

Barrier Options and Lumpy Dividends

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Abstract

We study the pricing of barrier options on stocks with lumpy dividends. By extending the European option methodology presented in Haug, Haug & Lewis (2003), we show that in the Black-Scholes model with a single dividend payment, barrier option prices can be expressed in terms of well-behaved one-dimensional integrals that can be evaluated very rapidly. With multiple dividend payments, the price integrals are more involved, but we show that for the down-and-out call option a simple approximation method gives accurate results.

Keywords: Barrier options, lumpy dividends, numerical integration, finite difference method, down-and-out call.

1 Introduction

Dividend payments impact stock price dynamics and consequently the pricing of equity derivatives. The common solution of approximating lumpy dividends with a

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continuous dividend yield works reasonably well for stock indices,¹ but the discreteness of dividends cannot be ignored for derivatives written on individual stocks. This issue becomes even more acute for barrier options than for plain-vanilla European options, essentially because dividend payments affect not only the stock price distribution at expiry, but also the probability of barrier crossings.

The literature contains a number of suggestions for how to adjust derivative prices for discrete dividends in the underlying stock, see for instance Beneder & Vorst (2001), Bos, Gairat & Shepeleva (2003), Chriss (1997) and Frishling (2002). However, as discussed extensively in Haug et al. (2003), there are various problems with all of these approaches, ranging from ill-defined stock prices to poor performance of approximations in particular situations. Haug et al. (2003) resolve these issues by proposing a rigorous, yet natural, method for pricing European-type derivatives in the case of a single dividend — we describe the approach in Section 2.

In Section 3, we extend the methodology to barrier options. The key observation is that while barrier options are path-dependent, they are only weakly so — given that a barrier option has not yet been knocked out, the price is a function only of time and the current stock price. We show that in the Black-Scholes model with a single dividend payment, the barrier option price is a one-dimensional integral of Black-Scholes-type functions against a log-normal density which is straightforward to evaluate numerically.

The case of a single dividend is the most relevant by far since dividends are usually paid yearly and most derivatives — especially exotic ones — have much shorter life-spans than that. For a point in case, so-called Turbo warrants typically expire within one year. These contracts are popular in many countries because they enable retail investors to acquire the payoff profile of a barrier option.²

Section 4 demonstrates that the proposed numerical integration method is significantly faster than a general purpose attack on the problem — solving the pricing PDE with the Crank-Nicolson finite difference scheme. In Section 5, we look at

¹It is important to account for the seasonality in dividend payments. To quote a practitioner who shall remain nameless: “In other words - don’t try using constant div-yield BS, even on indices”.

²Turbo warrants usually come in two forms — the long version is a regular down-and-out call (barrier lower than or equal to the strike) and the short version is a regular up-and-out put (barrier higher than or equal to the strike) — see Mahayni & Suchanecki (2006), Engelmann, Fengler, Nalholm & Schwendner (2007), Wilkens & Stoimenov (2007) and Wong & Chan (2008).

multiple dividends and show that an approximation approach similar to the one considered for plain-vanilla options in Haug et al. (2003) is feasible for the down-and-out call option.

2 European options for single dividends

This section reviews the approach from Haug et al. (2003) for pricing European-style contracts written on a stock that pays a single dividend.

Fix a time interval $[0, T]$ and consider the stock price for a company that pays the dividend d at a deterministic time $\tau \in (0, T)$ and that the size of the dividend is anticipated in the market prior to τ (“ $\mathcal{F}_{\tau-}$ -measurable” if you like). A simple arbitrage argument shows that the stock price will decrease by d at the same instant as the dividend is paid. Let $r \geq 0$ denote the interest rate and let $(S_t)_{t \in [0, T]}$ model the price of the stock under the risk-neutral measure:

