives the Black-Scholes formula by essentially the same calculations as those at the end of Section 4.

Using (33) we get the following expression for the hedge ratio

$$E^{\mathcal{R}} \left[\mathbf{1}_{\{K_0 \leq S_0\}} | K_T = K \right] = \Phi \left(\frac{\ln(S_0/K) + rT}{\sigma \sqrt{T}} + \frac{1}{2} \sigma \sqrt{T} \right).$$

8 A convergence proof

Cox et al. (1979) were the first to publish a paper with a formal convergence proof along the lines of this section. A much less known paper with the same result (and from the same year) is by Rendleman & Bartter (1979). But the use of binomial models for economic reasoning is much older, dating (at least) back to Arrow and Debreu in the 50'ies.

Let us consider the following situation: A stock today has a price of S_0 and can in the next period either go up to uS_0 or down to dS_0 . This happens with probabilities p and $1 - p$, respectively. In the economy there further exists a risk-free zero coupon bond maturing in the next period with (discretely compounded) interest rate r_d ($u > 1 + r_d > d > 0$, to avoid dominance), and a call-option on the stock with exercise price K . The situation is illustrated in Figure 1.

We are interested in hedging the option by trading a shares of stock and b bonds. A perfect hedge, i.e. an exact replication of the option's pay-off in every possible future state, is achieved by letting

$$a = \frac{C_u - C_d}{(u - d)S_0} \quad b = \frac{uC_d - dC_u}{(u - d)}$$

(notice that $a \approx \frac{\partial C}{\partial S}$, so the analogy to the continuous case is striking).

To prevent arbitrage opportunities the price of the hedge portfolio must be equal to

¹⁶For more on the duality see Dupire (1994) and Andreasen (1996).

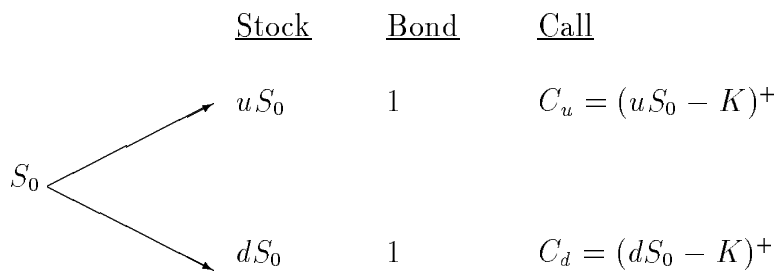


Figure 1: The One-Period Binomial Model

the present price of the call-option. Writing this out and reshuffling leads to

$$C_0 = R^{-1}(qC_u + (1 - q)C_d) \quad (35)$$

where $q = (R - d)/(u - d)$ and $R = 1 + r_d$. From this we see that the price of the call-option is the discounted expected future value where the expectation is under a measure that gives probability q of an 'up-jump'. Notice that the original probabilities do not enter the expression. This hedge argument is the key in Cox et al. (1979). Notice that we can write (35) as

$$\frac{C_0}{B_0} = E_0^{\mathcal{Q}} \left(\frac{C_T}{B_T} \right), \quad B_0 = R^{-T}$$

with obvious subscript notation and \mathcal{Q} denoting the measure naturally induced by q . So: Using the bond as numeraire, the call price is a \mathcal{Q} -martingale. In other words the notation is consistent with that of Section 5. At this point let us make an observation. If we let $q' = (uq)/(1 + r_d)$ then a direct inspection reveals that

$$\frac{C_0}{S_0} = E_0^{\mathcal{Q}'} \left(\frac{C_T}{S_T} \right)$$

with \mathcal{Q}' being the measure induced by q' . Again the notation is consistent: Using the stock as numeraire, the call price is a \mathcal{Q}' -martingale.

The argument is easily extended to a setting with n independent multiplicative binomial movements per unit of time ensuring us that the martingale pricing techniques of Section 4 carry over in a discrete setting. Using the arguments from Section 5 we can thus still write out the call price as

$$C_0 = S_0 \mathcal{Q}'(S_T^{(n)} > K) - K B_0 \mathcal{Q}(S_T^{(n)} > K) \quad (36)$$

where

$$\begin{aligned} S_T^{(n)} &= S_0 u^j d^{Tn-j}, & B_0 &= R^{-Tn}, \\ j &\stackrel{\mathcal{Q}}{\sim} \text{bi}(Tn, q), & j &\stackrel{\mathcal{Q}'}{\sim} \text{bi}(Tn, q'), \end{aligned}$$

and 'bi' denotes the binomial distribution. Again we claim that the call price at any time can only be a function of current stock price and time. This claim is then justified by our ability to exactly replicate the final pay-off which only depends on S_T by a dynamic trading strategy in the stock and the bond.

