

FinKont2: Exercises for Thursday February 5

Exercise 1.1: A Useful Result
Suppose that $X$ is normally distributed, $X \sim N(\mu, s^2)$, and that $a$, $l$, and $h$ ($\geq l$) are real numbers. Show (using for instance the same ideas as when the Black/Scholes formula was derived at the lectures) that

$$
E \left( e^{aX} 1_{l \leq X \leq h} \right) = e^{a\mu + \frac{1}{2} a^2 s^2} \left( \Phi \left( \frac{h - (\mu + as^2)}{s} \right) - \Phi \left( \frac{l - (\mu + as^2)}{s} \right) \right),
$$

where as usual $\Phi(\cdot)$ denotes the distribution function of the standard normal distribution.

Use the result to derive the Black-Scholes call-price formula. (Which roles are played by what?)

Exercise 1.2: Time-dependent Black/Scholes (Useful too)
In the base-case Black-Scholes model we assume that the interest rate and the volatility are constant. You don’t have to look very hard at data to see that this is not a particularly convincing assumption. In this exercise we look the generalization to deterministic but time-varying parameters. This can be thought of as a first-(or zero-)order adjustment. Perhaps not what we would ultimately want, . . .

Suppose first the base-case Black-Scholes assumption of constant volatility is maintained, but the interest rate is allowed to vary deterministically, ie. the bank account behaves as

$$d\beta(t) = r(t)\beta(t) dt,$$

where $r$ is some smooth function. Note (prove if you have to) that this ordinary differential equation (with 1 as initial condition) has the solution

$$\beta(t) = \exp \left( \int_0^t r(u) du \right).$$

Note also (same again . . . ) that the price of a zero coupon bond, $P(t; T)$ (= the price at time $t$ of an asset that pays 1 at time $T$), must satisfy

$$P(t; T) = \frac{\beta(t)}{\beta(T)}.$$
Derive a closed-form solution for the price of a call-option. Express the price in terms of (among other things, of course) zero coupon bond prices, i.e. no explicit \( r \)'s appearing.

Hint: It is convenient to work with the process \( S(t) = S(t)/P(t; T) \) (find the \( Q \)-dynamics of \( S \)) and then to use (show!) that

\[
\text{Call}(t) = P(t; T) \mathbb{E}_t^Q((\tilde{S}(T) - K)^+).
\]

Now suppose volatility is also made time-dependent (but deterministic), i.e.

\[
dS(t) = r(t)S(t)dt + \sigma(t)S(t)dW^Q(t),
\]

for some function \( \sigma \). Derive a closed-form solution for the price of a call-option.

Hint: It is still practical to work with \( \tilde{S} \). Put \( Y = \ln \tilde{S} \) and show that

\[
dY(t) = -\frac{1}{2}\sigma^2(t)dt + \sigma(t)dW^Q(t)
\]

Use this & Björk’s Lemma 4.15 to conclude that \( \tilde{S}(T) \) is still lognormal and find the appropriate parameters. The resulting formula should have the ordinary Black/Scholes \( \sigma^2 \)'s replaced by

\[
\frac{1}{T-t} \int_t^T \sigma^2(u)du.
\]

Exercise 1.3: Hedging Digital Options

At the lectures on Mon Feb 2, we looked at a discrete \( \Delta \)-hedging experiment for a call-option in the Black-Scholes model. On the course homepage, you can find the \( R \)-code, I used.

Alter the code to investigate the performance of hedging strategies for a digital option, i.e. an option whose pay-off is

\[
1_{\{S(T) > K\}}.
\]

What appears to be the convergence order (as the hedge-step length goes to 0) for the standard deviation of the (discounted) hedge error? And what does that say about the practical possibility to hedge digital options?

Exercise 1.4: Cheating Strategies

In this exercise we try to communicate in pseudo-code. (Actually, it’s functional \( R \)-code, but never mind.) That is a useful skill.

Consider trading like this:
V[1]<-100
for(i in 1:(nobs-1)){
  if (KFX[i+1]-KFX[i] > 0){
    hKFX[i]<-V[i]/KFX[i]
    hBANK[i]<-0
  }
  if(KFX[i+1]-KFX[i]<=0){
    hKFX[i]<-0
    hBANK[i]<-V[i]/bankaccount[i]
  }
  V[i+1]<-hKFX[i]*KFX[i+1] +hBANK[i]*bankaccount[i+1]
}

How does the value process for this strategy look? And which part of our usual and sensible assumptions is violated?

Would changing the logical conditions above to something like

if (bankaccount[i+1]-bankaccount[i] > 0)

lead to an allowable trading strategy?

Consider the trading strategy defined by

V[1]<-100
hKFX[1]<-0.5*V[1]/KFX[1]
hBANK[1]<-0.5*V[1]/bankaccount[1]

for(i in 2:nobs){
  V[i]<-hKFX[i-1]*KFX[i]+hBANK[i-1]*bankaccount[i]
  hKFX[i]<-0.5*V[i]/KFX[i]
  hBANK[i]<-hBANK[i-1]
}

Is it self-financing?