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dW_t, \text{ when } 0 \leq t < \tau \text{ or } \tau < t \leq T, \\ S_\tau &= S_{\tau-} - d, \text{ otherwise,} \end{aligned} \tag{1}$$

where $(W_t)_{t \in [0, T]}$ is a standard Brownian motion. Companies usually declare a dividend size D in advance, but for the model to be well defined one has to be careful and not assume that the actual dividend d is deterministic — if the stock price happens to be less than that constant at time $\tau-$, the company is unable to pay the declared dividend — we must have that $0 \leq d \leq S_{\tau-}$. This technicality, which is ignored for instance by Frishling (2002) and Bos et al. (2003), can be resolved by assuming that $d = d(S_{\tau-})$ is a deterministic function of the stock price the instant prior to the dividend payment. Haug et al. (2003) consider $d(s) = \min\{s, D\}$ (“the liquidator”) and $d(s) = D\mathbf{1}_{\{s > D\}}$ (“the survivor”) for a given deterministic $D \geq 0$. Another natural choice would be $d(s) = \alpha s$ for some constant α with $0 < \alpha < 1$ — a proportional dividend.

Let $C(t, s) = e^{-r(T-t)}\mathbf{E}[(S_T - K)^+ | S_t = s]$ denote the price at time t of a European call option with strike K and expiry T , given that $S_t = s$. At time τ , the one and only dividend has been paid, hence the stock price follows a geometric Brownian motion on $[\tau, T]$ — the price $C(\tau, s)$ is thus the usual Black-Scholes price. Now, to compute the price of the contract at time 0, simply condition on the information

available at time τ :

$$\begin{aligned}
e^{-rT} \mathbf{E}[(S_T - K)^+] &= e^{-r\tau} \mathbf{E}[e^{-r(T-\tau)} E[(S_T - K)^+ | S_\tau]] \\
&= e^{-r\tau} \mathbf{E}[C(\tau, S_\tau)] \\
&= e^{-r\tau} \mathbf{E}[C(\tau, S_{\tau-} - d(S_{\tau-}))].
\end{aligned} \tag{2}$$

The last expectation is a one-dimensional integral of the function $s \mapsto C(\tau, s - d(s))$ multiplied by a log-normal density and can easily be evaluated numerically.

The method is valid not only for the plain-vanilla European call option, but generally for contracts with payoff $f(S_T)$ at time T , provided that the function $s \mapsto \mathbf{E}[f(S_T) | S_\tau = s]$ is available in closed form, or at least is easy to compute. Moreover, it is valid for more general models than the geometric Brownian motion. As long as the density of the stock price at the instant prior to the dividend payment is available, then the expectation (2) can be computed by numerical integration.

3 Barrier options for single dividends

We now extend the approach described in the previous section to treat barrier options in the case of a single discrete dividend. For concreteness, we specialize in a down-and-out call option, but it is easy to see that similar calculations work for other contracts, in particular for the other single barrier options.

We compute the price at time 0 of a down-and-out call option with expiry T , strike price K , barrier L , with $0 \leq L \leq K$, written on the dividend-paying stock. This contract is a contingent claim with time- T payoff

$$(S_T - K)^+ \mathbf{1}_{\{\min_{0 \leq t \leq T} S_t > L\}}. \tag{3}$$

At time τ , the one and only dividend has been paid, hence the stock price follows a geometric Brownian motion on $[\tau, T]$. From Björk (2004, Proposition 18.17), we have that the price of the contract at time τ is $P(\tau, S_\tau)$, where

$$P(\tau, s) := \left(C(\tau, s) - \left(\frac{L}{s} \right)^p C\left(\tau, \frac{L^2}{s} \right) \right) \mathbf{1}_{\{s > L\}}, \tag{4}$$

with $p := \frac{2r}{\sigma^2} - 1$. As in the previous section, $C(\tau, s)$ denotes the usual no-dividend Black-Scholes price at time τ of a European call option with strike K and expiry T .

