ON RØRDAM'S CLASSIFICATION OF CERTAIN C^*-ALGEBRAS WITH ONE NON-TRIVIAL IDEAL, II

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Abstract
In this paper we extend the classification results obtained by Rørdam in the paper [15]. We prove a strong classification theorem for the unital essential extensions of Kirchberg algebras, a classification theorem for the non-stable, non-unital essential extensions of Kirchberg algebras, and we characterize the range in both cases. The invariants are cyclic six term exact sequences together with the class of some unit.

In the mid-nineties Rørdam considered the classification problem for essential extensions of Kirchberg algebras ([15]). It turned out that one has to consider three cases: the stable case, the unital case, and the non-stable, non-unital case. Using the associated six term exact sequence he solved the classification problem in the first case and characterized the range of the invariant.

Rørdam's article is quite outstanding both in the general classification theory and with respect to the specific classification problem for essential extensions of Kirchberg algebras.

The classification theory has mainly been concerned with *simple* C^*-algebras (especially when the positive cone of K_0 does not encode the ideal structure as in the purely infinite case), so Rørdam's work suggested what kind of invariants to use (in certain non-simple cases). However, his methods are not similar to the usual methods of classification theory and at several points the proofs are ad hoc. Therefore Rørdam's methods have not really been generalized until recently ([5]).

While Rørdam's article did solve the classification problem in the stable case, it did not seem possible to apply his argument neither to the other two cases nor to prove a lifting theorem in the stable case (i.e. every isomorphism on the level of the invariant lifts to an isomorphism on the level of the algebras). In extending the classification result for these particular cases, there has almost not been any progress until 2003.

Extending Rørdam's work, the lifting theorem in the stable case and a classification theorem in the unital case was proved in [4]. In the present paper, we will present the classification results and describe the ranges of the invariants for all the three cases — in the first two cases, we will even allow for lifting of isomorphisms. Thus giving a

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very satisfactory answer to the classification problem for this class of C^*-algebras.

1. Introduction

Consider an extension $e\colon \mathfrak{B} \hookrightarrow \mathfrak{E} \twoheadrightarrow \mathfrak{A}$, where $\mathfrak{A}$ and $\mathfrak{B}$ are non-zero C^*-algebras (note that we use the symbols $\hookrightarrow$ resp. $\twoheadrightarrow$ meaning an injective resp. surjective morphism). If $\mathfrak{B}$ is unital, then the Busby invariant is zero — so in this case e is strongly isomorphic to the direct sum extension $e_0\colon \mathfrak{B} \hookrightarrow \mathfrak{B} \oplus \mathfrak{A} \twoheadrightarrow \mathfrak{A}$. If $\mathfrak{A}$ is simple, then either the Busby invariant is zero or injective — so either e is strongly isomorphic to the direct sum extension e_0 or e is an essential extension. If both $\mathfrak{A}$ and $\mathfrak{B}$ are simple, then there are at most two cases:

(i) e is essential, $\mathfrak{E}$ has exactly one non-trivial ideal, and $\mathrm{Prim}(\mathfrak{E}) \cong \{0, 1\}$ with the topology $\{\emptyset, \{0, 1\}, \{0\}\}$;
(ii) e is strongly isomorphic to the direct sum extension, $\mathfrak{E}$ has exactly two non-trivial (orthogonal) ideals, and $\mathrm{Prim}(\mathfrak{E}) \cong \{0, 1\}$ with the discrete topology.

On the other hand, if a C^*-algebra $\mathfrak{E}$ has exactly one non-trivial ideal $\mathfrak{I}$, then the extension $e\colon \mathfrak{I} \hookrightarrow \mathfrak{E} \twoheadrightarrow \mathfrak{E}/\mathfrak{I}$ is essential, $\mathfrak{I}$ is non-unital, and $\mathfrak{I}$ and $\mathfrak{E}/\mathfrak{I}$ are simple.

Recall that $\mathscr{D}$ is a subcategory of a category $\mathscr{C}$ if every object in $\mathscr{D}$ is an object in $\mathscr{C}$, every morphism between objects in $\mathscr{D}$ is an morphism in $\mathscr{C}$, the identity morphism of every object in $\mathscr{D}$ is also the identity morphism of that object in $\mathscr{C}$, and $\mathscr{D}$ with the inherited composition is a category. We say that a subcategory $\mathscr{D}$ of $\mathscr{C}$ is full if for all objects X and Y in $\mathscr{D}$, every morphism in $\mathscr{C}$ from X to Y is in fact a morphism in $\mathscr{D}$.