For computational purposes (36) is often rewritten as

$$C_0 = S_0 \tilde{\Phi}(m; Tn, q') - K B_0 \tilde{\Phi}(m; Tn, q) \quad (37)$$

with m being the smallest non-negative integer greater than $\ln(K/(S_0 d^{Tn}))/\ln(u/d)$ and $\tilde{\Phi}(m; Tn, q')$ denoting the complementary binomial distribution function.

Now let anything with an 'n' on it refer to a binomial model with n moves per time unit. Our aim is to show that as n approaches infinity the call price in the n -model converges to that of the Black-Scholes model. Because of the decompositions and (14) and (36) and the distribution results (11)-(12) and (15)-(16) our main task is to choose the parameters of the binomial model such that

$$\ln S_T^{(n)} \stackrel{\mathcal{Q}}{\rightarrow} N(\ln S_0 + (r - \frac{\sigma^2}{2})T, \sigma^2 T) \quad (38)$$

$$\ln S_T^{(n)} \stackrel{\mathcal{Q}'}{\rightarrow} N(\ln S_0 + (r + \frac{\sigma^2}{2})T, \sigma^2 T). \quad (39)$$

Regarding interest rates we don't have much choice but to let $R_n = e^{r/n}$. This means that the key parameters we have to choose are the sizes of the up and down moves, u_n and d_n . A good choice is

$$\ln u_n = \frac{\sigma}{\sqrt{n}} \quad (40)$$

$$\ln d_n = -\frac{\sigma}{\sqrt{n}}. \quad (41)$$

With M_n and V_n denoting mean and variance of $\ln S_T^{(n)}$ we then have

$$M_n^{\mathcal{Q}} = \ln S_0 + Tn(q_n \ln u_n + (1 - q_n) \ln d_n)$$

$$V_n^{\mathcal{Q}} = Tnq_n(1 - q_n)(\ln u_n - \ln d_n)^2,$$

and likewise for \mathcal{Q}' . Remembering that $q_n = \frac{e^{r/n} - d_n}{u_n - d_n}$ allows us to rewrite M_n and V_n by Taylor expanding the exponential function to the second order. This reveals that

$$M_n^{\mathcal{Q}} \rightarrow \ln S_0 + (r - \frac{\sigma^2}{2})T$$

$$V_n^{\mathcal{Q}} \rightarrow \sigma^2 T,$$

and by similar calculations we get convergence of \mathcal{Q}' -moments. So the first and second moments converge, under the respective measures, and the jumps vanish in the limit. This allows us to invoke (basically) a Lindeberg-Feller version of the Central Limit Theorem (see e.g. Duffie (1992)) to confirm the validity of (38) and (39). Finally dominated convergence ensures that the elements of the binomial decomposition (36) converge to their continuous counterparts, which establishes the desired result.

We have not used the original probabilities for anything (except that we have implicitly assumed them to be non-zero). It is easy to see that we can add any term of order n or higher in (40) and (41) and still have the same \mathcal{Q} (and \mathcal{Q}') convergence results. This could, if we were such inclined, help us establish convergence of the underlying process.

The model described in this section illustrates the fundamentals of pricing by no arbitrage using only linear algebra. Therefore it is ideal for teaching purposes. It is not the most advanced model in the paper, but it is 'solid as a rock.' It is also very handy when we want initial price estimates for exotic derivatives in cases where it is unclear how more advanced methods work, if indeed they do.

9 The continuous-time CAPM

This section shows that one might also obtain the Black-Scholes formula in the continuous-time capital asset pricing model by Merton (1971). The derivation is basically taken from Ingersoll (1987) but a similar derivation appears in Cox & Rubinstein (1985). Suppose that the market in total contains N (non-dividend paying) risky assets that evolve according to the N -dimensional stochastic differential equation

$$dS_t = I_{S_t}(\mu dt + \Sigma dW_t^{\mathcal{P}}),$$

where I_S is the diagonal matrix with diagonal elements (S_1, \dots, S_N) , μ is an N -dimensional constant vector, Σ is a constant $N \times N$ matrix, and $W^{\mathcal{P}}$ is an N -dimensional Brownian motion under \mathcal{P} . For simplicity we will suppose that Σ has full rank. Suppose that there additionally exists a risk free asset paying a constant continuously compounded interest rate r . Consider an investor that maximises expected additive utility on some time horizon $[0, \tau]$,