The key observation that enables us to compute the price at time 0 of the claim (3) is that at time τ , given that the barrier has not yet been crossed, the value of

the option is precisely $P(\tau, S_\tau) = P(\tau, S_{\tau-} - d(S_{\tau-}))$. Consequently, at time 0 we can view the barrier option we are trying to price as a contingent claim with expiry τ and payoff

$$P(\tau, S_{\tau-} - d(S_{\tau-})) \mathbf{1}_{\{\min_{0 \leq t \leq \tau-} S_t > L\}}.$$

But this is just a down-and-out option with an unusual payoff written on a geometric Brownian motion — the stock price follows a geometric Brownian motion on $[0, \tau)$. It follows from Björk (2004, Theorem 18.8) that the price at time 0 of the claim is

$$P(0, S_0) := e^{-r\tau} \left(I(S_0) - \left(\frac{L}{S_0} \right)^p I \left(\frac{L^2}{S_0} \right) \right) \mathbf{1}_{\{S_0 > L\}}, \quad (5)$$

where

$$I(s) := \mathbf{E} \left[P \left(\tau, g \left(s e^{\left(r - \frac{\sigma^2}{2} \right) \tau + \sigma W_\tau} \right) \right) \mathbf{1}_{\left\{ g \left(s e^{\left(r - \frac{\sigma^2}{2} \right) \tau + \sigma W_\tau} \right) > L \right\}} \right], \quad (6)$$

$$g(s) := s - d(s).$$

Consider the dividend policy of “the liquidator”, i.e. $d(s) = \min\{s, D\}$ for some deterministic $D \geq 0$. The expectation (6) becomes

$$I(s) = \int_{c(s)}^{\infty} P \left(\tau, g \left(s e^{\left(r - \frac{\sigma^2}{2} \right) \tau + \sigma \sqrt{\tau} y} \right) \right) N'(y) dy,$$

$$c(s) := \frac{\log \left(\frac{L+D}{s} \right) - \left(r - \frac{\sigma^2}{2} \right) \tau}{\sigma \sqrt{\tau}},$$

where $N'(\cdot)$ denotes the density of the standard normal distribution. The two integrals in (5) have to be computed numerically, but this is easily done: the integrands are smooth and decay extremely fast as $y \rightarrow \infty$ due to the factor $N'(y)$. Greeks can be computed by evaluating the same kind of integrals.

4 Numerical experiments

In this section, we perform numerical experiments with the proposed numerical integration method for pricing a down-and-out call option. The method is benchmarked against a general purpose approach to the problem — solving the pricing PDE with a finite difference scheme. The computations are done using the software **R** on an ordinary work station and, unless stated explicitly, all experiments are conducted using the parameters in Table 1.

Quantity	Symbol	Value
call strike	K	100
knock-out barrier	L	95
initial spot	S_0	K
barrier option expiry	T	0.2
dividend date	τ	$T/2$
interest rate	r	0.05
stock volatility	σ	0.1

Table 1: Parameters in the numerical experiments.

Consider the price at time 0 of the down-and-out call option when the underlying stock pays a dividend $d = \min\{S_{\tau-}, D\}$ at time τ , where $D \geq 0$ and $0 < \tau < T$. Figure 1 shows the price as a function of D and τ . The dividend makes barrier crossings more likely since the contract is down-and-out and makes the option more out-of-the-money. So, the price is a decreasing function of D for each fixed τ . For $D = 0$, we obtain the usual no-dividend Black-Scholes price of 2.2981. Figure 2 shows the price as a function of τ for $D = 2, 5$ and 8. We see that the timing of the dividend is more important for large dividends and that the contract is cheaper the earlier the dividend is paid.

4.1 Comparison with the Crank-Nicolson scheme

Since the underlying has geometric Brownian motion dynamics everywhere except when the dividend is paid, the price at time t of the contract is given by $P(t, S_t)$, where $P(t, s)$ solves