In this paper we will be considering extensions of simple C^*-algebras. Let $\mathscr{E}$ denote the category of extensions $e\colon \mathfrak{B} \hookrightarrow \mathfrak{E} \twoheadrightarrow \mathfrak{A}$ of separable C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ with the morphisms being triples (ϕ_0, ϕ_1, ϕ_2) of *-homomorphisms such that the diagram

$$\begin{array}{cccccc} e\colon & \mathfrak{B} & \hookrightarrow & \mathfrak{E} & \twoheadrightarrow & \mathfrak{A} \\ & \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} \\ e'\colon & \mathfrak{B}' & \hookrightarrow & \mathfrak{E}' & \twoheadrightarrow & \mathfrak{A}' \end{array}$$

commutes. Consider the subcategory of the category of C^*-algebras consisting of all separable C^*-algebras with exactly one non-trivial ideal, and as morphisms we take all *-homomorphisms which map the non-trivial ideal into the non-trivial ideal. This category is equivalent to the full subcategory $\mathscr{E}_0$ of $\mathscr{E}$ consisting of all essential extensions $e\colon \mathfrak{B} \hookrightarrow \mathfrak{E} \twoheadrightarrow \mathfrak{A}$, where $\mathfrak{A}$ and $\mathfrak{B}$ are separable, simple C^*-algebras— and we will freely use this identification.

Now consider the subcategory $\mathscr{H}$ of the category of all complexes of Z-modules (i.e. abelian groups) consisting of exact sequences of countable Z-modules which are periodic with period six, i.e. the exact sequences $(M_n, \partial_n)_{n\in\mathsf{Z}}$ with each M_n a countable Z-module, and $M_n = M_{n+6}$, $\partial_n = \partial_{n+6}$ for all $n \in \mathsf{Z}$ (usually we visualize this as a cyclic six term exact sequence). A morphism from $(M_n, \partial_n)_{n\in\mathsf{Z}}$ to $(M'_n, \partial'_n)_{n\in\mathsf{Z}}$ is a collection, $(\alpha_n)_{n\in\mathsf{Z}}$, of Z-module homomorphisms such that for all $n \in \mathsf{Z}$

$$\begin{array}{ccc} M_n & \xrightarrow{\partial_n} & M_{n+1} \\ \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \alpha_{n+1}} \\ M'_n & \xrightarrow{\partial'_n} & M'_{n+1} \end{array}$$

commutes and $\alpha_n = \alpha_{n+6}$ for all $n \in \mathsf{Z}$. Now we define a functor $K_{\hexagon}$ from $\mathscr{E}$ to $\mathscr{H}$ by

$$M_{0+3i+6n} = K_i(\mathfrak{B}), \quad M_{1+3i+6n} = K_i(\mathfrak{E}), \quad M_{2+3i+6n} = K_i(\mathfrak{A}),$$

$$\partial_{0+3i+6n} = K_i(\iota), \quad \partial_{1+3i+6n} = K_i(\pi), \quad \partial_{2+3i+6n} = \delta_i,$$

for each $i = 0, 1$ and for each $n \in \mathsf{Z}$, for every extension $e\colon \mathfrak{B} \overset{\iota}{\hookrightarrow} \mathfrak{E} \overset{\pi}{\twoheadrightarrow} \mathfrak{A}$ in $\mathscr{E}$ (this is nothing but the standard cyclic six term exact sequence in K-theory). Define $K_{\hexagon}$ of morphisms in the obvious way (so $K_{\hexagon}((\phi_0, \phi_1, \phi_2))$ corresponds to the six-tuple $(K_0(\phi_0), K_0(\phi_1), K_0(\phi_2), K_1(\phi_0), K_1(\phi_1), K_1(\phi_2))$ of Z-module homomorphisms).

Let $\mathscr{E}_K$ be the full subcategory of $\mathscr{E}_0$ consisting of all essential extensions $e\colon \mathfrak{B} \hookrightarrow \mathfrak{E} \twoheadrightarrow \mathfrak{A}$, where $\mathfrak{A}$ and $\mathfrak{B}$ are Kirchberg algebras satisfying the UCT (a Kirchberg algebra is a separable, nuclear, simple, purely infinite C^*-algebra). By Zhang's dichotomy ([17]), $\mathfrak{B}$ is stable.

Rørdam proves in [15, Proposition 4.6] the following:

PROPOSITION 1.1. *Let $e\colon \mathfrak{B} \hookrightarrow \mathfrak{E} \twoheadrightarrow \mathfrak{A}$ be an object in $\mathscr{E}_K$. Then*

(i) $\mathfrak{E}$ is unital if and only if $\mathfrak{A}$ is unital and the Busby map $\tau\colon \mathfrak{A} \to \mathcal{Q}(\mathfrak{B})$ is unital.
(ii) $\mathfrak{E}$ is stable if and only if $\mathfrak{A}$ is non-unital (i.e. $\mathfrak{A}$ is stable).
(iii) $\mathfrak{E}$ is neither unital nor stable if and only if $\mathfrak{A}$ is unital but the Busby map $\tau\colon \mathfrak{A} \to \mathcal{Q}(\mathfrak{B})$ is not unital.

The purpose of this paper is to look at some functor F in each of these cases, and look at the following questions (here, we are using the language promoted by Elliott in [6]):

1. Is F a *classification functor*, i.e. do we have that $\mathsf{F}(e) \cong \mathsf{F}(e')$ implies $e \cong e'$ in $\mathscr{E}_K$ (for all extensions e and e' in the class considered in each case)?
2. Is F a *strong classification functor*, i.e. does there for each isomorphism $\alpha\colon \mathsf{F}(e) \to \mathsf{F}(e')$ exist an isomorphism $\Phi\colon e \to e'$ such that $\mathsf{F}(\Phi) = \alpha$ (for all extensions e and e' in the class considered in each case)? Clearly, this implies 1.