$$\mathbb{E}^{\mathcal{P}} \left[\int_0^{\tau} u(x_t, t) dt \right],$$

over consumption flow, x , and risky portfolio holdings vector, a , subject to the self-financing constraint or dynamic budget constraint (where $'$ denotes transposition)

$$\begin{aligned} dV_t &= a'_t dS_t + (V_t - a'_t S_t) r dt - x_t dt \\ &= (V_t \theta'_t (\mu - r\mathbf{1}) + rV_t - x_t) dt + V_t \theta'_t \Sigma dW_t^{\mathcal{P}} \end{aligned}$$

where V_t is the current wealth and θ is the N dimensional vector with elements $\theta_i = a_i S_i / V$. θ_i is the fraction of the investor's wealth invested in risky asset i .

Defining the indirect utility as

$$J(V_t, t) = \max_{(\theta_s, x_s)_{s \geq t}} \mathbb{E}_t^{\mathcal{P}} \left[\int_t^{\tau} u(x_s, s) ds \right],$$

we get the Bellman-Hamilton equation¹⁷

$$0 = \max_{\theta, x} u + \frac{\partial J}{\partial t} + (V \theta' (\mu - r\mathbf{1}) + rV - x) \frac{\partial J}{\partial V} + \frac{1}{2} V^2 \theta' \Sigma \Sigma' \theta \frac{\partial^2 J}{\partial V^2}.$$

The first order conditions imply that in optimum

$$\theta = - \frac{\partial J / \partial V}{V \partial^2 J / \partial V^2} (\Sigma \Sigma')^{-1} (\mu - r\mathbf{1}).$$

Observe that for all i, j the ratio θ_i / θ_j is independent of wealth and utility. So if all investors have additive separable utility they will all hold the same portfolio of risky assets. This means that the market portfolio of risky assets will be given by

$$\theta_M = k (\Sigma \Sigma')^{-1} (\mu - r\mathbf{1})$$

for some one-dimensional process k . The expected instantaneous excess return of the (risky) market portfolio is therefore

$$\mu_M - r = k (\mu - r\mathbf{1})' (\Sigma \Sigma')^{-1} (\mu - r\mathbf{1}),$$

and the local variance of the market return is

$$v_M^2 = k^2 (\mu - r\mathbf{1})' (\Sigma \Sigma')^{-1} (\mu - r\mathbf{1}).$$

The vector of local covariances between the market portfolio and instantaneous return of the assets is given by

$$c = k (\mu - r\mathbf{1}).$$

¹⁷For an intuitive proof of the Bellman-Hamilton equation see for example Ingersoll (1987).

Combining these equations we get

$$\mu_i = r + \frac{c_i}{v_M^2}(\mu_M - r).$$

Now suppose an option contract on S_i is introduced on the market in zero net supply. Since the market is dynamically complete the market equilibrium is not changed and the above expected return relation is still valid. If the option price is only a function of the underlying stock and time, the Ito formula implies that the local covariance of the return of the option with the market return can be written as

$$\frac{1}{C} \frac{\partial C}{\partial S} c_i$$

Therefore the expected instantaneous return of the option contract is given by

$$r + \frac{1}{C} \frac{\partial C}{\partial S} \frac{c_i}{v_M^2} (\mu_M - r).$$

Using the Ito formula on the option price, $C(S_i(t), t)$, yields that the instantaneous return of the option contract is given by

$$\frac{1}{C} \left[\frac{\partial C}{\partial t} + \mu_i S_i \frac{\partial C}{\partial S_i} + \frac{1}{2} \|\Sigma_i\|^2 S_i^2 \frac{\partial^2 C}{\partial S_i^2} \right].$$

Equating this to the return of the option and inserting the expected return of the underlying stock yields the partial differential equation

$$rC = \frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2},$$

where we have omitted the subscript on the stock and introduced the notation $\sigma^2 = \|\Sigma_i\|^2$. We have thereby derived the Black-Scholes partial differential equation in the context of the continuous-time CAPM. The formula for the European call can be calculated as in Section 3. This approach relies on the assumption that the option price is a function of time and current stock only. As shown in Section 3 this assumption can be justified by the Black-Scholes hedging argument that uniquely fixes the option price given no arbitrage possibilities. But here our argumentation is not based on a hedging argument but rather a risk-return relation. Therefore the above derivation shows the consistency of the Black-Scholes formula with the CAPM pricing relation.