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 P}{\partial s^2} + rs \frac{\partial P}{\partial s} - rP = 0$$

for $t \in [0, T]$, $t \neq \tau$, and $s > L$. The boundary and terminal conditions are given by

$$\begin{aligned} P(t, L) &= 0, \text{ for } t \in [0, T], \\ P(T, s) &= \max\{s - K, 0\}, \text{ for } s > L, \end{aligned} \tag{7}$$

and the behavior at the dividend payment date τ is given by the following *jump condition*:

$$P(\tau-, s) = P(\tau, s - d(s)).$$

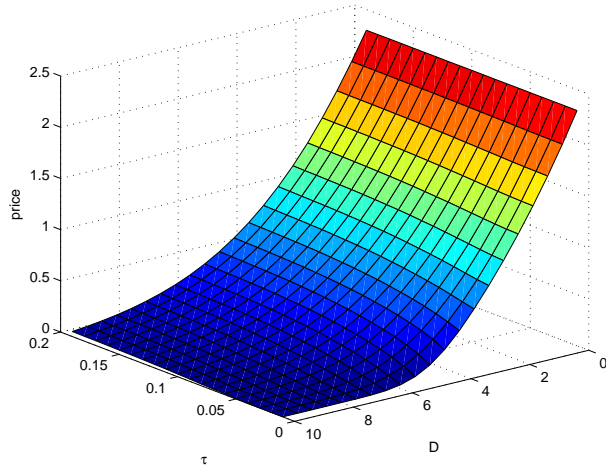


Figure 1: Price at time 0 of the down-and-out call option as a function of the dividend size D and payment time τ .

This PDE can be solved numerically with finite difference schemes — the domain $[0, T] \times [L, \infty)$ is truncated for some large s -value and discretized using some finite step sizes Δt and Δs , and the partial derivatives are approximated by finite differences. The Crank-Nicolson scheme is (at least locally) second-order accurate in both time and space and is frequently chosen in the finance literature and in practice. We refrain from describing it here and instead refer to Wilmott (2006) for a nice explanation of the method. He also discusses various implementational issues, including the need for interpolation when the dividend size is not an integer multiple of Δs . However, we note that there are potential pitfalls — Duffy (2004) discusses problems with the Crank-Nicolson scheme in situations with irregular boundary conditions — it is therefore important to investigate the convergence in our particular setting.

We choose the asset step proportional to the time step, $\Delta s = c\Delta t$. Using the analysis from Østerby (2008, Chapter 10) we find that the value $c = 12.5$ is optimal for computational efficiency — this choice gives errors of approximately the same size in the asset dimension and in the time dimension.

Table 2 reports the error for the prices computed with the Crank-Nicolson scheme relative to the exact price computed by numerical evaluation of (5). We note that the table confirms the second-order accuracy of the PDE solution method — doubling

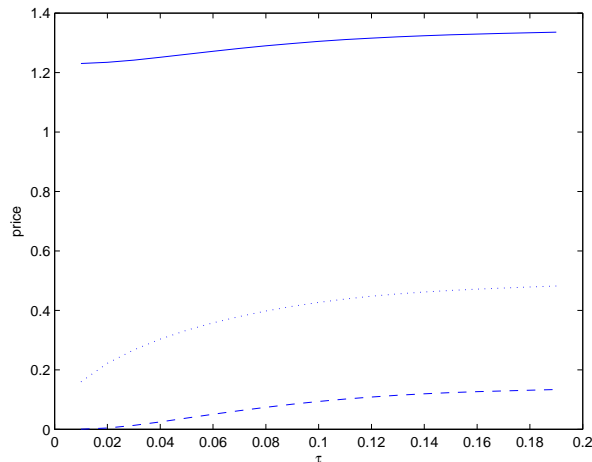


Figure 2: Price at time 0 of the down-and-out call option as a function of the payment time τ for $D = 2$ (solid), $D = 5$ (dotted) and $D = 8$ (dashed).

the number of asset and time steps reduces the error roughly by a factor of four. It also doubles the number and the size of the systems of linear equations that must be solved and hence quadruples computation time.³ Evaluating (5) takes only 0.02 seconds, so it is clear that the numerical integration method is significantly faster than the Crank-Nicolson scheme.

5 Approximation for several dividends

In this section, we discuss an approximation approach for pricing the down-and-out call option in the case of multiple dividends. It is analogous to the method considered for plain-vanilla options in Haug et al. (2003).