3. What is the range of the invariant F (in this context, this is only interesting when the answer to 1. is positive)?

REMARK 1.2. Note that for extensions all $e\colon \mathfrak{B} \hookrightarrow \mathfrak{E} \twoheadrightarrow \mathfrak{A}$ and $e'\colon \mathfrak{B}' \hookrightarrow \mathfrak{E}' \twoheadrightarrow \mathfrak{A}'$ in $\mathscr{E}_0$ we have that every isomorphism from $\mathfrak{E}$ to $\mathfrak{E}'$ canonically induces an isomorphism from e onto e' in $\mathscr{E}_0$. So whether we talk about isomorphisms between $\mathfrak{E}$ and $\mathfrak{E}'$ or between e and e' really does not matter.

2. Main results

The main theorems of this paper will be stated in this section. The proofs are in Sections 4, 5, and 6, while a few results needed in the proofs are in Section 3.

In [4, Theorem 11] there is a rather general metatheorem, which — in certain cases — allows us to deduce from a strong classification functor on stable algebras a classification functor on the unital algebras. With some mild extra conditions, we will prove that this is in fact a strong classification functor on the unital algebras:

THEOREM 2.1. *Let $\mathscr{C}$ be a subcategory of the category of C^*-algebras and let $\mathsf{F}\colon \mathscr{C} \to \mathscr{D}$ be a covariant functor defined on this subcategory. Assume that*

(i) For every C^*-algebra $\mathfrak{A}$ in $\mathscr{C}$, $\operatorname{Mat}_2(\mathfrak{A})$ and $\mathfrak{A} \otimes \mathbb{K}$ belong to $\mathscr{C}$, and the canonical embeddings $\kappa_1\colon \mathfrak{A} \to \operatorname{Mat}_2(\mathfrak{A})$ and $\kappa\colon \mathfrak{A} \to \mathfrak{A} \otimes \mathbb{K}$ induce isomorphisms $\mathsf{F}(\kappa_1)$ and $\mathsf{F}(\kappa)$.
(ii) For all stable C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ in $\mathscr{C}$, every isomorphism from $\mathsf{F}(\mathfrak{A})$ to $\mathsf{F}(\mathfrak{B})$ is induced by an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.
(iii) There exists a covariant functor G from $\mathscr{D}$ into the category of abelian groups such that $\mathsf{G} \circ \mathsf{F} = K_0$

Assume that $\mathfrak{A}$ and $\mathfrak{B}$ are unital, properly infinite, separable C^*-algebras in $\mathscr{C}$ (if $\mathfrak{A} \otimes \mathbb{K}$ and $\mathfrak{B} \otimes \mathbb{K}$ have the cancellation property, then we can omit the assumption on properly infiniteness and separability). Let there be given an isomorphism α from $\mathsf{F}(\mathfrak{A})$ onto $\mathsf{F}(\mathfrak{B})$, such that $\mathsf{G}(\alpha)$ maps $[\mathbb{1}_{\mathfrak{A}}]_0$ onto $[\mathbb{1}_{\mathfrak{B}}]_0$. Then the C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ are *-isomorphic.

If, moreover, for every C^*-algebra $\mathfrak{C}$ in $\mathscr{C}$, we have $\mathsf{F}(\operatorname{Ad} u|_{\mathfrak{C}}) = \mathrm{id}_{\mathsf{F}(\mathfrak{C})}$ for every unitary u in $\mathcal{M}(\mathfrak{C})$ (the multiplier algebra of $\mathfrak{C}$), then there exists a *-isomorphism $\phi\colon \mathfrak{A} \to \mathfrak{B}$ such that $\mathsf{F}(\phi) = \alpha$ provided that every *-isomorphism between algebras in $\mathscr{C}$ is an isomorphism in $\mathscr{C}$.

Rørdam proved in [15, Theorem 5.3] that $K_{\hexagon}$ is a classification functor for stable extensions in $\mathscr{E}_K$. Using Bonkat's thesis [2] and Kirchberg's isomorphism theorem for ideal filtrated KK-theory (see e.g. [8]), it was proven in [4] that $K_{\hexagon}$ is in fact a strong classification functor for stable extensions in $\mathscr{E}_K$.

THEOREM 2.2. *The functor $K_{\hexagon}$ restricted to the stable extensions in $\mathscr{E}_K$ is a strong classification functor. That is, for all stable extensions e and e' in $\mathscr{E}_K$, and for every isomorphism $\alpha\colon K_{\hexagon}(e) \to K_{\hexagon}(e')$ there exists an isomorphism Φ from e onto e' (in $\mathscr{E}_K$) such that $K_{\hexagon}(\Phi) = \alpha$.*

Moreover, Rørdam also characterized the range in this case ([15, Proposition 5.4]). We have added the (almost) trivial fact, that the extensions can be chosen to be essential. The argument of this is given in Section 5, p. 10.

