

Discrete Charms

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Pricing and hedging options discretely in time with transaction costs

Before 1973 and the *Journal of Political Economy* paper by Fischer Black and Myron Scholes, equity options were usually priced differently from the way that they are priced today. In those days the fair value of an option was the *present value of its expected payoff at expiry*. This sounds very familiar: isn't this how options are still valued? The answer lies in the important distinction between the "real world" and the "risk-neutral world". This distinction is related to the difference between the risk-free rate of return and the equity growth rate. The latter parameter does not affect the price of an option in the perfect Black-Scholes world, but can become important once we leave that world.

I shall begin by describing how equity options used to be valued. The notation is standard,

S = underlying asset price,

t = time,

$V(S,t)$ = value of the option.

As is common, I shall assume that S follows a lognormal random walk given by the stochastic differential equation

$$dS = \mu S dt + \sigma S dX,$$

where

μ = growth rate,

σ = volatility,

dX = Wiener process.¹

Before 1973 options were valued by setting the expected return on the option equal to the risk-free rate:

$$E[dV] = rV dt.$$

I shall not give the details, but this leads to a differential equation very similar to the Black-Scholes equation, except that it explicitly contains μ . It is difficult to justify this approach economically: the holder of the option is being rewarded for owning a risky, yet diversifiable, instrument.

In 1973 along came Black and Scholes with the mathematical analysis of the *hedged* option position. Their analysis was very similar to the above, yet had a crucial difference: they chose as their starting point a portfolio of one option and short a number Δ of the underlying asset,

$$\Pi = V - \Delta S.$$

Now we have

$$d\Pi = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} dt - \Delta dS,$$

from Itô's lemma. Now, by choosing

$$\Delta = \frac{\partial V}{\partial S}$$

there are no random terms in $d\Pi$. Thus $d\Pi$ is totally deterministic, we do not have to take expectations and we can now appeal to "no arbitrage" to say that such a riskless portfolio *must* earn the risk-free rate.² Black and Scholes found that

$$\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} - rV = 0. \quad (1)$$

This is the Black-Scholes equation.

The pre-1973 result and the Black-Scholes result are actually identical if we replace μ everywhere by r . This leads on to the observation of the late 1970s and early 1980s, put into a rigorous pure mathematical framework by Harrison and Kreps (1979) and Harrison and Pliska (1981), that the fair value of an option is the present value of the expected value at expiry *in a risk-neutral world*. The phrase in italics simply means that every time you see a μ replace it by r , or, more generally, no matter what appears in front of the dt in the stochastic differential equation for S replace it by rS (less the asset yield, where relevant).

Since 1973 no one has had to measure μ , which, in a sense, is a good thing since it is very difficult to measure statistically. For a

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model to contain μ is seen as a very bad point against it. Nevertheless, sometimes it *must* be measured. For example, although the price of an American option is independent of μ , some properties of this option *do* depend on μ . The expected time to exercise, termed the "fugit" by Mark Garman, is one of these.³

If you need to calculate the expected time to exercise then you must first determine μ , you cannot replace it by r . If you do use r , then you will get the wrong answer. The common use of "risk-neutral valuation" has led to many inaccuracies (some by otherwise well-respected people) in the academic literature. For example, it is often stated that the delta of a vanilla option may be interpreted as the probability of the option expiring in the money. This is not true. It is only true in a risk-neutral world, which, of course, is not the real world and does not exist. The probability of an option expiring in the money depends on μ ; the formula is easy to calculate.

The purpose of the above history lesson is to justify the assertion that μ must sometimes be measured. The discrete hedging of options is one of these times; I will need μ in what follows.

Discrete hedging

For the rest of this article I am going to address the question of hedging options *discretely* in time. The Black-Scholes analysis requires *continuous* hedging, which is possible in theory but impossible – even undesirable – in practice. The simplest model for discrete hedging is to re hedge at fixed intervals of time δt ; a strategy commonly used with δt ranging from one day to a week. Some of the ideas in this article can be found in Boyle and Emanuel (1980). They consider in detail the errors in following a pure Black-Scholes hedging strategy in discrete time. I shall find an improvement to this strategy.

The first step towards valuing an option is to choose a good model for the underlying. A sensible choice is

$$S = e^x$$

where

$$\delta x = \left(\mu - \frac{\sigma^2}{2} \right) \delta t + \sigma \phi \delta t^{1/2}. \quad (2)$$

This is a discrete-time version of the earlier continuous-time stochastic differential equation for S . Here ϕ is a random variable drawn from a

standardised normal distribution and the term $\phi \delta t^{1/2}$ replaces the earlier Wiener process. In principle, the ideas that I shall be describing do not depend on the random walk being lognormal and many models for S could be examined. In particular, ϕ need not be normal but could even be measured empirically. If historic volatility is to be used it should be measured at the same frequency as the rehedging takes place, that is, using data at intervals of δt . Note the use of δ to denote a discrete change in a quantity, this is to make a distinction with d , the earlier continuous changes.

