CHAPTER 26
jump diffusion

In this Chapter...

- the Poisson process for modeling jumps
- hedging in the presence of jumps
- how to price derivatives when the path of the underlying can be discontinuous
- jump volatility

26.1 INTRODUCTION

There is plenty of evidence that financial quantities, be they equities, currencies or interest rates, for example, do not follow the lognormal random walk that has been the foundation of almost everything in this book, and almost everything in the financial literature. We look at some of this evidence in a moment. One of the striking features of real financial markets is that every now and then there is a sudden unexpected fall or crash. These sudden movements occur far more frequently than would be expected from a Normally-distributed return with a reasonable volatility. On all but the shortest timescales the move looks discontinuous, the prices of assets have jumped. This is important for the theory and practice of derivatives because it is usually not possible to hedge through the crash. One certainly cannot delta hedge as the stock market tumbles around one’s ankles, and to offload all one’s positions will lead to real instead of paper losses, and may even make the fall worse.

In this chapter I explain classical ways of pricing and hedging when the underlying follows a jump-diffusion process.

26.2 EVIDENCE FOR JUMPS

Let’s look at some data to see just how far from Normal the returns really are. There are several ways to visualize the difference between two distributions, in our case the difference between the empirical distribution and the Normal distribution. One way is to overlay the two probability distributions. In Figure 26.1 we see the distribution of Xerox returns, from 1986 until 1997, normalized to unit standard deviation. The peak of the real distribution is clearly higher than the Normal distribution. Because both of these distributions have the same standard deviation then the higher peak must be balanced by fatter tails, it’s just that they would be too small to see on this figure. They may be too small to see here, but they are still very important.

This difference is typical of all markets, even typical of the changes in interest rates. The empirical distribution diverges from the Normal distribution quite markedly. The peak being
much higher means that there is a greater likelihood of a small move than we would expect from the lognormal random walk. More importantly, and concerning the subject of this chapter, the tails are much fatter. There is a greater chance of a large rise or fall than the Normal distribution predicts.

In Figure 26.2 is shown the difference between the cumulative distribution functions for the standardized returns of Xerox, and the Normal distribution. If you look at the figure you will see that there is more weight in the empirical distribution than the Normal distribution from about two standard deviations away from the mean. If you couple this likelihood of an extreme movement with the importance of an extreme movement, assuming perhaps that people are hedged against small movements, you begin to get a very worrying scenario.

The final picture that I plot, in Figure 26.3, is called a Quantile-Quantile or Q-Q plot. This is a common way of visualizing the difference between two distributions when you are particularly interested in the tails of the distribution. This plot is made up as follows. Rank the empirical returns in order from smallest to largest, call these $y_i$ with an index $i$ going from 1 to $n$. For the Normal distribution find the returns $x_i$ such that the cumulative distribution function at $x_i$ has value $i/n$. Now plot each point $(x_i, y_i)$. The better the fit between the two distributions, the closer the line is to straight. In the present case the line is far from straight, due to the extra weight in the tails.

Several theories have been put forward for the non-Normality of the empirical distribution. Three of these are

- Volatility is stochastic
- Returns are drawn from another distribution, a Pareto-Levy distribution, for example
- Assets can jump in value

There is truth in all of these. The first was the subject of Chapter 23. The second is a can of worms; moving away from the Normal distribution means throwing away 99% of current
theory and is not done lightly. But some of the issues this raises have to be addressed in jump diffusion (such as the impossibility of hedging). The third is the present subject.

26.3 POISSON PROCESSES

The basic building block for the random walks we have considered so far is continuous Brownian motion based on the Normally-distributed increment. We can think of this as adding to the return from one day to the next a Normally distributed random variable with variance proportional to the timestep. The extra building block we need for the jump-diffusion model for an asset price is the Poisson process.

We’ll be seeing this process again in another context in Chapter 43. A Poisson process \( dq \) is defined by

\[
dq = \begin{cases} 
0 & \text{with probability } 1 - \lambda dt \\
1 & \text{with probability } \lambda dt.
\end{cases}
\]

There is therefore a probability \( \lambda dt \) of a jump in \( q \) in the timestep \( dt \). The parameter \( \lambda \) is called the intensity of the Poisson process. The scaling of the probability of a jump with the size of the timestep is important in making the resulting process ‘sensible,’ i.e. there being a finite chance of a jump occurring in a finite time, with \( q \) not becoming infinite.

This Poisson process can be incorporated into a model for an asset in the following way:

\[
dS = \mu S dt + \sigma S dX + (J - 1) S dq.
\]

We assume that there is no correlation between the Brownian motion and the Poisson process. If there is a jump (\( dq = 1 \)) then \( S \) immediately goes to the value \( JS \). We can model a sudden 10% fall in the asset price by \( J = 0.9 \).

We can generalize further by allowing \( J \) to also be a random quantity. We assume that it is drawn from a distribution with probability density function \( P(J) \), again independent of the Brownian motion and Poisson process.

The random walk in \( \log S \) follows from (26.1):

\[
d(\log S) = (\mu - \frac{1}{2}\sigma^2) dt + \sigma dX + (\log J) dq.
\]

This is just a jump-diffusion version of Itô.

Opposite is a spreadsheet showing how to simulate the random walk for \( S \). In this simple example the stock jumps by 20% at random times given by a Poisson process.

26.4 HEDGING WHEN THERE ARE JUMPS

Now let us build up a theory of derivatives in the presence of jumps. Begin by holding a portfolio of the option and \( -\Delta \) of the underlying:

\[
\Pi = V(S, t) - \Delta S.
\]

The change in the value of this portfolio is

\[
d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \left( \frac{\partial V}{\partial S} - \Delta \right) dS + (V(JS, t) - V(S, t) - \Delta(J - 1)S) dq.
\]

Again, this is a jump-diffusion version of Itô.

\[
\]

Figure 26.4 Spreadsheet simulation of a jump-diffusion process.