THEOREM 2.3. *The range of $K_{\hexagon}$ restricted to the stable extensions in $\mathscr{E}_K$ is all the objects in $\mathscr{H}$. That is, for every cyclic six term exact sequence of countable abelian groups — i.e. $(M_n, \partial_n)_{n\in\mathsf{Z}}$ in $\mathscr{H}$ — there exists a stable extension e in $\mathscr{E}_K$ such that $K_{\hexagon}(e) \cong (M_n, \partial_n)_{n\in\mathsf{Z}}$ in $\mathscr{H}$.*

With this version of the metatheorem (Theorem 2.1), we are able to prove that the classification functor $(e\colon \mathfrak{B} \hookrightarrow \mathfrak{E} \twoheadrightarrow \mathfrak{A}) \mapsto (K_{\hexagon}(e), [\mathbb{1}_{\mathfrak{E}}]_0)$ from [4, Corollary 12] is in fact a strong classification functor.

THEOREM 2.4. *The functor $(e\colon \mathfrak{B} \hookrightarrow \mathfrak{E} \twoheadrightarrow \mathfrak{A}) \mapsto (K_{\hexagon}(e), [\mathbb{1}_{\mathfrak{E}}]_0)$ restricted to the unital extensions in $\mathscr{E}_K$ is a strong classification functor. That is, for all unital extensions $e\colon \mathfrak{B} \hookrightarrow \mathfrak{E} \twoheadrightarrow \mathfrak{A}$ and $e'\colon \mathfrak{B}' \hookrightarrow \mathfrak{E}' \twoheadrightarrow \mathfrak{A}'$ in $\mathscr{E}_K$, and for every isomorphism $\alpha = (\alpha_n)_{n\in\mathsf{Z}}\colon K_{\hexagon}(e) \to K_{\hexagon}(e')$ satisfying $\alpha_1([\mathbb{1}_{\mathfrak{E}}]_0) = [\mathbb{1}_{\mathfrak{E}'}]_0$ there exists an isomorphism Φ from e onto e' (in $\mathscr{E}_K$) such that $K_{\hexagon}(\Phi) = \alpha$.*

Using Rørdam's range result in the stable case, we characterize the range in the unital case.

THEOREM 2.5. *The range of the functor $(e\colon \mathfrak{B} \hookrightarrow \mathfrak{E} \twoheadrightarrow \mathfrak{A}) \mapsto (K_{\hexagon}(e), [\mathbb{1}_{\mathfrak{E}}]_0)$ restricted to the unital extensions in $\mathscr{E}_K$ is all the objects $(M_n, \partial_n)_{n\in\mathsf{Z}}$ in $\mathscr{H}$ together with one distinguished element $m_1 \in M_1$. That is, for every cyclic six term exact sequence of countable abelian groups — i.e. $(M_n, \partial_n)_{n\in\mathsf{Z}}$ in $\mathscr{H}$ — and for every element $m_1 \in M_1$ there exists a unital extension $e\colon \mathfrak{B} \hookrightarrow \mathfrak{E} \twoheadrightarrow \mathfrak{A}$ in $\mathscr{E}_K$ such that $(K_{\hexagon}(e), [\mathbb{1}_{\mathfrak{E}}]_0) \cong ((M_n, \partial_n)_{n\in\mathsf{Z}}, m_1)$.*

Using the methods invented by Rørdam in [15] and a more recent result of Elliott and Kucerovsky [7], we are able to arrive at a classification functor in the non-stable, non-unital case. There is no obvious way to deduce from our proof that this is a stong classification functor (though, we believe that this is the case).

THEOREM 2.6. *The functor $(e\colon \mathfrak{B} \hookrightarrow \mathfrak{E} \twoheadrightarrow \mathfrak{A}) \mapsto (K_{\hexagon}(e), [\mathbb{1}_{\mathfrak{A}}]_0)$ restricted to the non-stable, non-unital extensions in $\mathscr{E}_K$ is a classification functor. That is, for all non-stable, non-unital extensions $e\colon \mathfrak{B} \hookrightarrow \mathfrak{E} \twoheadrightarrow \mathfrak{A}$ and $e'\colon \mathfrak{B}' \hookrightarrow \mathfrak{E}' \twoheadrightarrow \mathfrak{A}'$ in $\mathscr{E}_K$, and for every isomorphism $\alpha = (\alpha_n)_{n\in\mathsf{Z}}\colon K_{\hexagon}(e) \to K_{\hexagon}(e')$ satisfying $\alpha_2([\mathbb{1}_{\mathfrak{A}}]_0) = [\mathbb{1}_{\mathfrak{A}'}]_0$ there exists an isomorphism Φ from e onto e' (in $\mathscr{E}_K$).*

Using Rørdam's range result in the stable case, we are also able to characterize the range in the non-stable, non-unital case.

