After these considerations we now turn to the second object of the paper: the implementation of the described theory.

**Numerical Implementation**

We now consider the construction of a model that fits a discrete set of observed European option prices in the sense that the option prices of the model lie in between the bid/ask spread of the marketed options.

According to Result 1 we can derive a unique local volatility surface that supports a given smooth double continuum, in strike and maturity, of option prices. First we need this smooth double continuum of option prices. This requires interpolation between the observed strikes and maturities but in most cases it also requires extrapolation into areas where we do not have observations. We will for example never have observations of option prices with maturities all the way down to current time. This is not unlike estimating the first and the last part of the term structure of interest rates.

We prefer to do the interpolation and extrapolation in the space of implied (Black-Scholes) volatilities rather than directly in option prices for two reasons: First, it is easier to relate to implied volatilities than to the prices themselves, which is probably also why traders often prefer to quote option prices in terms of implied volatilities. Secondly, it is easier to smooth, interpolate, and extrapolate in a space where the function that has to be interpolated is rather (though not at all perfectly) flat. Presumably this will be the case for the surface of implied volatilities.

When the identification of the implied volatility surface is done, we need to convert these into local volatilities. Defining \( \bar{\sigma}(T,K) \) to be the time 0 (Black-Scholes) implied volatility function of strike, \( K \), and maturity date, \( T \), Result 5 gives the relation between implied Black-Scholes volatility and local volatility.\(^9\)

**Result 5: The Relation between Implied and Local Volatility.**

The local volatility function is related to the Black-Scholes implied volatility by the equation

\[
\frac{1}{2} \sigma(T,K)^2 K^2 = \left( \frac{\mathcal{d}^2}{\mathcal{d}R^2} + \mathcal{d}T + \frac{(r - q)K}{\bar{\sigma}_K} \right)_{(T,K)}
\]

(34)

\[
d(T,K) = \frac{1}{\bar{\sigma}(T,K)\sqrt{T}} \ln \frac{S(t,0;T)}{K B(0;T)} + \frac{1}{2} \sigma(T,K)^2 \sqrt{T}.
\]

\(^9\) Result 5 was simultaneously derived by Andersen (1998).

where subscripts denote partial derivatives.

The proof of Result 5 is in the appendix.

The local volatilities are given as differentials of the implied volatility surface, whereas if we want to convert local volatilities into Black-Scholes implied volatilities we have to perform (not straightforward) integration, or solve the partial differential equation (10). For the term structure of interest rates we have a similar relation: Forward rates can be obtained from yield-to-maturity rates by differentiation, but we have to perform integration to get the yield-to-maturity rates from forward rates.

Our implementation is rather simple and does not require any iteration or search for an optimal or feasible solution once we have identified or rather estimated the surface of implied volatilities of the market. It can be summarized in the following steps:

i. Convert bid and ask option prices into implied volatilities.

ii. Smooth a surface of implied volatilities in the \((T,K)\) space between bid and ask volatilities.

iii. Use Result 5 to convert implied volatilities to local volatilities.

iv. Using Result 1 through 3 we simultaneously solve for all the “Greeks” of the marketed options by a forward running implicit finite difference scheme.

v. The scheme constructed in (iv.) can now be solved backwards to obtain prices and hedge ratios of non-marketed claims.

In the above we do not necessarily need to compute the “Greeks”. Step (iv.) might be skipped and we might go directly from step (iii.) to step (v.).

We now illustrate our procedure by a numerical example.\(^{10}\)

The input data are bid and ask option prices, interest rates, dividend yields and the spot. In the example below the option prices used as input are based on median bid/ask quotes of S&P 500 call option prices over the day 90.03.19. The interest rates and the dividend yields are backed out from the put-call parity.

Suppose we have a table of bid/ask option price implied volatilities, interest rates and dividend yields as the one below. An empty cell means that there is no observation for this particular time to maturity and strike.