Assume that we are facing N dividends d_1, \dots, d_N paid at times $0 < \tau_1 < \dots < \tau_N < T$ and that we want to compute the price at time $t < \tau_1$ of the down-and-out call option. Denote this price by $P(t, S_t; K, \sigma, \tau_1, \dots, \tau_N)$, where we explicitly note the dependence on $K, \sigma, \tau_1, \dots, \tau_N$. Analogous to the single-dividend case, we assume that the stock price follows a geometric Brownian motion in between the

³Computation time grows linearly in matrix-size only because the sparse structure is exploited — general inversion is $O(\text{size}^3)$, so doubling the number of time and asset steps would increase the computation time 16-fold.

ΔS	ε	Time
0.250	$2.56 \cdot 10^{-04}$	0.2 s
0.125	$1.07 \cdot 10^{-04}$	0.61 s
0.083	$4.82 \cdot 10^{-05}$	1.27 s
0.063	$2.70 \cdot 10^{-05}$	2.19 s
0.042	$1.20 \cdot 10^{-05}$	4.72 s
0.031	$6.75 \cdot 10^{-06}$	8.22 s

Table 2: Relative Crank-Nicolson error and computation time as a function of the asset step size, ΔS , for the price at time 0 of the down-out-call option. The time step size is chosen proportional to the asset step size, $\Delta t = c\Delta S$, with $c = 12.5$. The relative error is defined as $\varepsilon = |P_{CN} - P|/P$, where P_{CN} is the Crank-Nicolson price and P is the exact price calculated by numerical evaluation of (5). Computing P by numerical integration takes 0.02 s.

payment times and that $S_{\tau_k} = S_{\tau_k-} - d_k$ for $k = 1, \dots, N$, where $d_k = d_k(S_{\tau_k-}) = \min\{S_{\tau_k-}, D_k\}$ for some constants $D_k \geq 0$. We write $g_k(s) := s - \min\{s, D_k\}$.

For $N = 0$, in the case of no dividends, we have the closed-form Black-Scholes price $P(t, S_t; K, \sigma)$ and we showed above how to handle the case $N = 1$ — the price $P(t, S_t; K, \sigma, \tau_1)$ can be written as an expression involving integrals of the function $P(\tau_1, g_1(\cdot); K, \sigma)$. Recursively, we could in principle handle the case $N = 2$ by similarly expressing the price $P(t, S_t; K, \sigma, \tau_1, \tau_2)$ in terms of integrals of the function $P(\tau_1, g_1(\cdot); K, \sigma, \tau_2)$ and so on, but this would amount to evaluating N -dimensional integrals in the case of N dividends.

Consider the case $N = 2$. The whole problem is that we do not have the function $P(\tau_1, \cdot; K, \sigma, \tau_2)$ in closed form — if we had, the integrals would be one-dimensional, hence fast and easy to evaluate. To proceed, we attempt to approximate the function $P(\tau_1, \cdot; K, \sigma, \tau_2)$ with $P(\tau_1, \cdot; K_1^{adj}, \sigma_1^{adj})$ for suitable choices of the adjusted strike K_1^{adj} and volatility σ_1^{adj} . First,

$$P(\tau_1, s; K, \sigma, \tau_2) \approx s - e^{-r(\tau_2-\tau_1)}D_2 - e^{-r(T-\tau_1)}K$$

for large values of s , so $K_1^{adj} = K + e^{r(T-\tau_2)}D_2$ assures the correct behavior as $s \rightarrow \infty$. Next, we choose σ_1^{adj} such that

$$P(\tau_1, S_t; K, \sigma, \tau_2) = P(\tau_1, S_t; K_1^{adj}, \sigma_1^{adj}),$$