THEOREM 2.7. *The range of the functor* $(e\colon \mathfrak{B} \hookrightarrow \mathfrak{E} \twoheadrightarrow \mathfrak{A}) \mapsto (K_{\hexagon}(e), [\mathbb{1}_{\mathfrak{A}}]_0)$ *restricted to the non-stable, non-unital extensions in* $\mathscr{E}_K$ *is all the objects* $(M_n, \partial_n)_{n\in\mathsf{Z}}$ *in* $\mathscr{H}$ *together with one distinguished element* $m_2 \in M_2$*. That is, for every cyclic six term exact sequence of countable abelian groups — i.e.* $(M_n, \partial_n)_{n\in\mathsf{Z}}$ *in* $\mathscr{H}$ *— and for every element* $m_2 \in M_2$ *there exists a non-stable, non-unital extension* $e\colon \mathfrak{B} \hookrightarrow \mathfrak{E} \twoheadrightarrow \mathfrak{A}$ *in* $\mathscr{E}_K$ *such that* $(K_{\hexagon}(e), [\mathbb{1}_{\mathfrak{A}}]_0) \cong ((M_n, \partial_n)_{n\in\mathsf{Z}}, m_2)$.

QUESTION 1. *Is the functor* $(e\colon \mathfrak{B} \hookrightarrow \mathfrak{E} \twoheadrightarrow \mathfrak{A}) \mapsto (K_{\hexagon}(e), [\mathbb{1}_{\mathfrak{A}}]_0)$ *in the preceding theorem a strong classification functor?*

QUESTION 2. *To what extent do we have lifting of homomorphisms?*

QUESTION 3. *Do we have uniqueness of automorphisms, isomorphisms, or homomorphisms?*

QUESTION 4. *In* [13] *the purely infinite Cuntz-Krieger algebras with finitely many ideals are classified up to stable isomorphism by an invariant naturally extending* $K_{\hexagon}$*. Does every isomorphism on the invariant level lift to a* **-isomorphism (in this particular case)? If this is true, then we would get a strong classification functor for the unital case from Theorem 2.1. This would give a better understanding of the relationship between the two-sided and one-sided shifts of finite type, and the corresponding Cuntz-Krieger algebras.*

3. Prerequisites

In this section we will state some results and prove some lemmas we will need in the proofs of the main theorems.

REMARK 3.1. An ideal $\mathfrak{I}$ in a C^*-algebra $\mathfrak{A}$ is called essential if $a\mathfrak{I} = \{0\} \Rightarrow a = 0$ (or equivalently, if $\mathfrak{I}a = \{0\} \Rightarrow a = 0$) — and this is the case if and only if $\mathfrak{I} \cap \mathfrak{J} \neq \{0\}$ for every non-zero ideal $\mathfrak{J}$ in $\mathfrak{A}$.

Recall that the multiplier algebra $\mathcal{M}(\mathfrak{A})$ of a C^*-algebra $\mathfrak{A}$ is the largest unital C^*-algebra which contains $\mathfrak{A}$ as an essential ideal, i.e. if $\mathfrak{A}$ is embedded as an essential ideal in a unital C^*-algebra $\mathfrak{B}$, then the embedding $\mathfrak{A} \hookrightarrow \mathcal{M}(\mathfrak{A})$ can be uniquely extended to an (injective) *-homomorphism $\mathfrak{B} \to \mathcal{M}(\mathfrak{A})$ — moreover, this embedding is unital; to see this, notice that if $p \in \mathcal{M}(\mathfrak{A})$ acts as an identity on $\mathfrak{A}$, then $\mathfrak{A}(\mathbb{1}_{\mathcal{M}(\mathfrak{A})} - p) = \{0\}$, which implies that $\mathbb{1}_{\mathcal{M}(\mathfrak{A})} - p = 0$.

If $\mathfrak{A}$ and $\mathfrak{B}$ are C^*-algebras, then we have canonical embeddings $\mathfrak{A} \otimes_* \mathfrak{B} \subseteq \mathcal{M}(\mathfrak{A}) \otimes_* \mathcal{M}(\mathfrak{B}) \subseteq \mathcal{M}(\mathfrak{A} \otimes_* \mathfrak{B})$ (see e.g. [12, Lemma 11.12]). By the above comments, the latter embedding is unital.

We call a functor F on the category of C^*-algebras *stable* if $\mathsf{F}(\kappa)$ is an isomorphism whenever $\kappa\colon \mathfrak{A} \to \mathfrak{A} \otimes \mathbb{K}$ is the canonical embedding.

The following lemma is well known. Since we were not able to find it in the litterature, we include the short proof.

Lemma 3.2. *Let $\mathfrak{A}$ be a C^*-algebra, let u be a unitary in $\mathcal{M}(\mathfrak{A})$, and let F be a stable, homotopy invariant functor on the category of C^*-algebras. Then $\mathsf{F}(\operatorname{Ad} u|_{\mathfrak{A}}) = \mathrm{id}_{\mathsf{F}(\mathfrak{A})}$. In particular, $K_i(\operatorname{Ad} u|_{\mathfrak{A}}) = \mathrm{id}_{K_i(\mathfrak{A})}$ for $i = 0, 1$.*

Proof. Let $\kappa\colon \mathfrak{A} \to \mathfrak{A} \otimes \mathbb{K}$ be the canonical embedding $a \mapsto a \otimes e_{11}$, where e_{11} is a minimal projection. Let

$$w = u \otimes \mathbb{1}_{\mathbb{B}} \in \mathcal{M}(\mathfrak{A}) \otimes_* \mathbb{B} \subseteq \mathcal{M}(\mathfrak{A} \otimes \mathbb{K}).$$

Since the embedding is unital, it is evident that w is a unitary in $\mathcal{M}(\mathfrak{A} \otimes \mathbb{K})$.

