30. (Delta-hedging). The following derivation of the Black-Scholes equation is very popular in the finance literature. We will suppose, as usual, that an asset price follows a geometric Brownian motion. That is, there are parameters $\mu, \sigma$, such that

$$dS_t = \mu S_t dt + \sigma S_t dB_t.$$ 

Suppose that we are trying to value a European option based on this asset. Let us denote the value of the option at time $t$ by $V(t, S_t)$. We know that at time $T$, $V(T, S_T) = f(S_T)$, for some function $f$.

a. Using Itô’s formula express $V$ as the solution to a stochastic differential equation.

b. Suppose that a portfolio, whose value we denote by $\pi$, consists of one option and a (negative) quantity $-\delta$ of the asset. Assuming that the portfolio is self-financing, find the stochastic differential equation satisfied by $\pi$.

c. Find the value of $\delta$ for which the portfolio you have constructed is ‘instantaneously riskless’, that is for which the stochastic term vanishes.

d. An instantaneously riskless portfolio must have the same rate of return as the risk free interest rate. Use this observation to find a (deterministic) partial differential equation for the $V(t, x)$. Notice that this is the Black-Scholes equation obtained in Question 29.

e. Now for the crunch: is the portfolio that you have constructed self-financing?

(a) Write $U_t = v'(t, S_t)$, where $v$ is a solution of the Black-Scholes equation. Use Itô’s formula to write down a stochastic differential equation for $U$.

(b) Find an expression for $dV_t \equiv d(U_t S_t) - dv(t, S_t)$.

(c) If the trading strategy is self-financing then it must satisfy $dV_t = v'(t, S_t)S_t dt - dv(t, S_t)$. Use the stochastic differential equation for $V_t$ and the Black-Scholes differential equation (differentiated with respect to the $x$ variable) to show that this is equivalent to

$$\sigma S_t^2 v''(t, S_t) (dB_t + \sigma^{-1}(\mu - r) dt) = \sigma S_t^2 v''(t, S_t) dW_t = 0.$$ 

Here $W_t = B_t + \sigma^{-1}(\mu - r) t$ is a Brownian motion under the risk neutral measure.

(d) Deduce that the portfolio is not self-financing.

You have shown that the delta-hedging argument is mathematically unsatisfactory in the continuous time setting. It should be emphasized that it is unquestionable in the discrete time setting. All the same, in continuous time it does lead to the right value of the option. The reason is that the additional cost that might be associated with this trading strategy is a martingale under the risk neutral measure.