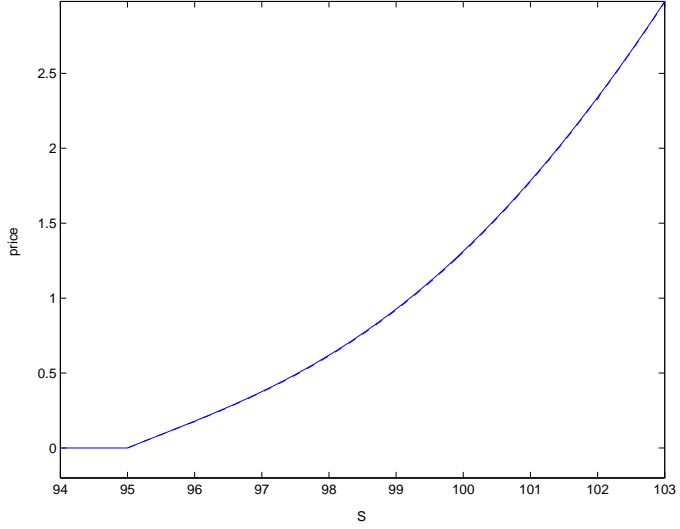


Figure 3: The price of the down-and-out call option for dividends $D_1 = D_2 = 1$ paid at times $\tau_1 = T/3$ and $\tau_2 = 2T/3$, computed by using the approximative numerical integration method (solid) and the Crank-Nicolson scheme (dashed) with very small step sizes.

and so it only remains to compute $P(t, S_t; K, \sigma, \tau_1, \tau_2)$ by evaluating one-dimensional integrals of the function $P(\tau_1, g_1(\cdot); K_1^{adj}, \sigma_1^{adj})$.

For $N > 2$, we just continue this procedure recursively: at the k th step, $k = 1, \dots, N$, compute an approximation of the time- τ_{N-k} price $P(\tau_{N-k}, S_t; K, \sigma, \tau_{N-k+1}, \dots, \tau_N)$ by integrating $P(\tau_{N-(k-1)}, g_{N-(k-1)}(\cdot); K_{k-1}^{adj}, \sigma_{k-1}^{adj})$. By convention, $K_0^{adj} = K$, $\sigma_0^{adj} = \sigma$, and $\tau_0 = 0$. Adjust the strike to

$$K_k^{adj} := K + \sum_{j=k+1}^N e^{r(T-\tau_j)} D_j = K_{k-1}^{adj} + e^{r(T-\tau_{k+1})} D_{k+1}$$

and choose the adjusted volatility σ_k^{adj} such that

$$P(\tau_{N-k}, S_t; K, \sigma, \tau_{N-k+1}, \dots, \tau_N) = P(\tau_{N-k}, S_t; K_k^{adj}, \sigma_k^{adj}),$$

and iterate.

Figure 3 compares the price computed with the described approximative method to the result from the Crank-Nicolson scheme with very small step sizes. We consider the case $N = 2$ with dividends $D_1 = D_2 = 1$ paid at times $\tau_1 = T/3$ and $\tau_2 = 2T/3$. The approximation works nicely — relative errors are in the order of 0.1%-1%,

except very close to the barrier, where the relative errors are larger. Running the approximative method takes only 0.05 seconds, so it is a fast alternative to the Crank-Nicolson method.

6 Conclusion

We described an extension of the method presented in Haug et al. (2003) for pricing the down-and-out call option in the Black-Scholes model when the underlying pays a discrete, deterministic cash dividend. The calculations above can be performed in exactly the same way for all the other single-barrier options (down or up, call or put, in or out). The method is also valid for one-touch options and similar variants. Knock-out options with rebates are also straightforward to handle — one can use the same decomposition as for pricing in the no-dividend case — write the contract as a sum of the corresponding knock-out option without the rebate and a digital knock-in contract, see Poulsen (2006, Section 6.1). The method is benchmarked against solving the pricing PDE with a Crank-Nicolson scheme — as expected, our proposed method was significantly faster.

We also described an approximation method in the case of multiple dividends for the down-and-out call option. Unfortunately, the same approach does not seem to be feasible for the other single barrier options — in this case, we are still limited to trees or finite difference schemes. Luckily, the case of multiple dividends is not paramount in practice — the vast majority of traded barrier options have expiries shorter than three months and dividends are usually paid less frequently than that.

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