Now the unitary group of $\mathfrak{A} \otimes \mathbb{K}$ is path-connected in the strict topology, i.e. there is a strictly continuous path $(u_t)_{t\in[0,1]}$ of unitaries in $\mathcal{M}(\mathfrak{A} \otimes \mathbb{K})$ with $u_0 = \mathbb{1}_{\mathcal{M}(\mathfrak{A}\otimes\mathbb{K})}$ and $u_1 = w$ (by [1, Proposition 12.2.2]). For each $t \in [0,1]$, let $\phi_t = \operatorname{Ad} u_t|_{\mathfrak{A}\otimes\mathbb{K}}$. Let $(t_n)_{n\in\mathsf{N}}$ be a sequence in $[0,1]$ converging to $t_0 \in [0,1]$ and let $x \in \mathfrak{A} \otimes \mathbb{K}$ be given. We want to prove that $\phi_{t_n}(x)$ converges to $\phi_{t_0}(x)$. Because the positive elements linearily span the C^*-algebra, we may assume that x is positive. Since $\lim_{n\to\infty}(\|\sqrt{x}(u_{t_n} - u_{t_0})\| + \|(u_{t_n} - u_{t_0})\sqrt{x}\|) = 0$, it follows from the continuity of the product, that $\lim_{n\to\infty} \|u_{t_n} x u_{t_n}^* - u_{t_0} x u_{t_0}^*\| = 0$. Hence $\mathrm{id}_{\mathfrak{A}\otimes\mathbb{K}}$ is homotopic to $\operatorname{Ad} w|_{\mathfrak{A}\otimes\mathbb{K}}$. Consequently,

$$\mathsf{F}(\operatorname{Ad} w|_{\mathfrak{A}\otimes\mathbb{K}}) = \mathsf{F}(\mathrm{id}_{\mathfrak{A}\otimes\mathbb{K}}) = \mathrm{id}_{\mathsf{F}(\mathfrak{A}\otimes\mathbb{K})}.$$

Since $\operatorname{Ad} w|_{\mathfrak{A}\otimes\mathbb{K}} \circ \kappa = \kappa \circ \operatorname{Ad} u|_{\mathfrak{A}}$ and $\mathsf{F}(\kappa)$ is an isomorphism, we have that $\mathsf{F}(\operatorname{Ad} u|_{\mathfrak{A}}) = \mathrm{id}_{\mathsf{F}(\mathfrak{A})}$. It is well known that K_0 and K_1 are stable and homotopy invariant.

We say that a sub-C^*-algebra $\mathfrak{B}$ of a C^*-algebra $\mathfrak{A}$ is *full* if the ideal generated by $\mathfrak{B}$ is $\mathfrak{A}$. We say that a projection p in $\mathcal{M}(\mathfrak{A})$ is *full* if the hereditary corner $p\mathfrak{A}p$ is a full sub-C^*-algebra of $\mathfrak{A}$. Brown proved the corollary below for the contravariant functor $\operatorname{Ext}(-)$ ([3, Corollary 2.10]). With essentially the same proof, we get the analogous result for K-theory:

Corollary 3.3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be σ-unital C^*-algebras, and assume that $\mathfrak{B}$ is a full hereditary subalgebra of $\mathfrak{A}$. Then the inclusion map $\iota\colon \mathfrak{B} \hookrightarrow \mathfrak{A}$ induces isomorphisms $K_0(\iota)$ and $K_1(\iota)$ in K-theory.*

Proof. The proofs of Corollaries 2.7 and 2.10 in [3] carry over if we use the preceding lemma instead of the reference '[2,3.11]' in the end of each proof.

Lemma 3.4. *Let $\mathscr{C}$ be a subcategory of the category of C^*-algebras, and let $\mathsf{F}\colon \mathscr{C} \to \mathscr{D}$ be a covariant functor. Assume that*

- *For every $\mathfrak{A}$ in $\mathscr{C}$, the C^*-algebra $\operatorname{Mat}_2(\mathfrak{A})$ is an object in $\mathscr{C}$, the standard embedding $\kappa_1\colon \mathfrak{A} \to \operatorname{Mat}_2(\mathfrak{A})$ is a morphism in $\mathscr{C}$, and $\mathsf{F}(\kappa_1)$ is an isomorphism.*
- *For every $\mathfrak{A}$ in $\mathscr{C}$ and every unitary u in $\mathcal{M}(\mathfrak{A})$, $\operatorname{Ad} u|_{\mathfrak{A}}$ is an automorphism in $\mathscr{C}$ and $\mathsf{F}(\operatorname{Ad} u|_{\mathfrak{A}}) = \operatorname{id}_{\mathsf{F}(\mathfrak{A})}$.*

Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras in $\mathscr{C}$, let $\varphi\colon \mathfrak{A} \to \mathfrak{B}$ be a morphism in $\mathscr{C}$, let $v \in \mathcal{M}(\mathfrak{B})$ be a partial isometry satisfying $v^*v\varphi(a) = \varphi(a) = \varphi(a)v^*v$ for all $a \in \mathfrak{A}$, and define $\psi\colon \mathfrak{A} \to \mathfrak{B}$ by $\psi(a) = v\varphi(a)v^*$. Then ψ is a *-homomorphism and $\mathsf{F}(\varphi) = \mathsf{F}(\psi)$, provided that ψ is a morphism in $\mathscr{C}$.