From the last equations it is tempting to conclude that the Black-Scholes formula or a preference-free pricing formula can be derived in the context of any linear factor model of expected asset return like the continuous-time CAPM. This is only true

though if the market additionally is dynamically complete or effectively complete. In general, if a new asset is introduced in an incomplete economy, a new equilibrium will be the outcome and the prices of the existing assets will change. As mentioned, an exception to this is when an incomplete market is effectively complete. An example of this situation is the consumption based capital asset pricing model described in Merton (1973*a*) where the state variables determining the investment opportunity set are spanned by the marketed assets.¹⁸

10 A representative investor approach

In this section we show that the Black-Scholes formula can be derived in a model where the continuous-trade assumption is replaced by the assumption of a representative investor with power utility. The approach was introduced by Rubinstein (1976).

First, let us consider a one-period model where trading can be performed at the times $0, T$. Suppose an agent maximises expected utility of terminal consumption

$$\mathbb{E}^{\mathcal{P}} [u(x_T)]$$

subject to the budget constraint

$$\begin{aligned} x_T &= a' S_T \\ V_0 &= a' S_0, \end{aligned}$$

where a is the vector of portfolio holdings, S is the vector of prices of the marketed assets, and V_0 is the initial wealth. Forming the Lagrangian yields the first order condition

$$S_0 = \lambda^{-1} \mathbb{E}^{\mathcal{P}} [u'(x_T) S_T],$$

where the prime denotes the first derivative and λ is the Lagrange multiplier of the budget constraint. Specifically we get for the risk-free asset

$$e^{-rT} = \lambda^{-1} \mathbb{E}^{\mathcal{P}} [u'(x_T)].$$

Combining these equations we get the valuation equation

$$S_0 = e^{-rT} \mathbb{E}^{\mathcal{P}} \left[\frac{u'(x_T)}{\mathbb{E}^{\mathcal{P}} [u'(x_T)]} S_T \right].$$

¹⁸See also Christensen, Graversen & Miltersen (1996).

With these preliminaries let us now assume that the market has a representative investor with power utility function

$$u(x) = \frac{x^{1+\gamma}}{1+\gamma},$$

with $\gamma < 0$, i.e. $-\gamma$ is the constant relative risk aversion of the representative investor. Now we redefine the notation; let S be the price of one particular stock, and V_0 be initial aggregate wealth.

Assume that aggregate consumption at time T and the time T stock price are jointly log-normally distributed, so that we can write

$$\begin{aligned} S_T &= S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma W_T^{\mathcal{P}}} \\ x_T &= V_0 e^{(\mu_x - \frac{1}{2}\sigma_x^2)T + \sigma_x W_{x,T}^{\mathcal{P}}}, \end{aligned}$$

where $W^{\mathcal{P}}, W_x^{\mathcal{P}}$ are \mathcal{P} -Brownian motions with constant correlation ρ .

Note that

$$\frac{u'(x_T)}{\mathbb{E}^{\mathcal{P}}[u'(x_T)]} = e^{-\frac{1}{2}\gamma^2\sigma_x^2 T + \gamma\sigma_x W_{x,T}^{\mathcal{P}}}.$$

For the market to be in equilibrium we must have that the valuation equation holds for the stock. Inserting the above in the valuation equation yields

$$\begin{aligned} S_0 &= S_0 e^{-rT} \mathbb{E}^{\mathcal{P}} \left[e^{(\mu - \frac{1}{2}(\sigma^2 + \gamma^2\sigma_x^2))T + \sigma W_T^{\mathcal{P}} + \gamma\sigma W_{x,T}^{\mathcal{P}}} \right] \\ &= S_0 e^{(\mu + \sigma\gamma\sigma_x\rho - r)T}, \end{aligned}$$

so

$$\mu = r - \gamma\rho\sigma\sigma_x. \quad (42)$$

Now we want to evaluate a call-option on S_T with strike K . Using the valuation equation and the above derivations we get:

$$\begin{aligned} C_0 &= e^{-rT} \mathbb{E}^{\mathcal{P}} \left[e^{-\frac{1}{2}\gamma^2\sigma_x^2 T + \gamma\sigma_x W_{x,T}^{\mathcal{P}}} (S_T - K)^+ \right] \\ &= S_0 \mathbb{E}^{\mathcal{P}} \left[e^{-\frac{1}{2}(\sigma^2 + 2\gamma\rho\sigma\sigma_x + \gamma^2\sigma_x^2)T + \sigma W_T^{\mathcal{P}} + \gamma\sigma_x W_{x,T}^{\mathcal{P}}} \mathbf{1}_{\{S_T \geq K\}} \right] \\ &\quad - K e^{-rT} \mathbb{E}^{\mathcal{P}} \left[e^{-\frac{1}{2}\gamma^2\sigma_x^2 T + \gamma\sigma_x W_{x,T}^{\mathcal{P}}} \mathbf{1}_{\{S_T \geq K\}} \right] \end{aligned}$$