As in Black-Scholes we construct a hedged portfolio

$$\Pi = V - \Delta S, \quad (3)$$

with Δ to be chosen. We no longer have Itô's lemma, since we are in discrete time, but we still have the Taylor series expansion. Thus, it is very simple matter to derive $\delta \Pi$ as a power series in δt and δx . On substituting for δx from (2) this expression becomes

$$\begin{aligned} \delta \Pi = & \delta t^{1/2} A_1(\phi, \Delta) + \delta t A_2(\phi, \Delta) \\ & + \delta t^{3/2} A_3(\phi, \Delta) + \delta t^2 A_4(\phi, \Delta) + \dots \end{aligned} \quad (4)$$

Here V and its derivatives appear in each of the A_i s, but only the dependence on ϕ and Δ is shown. For example, the first term A_1 is given by

$$A_1(\phi, \Delta) = \sigma \phi S \left(\frac{\partial V}{\partial S} - \Delta \right).$$

This expansion in powers of $\delta t^{1/2}$ can be continued indefinitely; I shall stop at the order shown above. Since time is measured in units of a year, δt is small but not zero.

Now I can state the very simple hedging strategy and valuation policy:

- choose Δ to minimise the variance of $\delta \Pi$;
- value the option by setting the expected return on Π equal to the risk-free rate.

The first of these, the hedging strategy, is easy to justify: the portfolio is, after all, hedged so as to reduce risk. But how can I justify the second, the portfolio is not riskless? The argument for the latter, the valuation policy, is that since options are typically valued according to Black-Scholes yet are discretely hedged, the second assumption is already being used, but with an inferior choice for Δ .

Choosing the best Δ

The variance of $\delta \Pi$ is easily calculated from (4)

since

$$\text{var}[\delta\Pi] = E[\delta\Pi^2] - (E[\delta\Pi])^2 \quad (5)$$

In taking the expectations of $\delta\Pi$ and $\delta\Pi^2$ to calculate (5) all of the ϕ terms are integrated out leaving the variance of $\delta\Pi$ as a function of V , its derivatives, and, most importantly, Δ . Then, to minimise the variance by correctly choosing Δ we find the value of Δ for which

$$\frac{\partial}{\partial\Delta} \text{var}[\delta\Pi] = 0.$$

The result is that the optimal Δ is given by

$$\Delta = \frac{\partial V}{\partial S} + \delta t(\dots) \quad (6)$$

The first term will be recognised as the Black-Scholes delta. The second term, which I shall give explicitly in a moment, is the correction to the Black-Scholes delta that gives a better reduction in the variance of $\delta\Pi$, and thus a reduction in risk. This term contains V and its derivatives.⁴

Pricing the option

Having chosen the best Δ , I now derive the pricing equation. The option should not be valued at the Black-Scholes value since that assumes perfect hedging and no risk, the fair value to an imperfectly hedged investor may be different. I find that the option value to the investor is equal to the Black-Scholes value plus a correction which prices the hedging risk.

The pricing policy I adopt has been stated as equating the expected return on the discretely hedged portfolio with the risk-free rate. This may be written as

$$E[\delta\Pi] = \left(r\delta t + \frac{r^2\delta t^2}{2} + \dots \right) \Pi \quad (7)$$

This is slightly different from the usual right-hand side, but simply represents a consistent higher order correction to exponential growth:

$$e^{r\delta t} = 1 + r\delta t + \frac{r^2\delta t^2}{2} + \dots$$

Now substitute (3) and (4) into (7) to get the equation:

$$\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} - rV + \delta t(\dots) = 0 \quad (8)$$

Again the first part of the equation is that derived by Black and Scholes and the second is a correction to allow for the imperfect hedge; it contains V and its derivatives. I shall give the second term shortly.

The adjusted Δ and option value

The as yet undisclosed terms in parentheses in (6) and (8) contain V and its derivatives, up to the second derivative with respect to t and up to the fourth with respect to S . However, since the adjusted option price is clearly close in value to the Black-Scholes price I can put the Black-Scholes value into the terms in parentheses without any reduction in accuracy. This amounts to solving (6) and (8) iteratively.