Proof. Clearly, ψ is linear and *-preserving. Also

$$\psi(aa') = v\varphi(aa')v^* = v\varphi(a)v^*v\varphi(a')v^* = \psi(a)\psi(a')$$

for all $a, a' \in \mathfrak{A}$.

Let

$$\begin{aligned} u = \begin{pmatrix} v & \mathbb{1} - vv^* \\ \mathbb{1} - v^*v & v^* \end{pmatrix} \in &\operatorname{Mat}_2(\mathcal{M}(\mathfrak{B})) = \operatorname{Mat}_2(\mathsf{C}) \otimes \mathcal{M}(\mathfrak{B}) \\ &\subseteq \mathcal{M}(\operatorname{Mat}_2(\mathsf{C}) \otimes \mathfrak{B}) = \mathcal{M}(\operatorname{Mat}_2(\mathfrak{B})), \end{aligned}$$

and recall from Remark 3.1 that the above inclusion is a unital embedding. Then $u^* = \left(\begin{smallmatrix} v^* & \mathbb{1}-v^*v \\ \mathbb{1}-vv^* & v \end{smallmatrix}\right)$, and a short calculation shows that u is a unitary in $\mathcal{M}(\operatorname{Mat}_2(\mathfrak{B}))$. Let $\kappa_1\colon \mathfrak{B} \to \operatorname{Mat}_2(\mathfrak{B})$ be the canonical embedding. Then

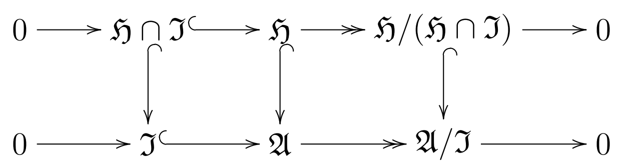
$$\begin{aligned} \operatorname{Ad} u \circ \kappa_1 \circ \varphi(a) &= u \begin{pmatrix} \varphi(a) & 0 \\ 0 & 0 \end{pmatrix} u^* = \begin{pmatrix} v\varphi(a) & 0 \\ \varphi(a) - v^*v\varphi(a) & 0 \end{pmatrix} u^* \\ &= \begin{pmatrix} v\varphi(a) & 0 \\ 0 & 0 \end{pmatrix} u^* = \begin{pmatrix} v\varphi(a)v^* & v\varphi(a) - v\varphi(a)v^*v \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} v\varphi(a)v^* & 0 \\ 0 & 0 \end{pmatrix} = \kappa_1 \circ \psi(a) \end{aligned}$$

for all $a \in \mathfrak{A}$. So by the assumption, we have that

$$\begin{aligned} \mathsf{F}(\kappa_1) \circ \mathsf{F}(\psi) &= \mathsf{F}(\kappa_1 \circ \psi) = \mathsf{F}(\operatorname{Ad} u|_{\operatorname{Mat}_2(\mathfrak{B})} \circ \kappa_1 \circ \varphi) \\ &= \mathsf{F}(\operatorname{Ad} u|_{\operatorname{Mat}_2(\mathfrak{B})}) \circ \mathsf{F}(\kappa_1) \circ \mathsf{F}(\varphi) = \mathsf{F}(\kappa_1) \circ \mathsf{F}(\varphi). \end{aligned}$$

Since $\mathsf{F}(\kappa_1)$ is an isomorphism, $\mathsf{F}(\psi) = \mathsf{F}(\varphi)$.

Lemma 3.5. *Let $\mathfrak{I}$ be a non-trivial ideal in the C^*-algebra $\mathfrak{A}$, and assume that $\mathfrak{H}$ is a hereditary sub-C^*-algebra of $\mathfrak{A}$. Then we have the following commutative diagram:*



with exact rows, and the vertical arrows being the inclusions, in particular, the map from $\mathfrak{H}/(\mathfrak{H}\cap\mathfrak{I})$ to $\mathfrak{A}/\mathfrak{I}$ is given by $h+\mathfrak{H}\cap\mathfrak{I}\mapsto h+\mathfrak{I}$. Moreover, $\mathfrak{H}\cap\mathfrak{I}$ and $\mathfrak{H}/(\mathfrak{H}\cap\mathfrak{I})$ are hereditary sub-C^*-algebras of $\mathfrak{I}$ and $\mathfrak{A}/\mathfrak{I}$, resp.

If, in addition, $\mathfrak{I}$ and $\mathfrak{A}/\mathfrak{I}$ are simple, $\mathfrak{H}$ is a full sub-C^*-algebra of $\mathfrak{A}$, and $\mathfrak{H}\cap\mathfrak{I}$ is non-zero, then $\mathfrak{H}\cap\mathfrak{I}$ and $\mathfrak{H}/(\mathfrak{H}\cap\mathfrak{I})$ are full hereditary sub-C^*-algebras of I and $\mathfrak{A}/\mathfrak{I}$, resp.