If there is no jump at time \( t \) so that \( dq = 0 \), then we could have chosen \( \Delta = \partial V/\partial S \) to eliminate the risk. If there is a jump and \( dq = 1 \) then the portfolio changes in value by an \( O(1) \) amount, that cannot be hedged away. In that case perhaps we should choose \( \Delta \) to minimize the variance of \( d\Pi \). This presents us with a dilemma. We don’t know whether to hedge the small(ish) diffusive changes in the underlying which are always present, or the large moves which happen rarely. Let us pursue both of these possibilities.

26.5 HEDGING THE DIFFUSION

If we choose

\[
\Delta = \frac{\partial V}{\partial S}
\]
we are following a Black–Scholes type of strategy, hedging the diffusive movements. The change in the portfolio value is then

\[
d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \left( V(S, t) - V(S, t) - (S - J) \frac{\partial V}{\partial S} \right) dq.
\]

The portfolio now evolves in a deterministic fashion, except that every so often there is a non-deterministic jump in its value. It can be argued (see Merton, 1976) that if the jump component of the asset price process is uncorrelated with the market as a whole, then the risk in the discontinuity should not be priced into the option. Diversifiable risk should not be rewarded. In other words, we can take expectations of this expression and set that value equal to the risk-free return from the portfolio. This is not completely satisfactory, but is a common assumption whenever there is a risk that cannot be fully hedged; default risk is another example of this. If we take such an approach then we arrive at

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV + \lambda E[V(S, t) - V(S, t)] - \lambda \frac{\partial V}{\partial S} SE(J - 1) = 0. \tag{26.3}
\]

\(E[\cdot]\) is the expectation taken over the jump size \(J\), which can also be written

\[
E[\lambda] = \int \lambda P(J) dJ,
\]

where \(P(J)\) is the probability density function for the jump size.

This is a pricing equation for an option when there are jumps in the underlying. The important point to note about this equation, that makes it different from others we have derived, is its non-local nature. That is, the equation links together option values at distant \(S\) values, instead of just containing local derivatives. Naturally, the value of an option here and now depends on the prices to which it can instantaneously jump. When \(\lambda = 0\) the equation reduces to the Black–Scholes equation.

There is a simple solution of this equation in the special case that the logarithm of \(J\) is Normal distributed. If the logarithm of \(J\) is Normally distributed with standard deviation \(\sigma'\) and if we write

\[
k = E[J - 1]
\]

then the price of a European non-path-dependent option can be written as

\[
\sum_{n=1}^{\infty} \frac{1}{n!} e^{-\lambda'(T-t)} \lambda'(T-t)^n V_{BS}(S,t;\sigma',r).
\]

In the above

\[
\lambda' = \lambda(1 + k), \quad \sigma'^2 = \sigma^2 + \frac{n \sigma^2}{T-t} \quad \text{and} \quad r = r - \frac{n \log(1 + k)}{T-t},
\]

and \(V_{BS}\) is the Black–Scholes formula for the option value in the absence of jumps. This formula can be interpreted as the sum of individual Black–Scholes values, each of which assumes that there have been \(n\) jumps, and they are weighted according to the probability that there will have been \(n\) jumps before expiry.

If one does not make the assumption that jumps should not be priced in, then one has to play around with concepts such as the market price of risk.

26.6 Hedging the Jumps

In the above we hedged the diffusive element of the random walk for the underlying. Another possibility is to hedge both the diffusion and jumps as much as we can. For example, we could choose \(\Delta\) to minimize the variance of the hedged portfolio. After all, this is ultimately what hedging is about.

The change in the value of the portfolio with an arbitrary \(\Delta\) is, to leading order,

\[
d\Pi = \left( \frac{\partial V}{\partial S} - \Delta \right) dS + (-\Delta(J - 1) + V(JS, t) - V(S, t)) dq + \ldots.
\]

The variance in this change, which is a measure of the risk in the portfolio, is

\[
\text{var}[d\Pi] = \left( \frac{\partial V}{\partial S} - \Delta \right)^2 \sigma^2 S^2 dt + \lambda E[(\Delta(J - 1) + V(JS, t) - V(S, t))^2] dt + \ldots.
\]

This is minimized by the choice

\[
\Delta = \frac{\lambda E[(J - 1) + V(JS, t) - V(S, t)]}{\lambda SE(J - 1)^2 + \sigma^2 S}.
\]

(To see this, differentiate (26.4) with respect to \(\Delta\) and set the resulting expression equal to zero.)

If we value the options as a pure discounted real expectation under this best-hedge strategy, then we find that

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV + \lambda E \left[ (V(JS, t) - V(S, t)) \left( 1 - \frac{J - 1}{d} (\mu + \lambda k - r) \right) \right] = 0.
\]

where

\[
d = \lambda E(J - 1)^2 + \sigma^2.
\]

When \(\lambda = 0\) this collapses to the Black–Scholes equation. At the other extreme, when there is no diffusion, so that \(\sigma = 0\), we have

\[
\Delta = \frac{E(J - 1)(V(JS, t) - V(S, t))}{SE(J - 1)^2}
\]

and

\[
\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} - rV + \lambda E \left[ (V(JS, t) - V(S, t)) \left( 1 - J - 1 d (\mu + \lambda k - r) \right) \right] = 0.
\]

All of the pricing equations we have seen in this chapter are integro-differential equations. (The integral nature is due to the expectation taken over the jump size.) Because of the convolution nature of these equations they are candidates for solution by Fourier transform methods.