The result is that the adjusted option price satisfies

$$\begin{aligned} \frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} - rV \\ + \delta t \frac{(\mu - r)}{2} (3(\mu - r) + \sigma^2) S^2 \frac{\partial^2 V}{\partial S^2} = 0 \end{aligned} \quad (9)$$

and the better Δ is given by

$$\Delta = \frac{\partial V}{\partial S} + \delta t \left(\mu - r + \frac{\sigma^2}{2} \right) S \frac{\partial^2 V}{\partial S^2}.$$

These equations contain μ explicitly. There is no such thing as "perfect hedging" in the real world. In practice the investor is *necessarily* exposed to risk in the underlying, and this manifests itself in the appearance of the drift of the asset price.

Notice how both the second derivative terms in (9) have coefficients of the form $S^2 \times \text{constant}$. Therefore the correction to the option price can very easily be achieved by adjusting the volatility, however it is measured, and using the value σ^* where

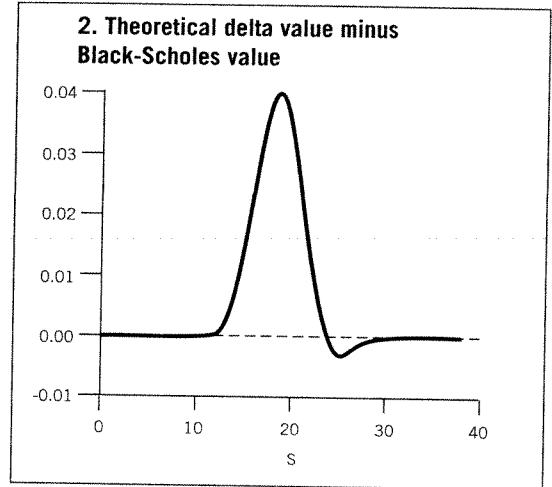
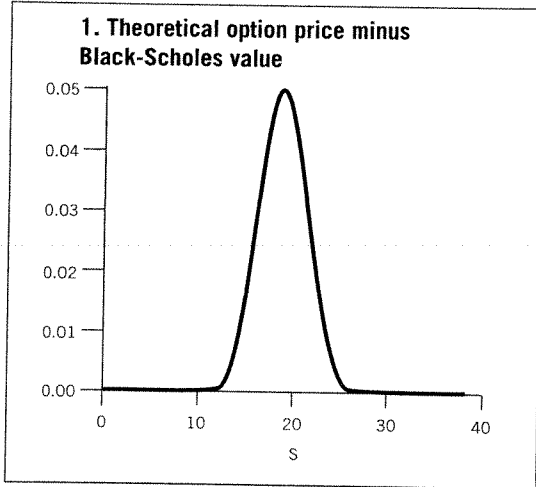
$$\sigma^* = \sigma \left(1 + \frac{\delta t}{2\sigma^2} (\mu - r)(3(\mu - r) + \sigma^2) \right).$$

This is similar to the volatility adjustment when there are transaction costs (see Whalley and Wilmott, 1993) but this time the adjustment is symmetric for long and short positions.

Is this volatility effect important? Fortunately, in most cases it is not. With typical values for the parameters and daily rehedging there is a volatility correction of 1–2%. In trending markets, however, when large μ can be experienced, this correction can reach 5–10%, a value that cannot be ignored.

More importantly, in trending markets the corrected Δ will give a better risk reduction since it is in effect an anticipatory hedge: the variance is minimised for the *next* rehedged.

Since the option should be valued with a modified volatility, the difference between the adjusted option value and the Black-Scholes



value is proportional to the option vega, its derivative with respect to σ . Figure 1 shows the difference between the adjusted option value and the Black-Scholes value for a call option with exercise price 20, time to expiry one year, $\delta t=0.1$, $\mu=0.15$, $r=0.085$ and $\sigma=0.1$.

Figure 2 shows the difference between the anticipatory Δ and the Black-Scholes delta for the same parameter values.