By introducing the joint density of $(W^{\mathcal{P}}, W_x^{\mathcal{P}})_T$ we could calculate the expectations to give us the Black-Scholes formula. But it is much easier to make use of the change of measure induced by the Girsanov factors under the expectations.

Define two new equivalent probability measure \mathcal{Q}' and \mathcal{Q} by the Radon-Nikodym derivatives

$$\begin{aligned} \frac{d\mathcal{Q}'}{d\mathcal{P}} &= e^{-\frac{1}{2}(\sigma^2 + 2\gamma\rho\sigma\sigma_x + \gamma^2\sigma_x^2)T + \sigma W_T^{\mathcal{P}} + \gamma\sigma_x W_{x,T}^{\mathcal{P}}} \\ \frac{d\mathcal{Q}}{d\mathcal{P}} &= e^{-\frac{1}{2}\gamma^2\sigma_x^2 T + \gamma\sigma_x W_{x,T}^{\mathcal{P}}}. \end{aligned}$$

Using these probability measures we can write

$$C_0 = S_0 \mathcal{Q}'(S_T > K) - e^{-rT} K \mathcal{Q}(S_T > K).$$

The Girsanov Theorem together with relation (42) imply that

$$\begin{aligned} S_T &= S_0 e^{rT + \frac{1}{2}\sigma^2 T + \sigma W_T^{\mathcal{Q}'}} \\ &= S_0 e^{rT - \frac{1}{2}\sigma^2 T + \sigma W_T^{\mathcal{Q}}}, \end{aligned}$$

where $W^{\mathcal{Q}'}$, $W^{\mathcal{Q}}$ are some standard normal Brownian motions under the two respective probability measures.¹⁹ Using this we immediately obtain the Black-Scholes formula. In this section the assumption of continuous trade was replaced by the assumption of existence of a representative investor. Unless investors have identical or very similar preferences a representative investor is in general not guaranteed to exist in an incomplete market like the one analysed. Even if a representative investor exists, the market equilibrium, prices of existing assets, and the representative preferences might change when a new asset (in this case the option) is introduced on an incomplete market. Despite these drawbacks this approach is widely used in models of incomplete markets.

11 Discussion

Economics has been described as the only field where people can win Nobel Prizes for saying the exact opposite things. This paper has shown that in the subset of economics known as 'financial mathematics' there is a very high degree of consistency between models and approaches.

We did this by showing that as special cases they could produce the Black-Scholes formula which, despite its widespread recognition and use, certainly is no trivial result. Some of the methods seemed different (compare the PDEs of Section 3 to the SDEs of Section 4) - but were in fact very similar. Some approaches seemed at a first glance to offer little extra (comparing Section 4 to Sections 5 and 7 it is unclear what could possibly be the benefit of 'counting in units of the stock' or 'letting time run backwards' in a time-homogeneous model) - but they did. One model was very intuitive (Section 8) - one was very much the opposite (Section 6). Two models (Sections 9 and 10) build on the long-honoured economic concept of utility

¹⁹Notice: The correlation between the two coordinates is ρ no matter which of the two measures (\mathcal{Q} or \mathcal{Q}') we use.

maximisation, and produced the Black-Scholes formula as a special case when utility functions and/or distributions of returns were restricted.

Still, after eight proofs of the Black-Scholes formula skeptics could ask if the finance community has not progressed beyond that result. We believe that it has, and that the criticism is unreasonable. We have illustrated that by, in each of the Sections 3 through 10 outlining the applicability of the particular model or approach to more general cases than the one originally considered by Black, Scholes and Merton. This included both more advanced dynamics of fundamentals and contractually more sophisticated derivatives. Hence we hope to have convinced the reader that each method has validity beyond the basic setting, which in turn justifies the research done in the past, as well as the research that will continue for a long time. If the reader was already aware of this, we hope to have provided 'a couple of cheap thrills.'

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