The interest rates and dividend yields in the above table are related to the discount factors and the factors of accumulated dividends by

\[
B(0;T) = \exp(-T R(0;T))
\]

\[
D(0;T) = \exp(-T Q(0;T))
\]

(35)

We perform cubic splines in the rates \( R, Q \), and then find \( r, q \) by the relations

\[
r(T) = R(0;T) + T \frac{\partial R(0;T)}{\partial T}
\]

\[
q(T) = Q(0;T) + T \frac{\partial Q(0;T)}{\partial T}
\]

(36)

\(^{10}\) The data set was kindly provided by Jesper Jackwerth and Mark Rubinstein.
By taking derivatives of the above formulae we get:

\[
\frac{\partial C(T,K)}{\partial T} = -r(T)C(T,K) + B(0;T) \frac{\partial}{\partial T} \int_0^\infty \phi(T,x) dx
\]

\[
\frac{\partial C(T,K)}{\partial K} = -B(0;T) \int_0^\infty \phi(T,y) dy
\]

\[
\frac{\partial^2 C(T,K)}{\partial K^2} = B(0;T) \phi(T,K)
\]

(67)

By rearranging this and inserting in the forward equation the forward PDE obtains.

Proof of Result 5

Like in the previous proofs we fix current time to be 0 without loss of generality.

The implied volatilities satisfy:

\[C(0, S, T, K) = W(T, K, \sigma(T, K))\]

(68)

where \(W(\cdot)\) is the Black-Scholes formula:

\[W(T, K; \nu) = D(0; T) S(0; T) K \Phi(d - \nu \sqrt{T})\]

\[d = \frac{1}{\nu} \ln \frac{S(0; T)}{K} + \frac{1}{2} \nu \sqrt{T}\]

(69)

This means that we can write:

\[
\frac{\partial C}{\partial T} = \frac{\partial W}{\partial T} + \frac{\partial W}{\partial \nu} \sigma_T
\]

\[
\frac{\partial C}{\partial K} = \frac{\partial W}{\partial K} + \frac{\partial W}{\partial \nu} \sigma_K
\]

\[
\frac{\partial^2 C}{\partial K^2} = \frac{\partial^2 W}{\partial K^2} + 2 \frac{\partial^2 W}{\partial \nu \partial K} \sigma_K + \frac{\partial W}{\partial \nu} \sigma_K
\]

\[
\frac{\partial^2 C}{\partial K^2} = \frac{\partial^2 W}{\partial K^2} + 2 \frac{\partial^2 W}{\partial \nu \partial K} \sigma_K + \frac{\partial W}{\partial \nu} \sigma_K
\]

(70)

where subscripts denote partial derivatives.

By Result 1 we have that:

\[q(T)W = -\frac{\partial W}{\partial T} - (r(T) - q(T)) K \frac{\partial W}{\partial K} + \frac{1}{2} \nu^2 K^2 \frac{\partial^2 W}{\partial K^2}\]

(71)

So by plugging the derivatives of \(C(\cdot)\) into Result 1 we get:

\[
\frac{1}{2} \sigma^2 K^2 \frac{\partial^2 W}{\partial K^2} + \frac{\partial W}{\partial \nu} \sigma_T + (r - q) K \frac{\partial W}{\partial \nu} \sigma_K
\]

\[
= \frac{1}{2} \sigma(T, K)^2 K^2 \left[ \frac{\partial^2 W}{\partial K^2} + 2 \frac{\partial^2 W}{\partial \nu \partial K} \sigma_K + \frac{\partial W}{\partial \nu} \sigma_K \right]
\]

\[
= \frac{1}{2} \sigma(T, K)^2 K^2 \left( \frac{\partial^2 W}{\partial K^2} + \frac{\partial^2 W}{\partial \nu \partial K} \sigma_K + \frac{\partial W}{\partial \nu} \sigma_K \right)^2 + \frac{\partial W}{\partial \nu} \sigma_K
\]

(72)

The last equality follows since differentiation of the Black-Scholes formula shows that:

\[
\frac{\partial W}{\partial \nu} = \frac{\partial W}{\partial \nu} = 1
\]

\[
\frac{\partial W}{\partial \sigma_T} = \frac{\partial W}{\partial \sigma_T} = \frac{d}{\sigma \sqrt{T}}
\]

\[
\frac{\partial W}{\partial \sigma_K} = \frac{d}{\sigma \sqrt{T}}
\]

(73)

This concludes the proof.