Choosing the hedging frequency

Is there a best choice for δt , the time between rehedges? Clearly, the smaller δt the less risk in the hedged portfolio and the closer the hedging strategy and option value are to Black-Scholes. On the down side, the smaller δt the larger the accumulated cost of transacting. Leland (1985) has derived a simple model for pricing options in the presence of transaction costs (see Whalley and Wilmott, 1993, for a discussion of several cost models). The result of this model is that vanilla options should be valued using *another* adjusted volatility, one that incorporates the parameter k , where kNS is the cost of buying/selling N shares. In this case, the volatility adjustment is different for long and short option

positions. Combining the results of Leland with those above, it is a simple matter to find that long positions should be valued with

$$\hat{\sigma} = \sigma \left(1 + \frac{\delta t}{2\sigma^2} (\mu - r)(3(\mu - r) + \sigma^2) - \frac{k}{\sigma} \sqrt{\frac{2}{\pi \delta t}} \right)$$

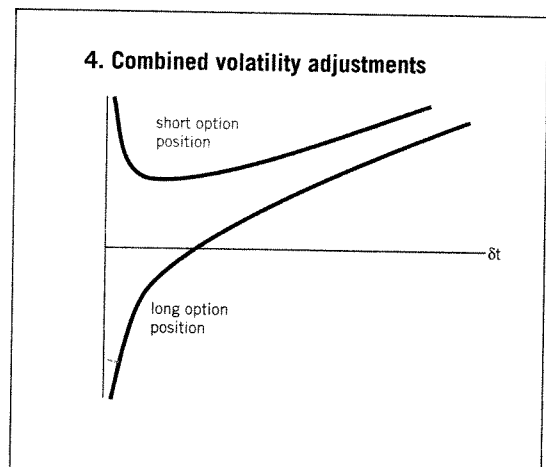
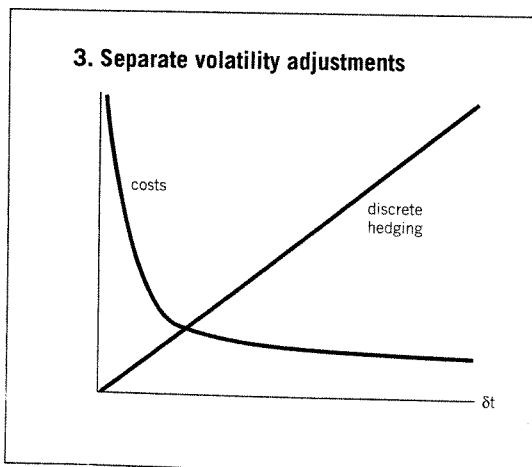
and short positions with

$$\hat{\sigma} = \sigma \left(1 + \frac{\delta t}{2\sigma^2} (\mu - r)(3(\mu - r) + \sigma^2) + \frac{k}{\sigma} \sqrt{\frac{2}{\pi \delta t}} \right)$$

The two volatility adjustment terms are shown in Figure 3. The discrete hedging term is linear in the time step δt whereas the cost term is singular at $\delta t=0$; there is no sensible option price in the limit of continuous hedging when there are costs.

Assuming that $\mu > r$, as is usually the case, Figure 4 shows the combination of the two adjustment terms for both long and short positions.

Does this figure suggest that there is an optimal δt ? This depends on the definition of optimal, on option prices in the market and on their spreads. And, of course, it must always be remembered that the smaller δt , the less the



risk. Nevertheless, the figure does show some interesting features. For example, for long option positions the choice $\delta t = \delta t^*$ where

$$\delta t^* = \left(\frac{\sqrt{8}}{\pi} \frac{k\sigma}{(\mu - r)(3(\mu - r) + \sigma^2)} \right)^{2/3}$$

gives an option value that is exactly the Black-Scholes value! This exploits the cancellation of the discrete hedging and transaction cost effects.

For a short position the choice

$$\delta t = 2^{-2/3} \delta t^*$$

minimises the difference between the adjusted option value and the Black-Scholes value. With typical parameter values this gives a rehedg approximately every week. Rehedging at this frequency the two effects of

discrete hedging and transaction costs are of the same order. The volatility adjustment adds up to approximately 5%.

Conclusion

The mathematical tools I have used in this article are very basic, little more than elementary probability theory together with Taylor series. Yet the results, especially for the "better Δ ", are important.

One aspect of the ideas that I have only hinted at is that they are applicable to more general random walks than the lognormal. I encourage the interested reader to do the analysis for the general case, and, in particular, to discretely hedge and value an option when the statistical properties of the underlying asset are only known empirically!

1 For further details of random walks in finance, see Wilmott, Dewynne and Howison, 1993.

2 The principle of no arbitrage only works in two types of asset price model, the continuous time/continuous asset price stochastic differential equation model, and the discrete time/discrete asset price binomial model.

3 See Garman, "Semper Tempus Fugit", in From Black-

Scholes to Black Holes (1993).

4 Although there are $\delta t^{1/2}$ terms in (4) there are no such terms in (6) because of the symmetry properties of ϕ . If ϕ were chosen to be other than a standardised normal random variable then $\delta t^{1/2}$ terms could well occur in the equivalent expression for the best Δ .

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