Proof. Clearly $\mathfrak{H}\cap\mathfrak{I}$ is an ideal in $\mathfrak{H}$. It is routine to show, that the map from $\mathfrak{H}/(\mathfrak{H}\cap\mathfrak{I})$ into $\mathfrak{A}/\mathfrak{I}$ is a well-defined, injective *-homomorphism — in fact, $\mathfrak{H}/(\mathfrak{H}\cap\mathfrak{I})$ is canonically isomorphic to the sub-C^*-algebra $(\mathfrak{H}+\mathfrak{I})/\mathfrak{I}$ of $\mathfrak{A}/\mathfrak{I}$. Commutativity of the two squares are now obvious.

By the definition of a hereditary sub-C^*-algebra, it is trivial to show that $\mathfrak{H}\cap\mathfrak{I}$ is a hereditary sub-C^*-algebra of $\mathfrak{I}$. If $h,h'\in\mathfrak{H}$ and $a\in\mathfrak{A}$ are arbitrary elements, then $(h+\mathfrak{I})(a+\mathfrak{I})(h'+\mathfrak{I})=hah'+\mathfrak{I}$ in $\mathfrak{A}/\mathfrak{I}$, and $hah'\in\mathfrak{H}$ because $\mathfrak{H}$ is hereditary. Consequently, $\mathfrak{H}/(\mathfrak{H}\cap\mathfrak{I})\cong(\mathfrak{H}+\mathfrak{I})/\mathfrak{I}$ is considered as a hereditary sub-C^*-algebra of $\mathfrak{A}/\mathfrak{I}$.

Assume that $\mathfrak{I}$ and $\mathfrak{A}/\mathfrak{I}$ are simple, $\mathfrak{H}$ is a full sub-C^*-algebra of $\mathfrak{A}$, and $\mathfrak{H}\cap\mathfrak{I}$ is non-zero. Since $\mathfrak{H}$ is full in $\mathfrak{A}$, we have that $\mathfrak{H}\cap\mathfrak{I}\subsetneq\mathfrak{H}$. Consequently, $\mathfrak{H}/(\mathfrak{H}\cap\mathfrak{I})$ is non-zero. Being non-zero hereditary sub-C^*-algebras of simple C^*-algebras, $\mathfrak{H}$ and $\mathfrak{H}/(\mathfrak{H}\cap\mathfrak{I})$ are full.

4. The classification theorem — the unital case

In this section we will extend the proofs in [4] to prove the metatheorem, Theorem 2.1, and the classification result in the unital case, Theorem 2.4.

Proof of Theorem 2.1. The first part is proved in [4, Theorem 11]. So assume that $\mathfrak{A}$, $\mathfrak{B}$, and α are as in the theorem. Assume furthermore that for every C^*-algebra $\mathfrak{C}$ in $\mathscr{C}$, we have $\mathsf{F}(\operatorname{Ad}u|_{\mathfrak{C}})=\mathrm{id}_{\mathsf{F}(\mathfrak{C})}$ for all u in $\mathcal{M}(\mathfrak{C})$.

In the proof of [4, Theorem 11] we found a *-isomorphism $\phi\colon\mathfrak{A}\otimes\mathbb{K}\to\mathfrak{B}\otimes\mathbb{K}$ such that $\mathsf{F}(\phi)=\mathsf{F}(\kappa')\circ\alpha\circ\mathsf{F}(\kappa)^{-1}$, where $\kappa\colon\mathfrak{A}\to\mathfrak{A}\otimes\mathbb{K}$ and $\kappa'\colon\mathfrak{B}\to\mathfrak{B}\otimes\mathbb{K}$ are the canonical embeddings (corresponding to the minimal projection e_{11}). We found a partial isometry $v\in\mathcal{M}(\mathfrak{B}\otimes\mathbb{K})$ such that $v^*v=\mathbb{1}_{\mathcal{M}(\mathfrak{B}\otimes\mathbb{K})}$ and $vv^*=\mathbb{1}_{\mathcal{M}(\mathfrak{B})}\otimes e_{11}$, and we showed that $\psi(x)=v(\phi\circ\kappa)(x)v^*$ for $x\in\mathfrak{A}$, is an *-isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}\otimes e\cong\mathfrak{B}$. So there exists a unique *-isomorphism $\psi_0\colon\mathfrak{A}\to\mathfrak{B}$ such that $\psi=\kappa'\circ\psi_0$. We claim that $\mathsf{F}(\psi_0)=\alpha$.

By Lemma 3.4

$$\mathsf{F}(\kappa')\circ\mathsf{F}(\psi_0)=\mathsf{F}(\psi)=\mathsf{F}(\phi\circ\kappa)=\mathsf{F}(\kappa')\circ\alpha.$$

Since $\mathsf{F}(\kappa')$ is an isomorphism, it follows that $\mathsf{F}(\psi_0)=\alpha$.

Proof of Theorem 2.4. This is a direct consequence of Lemma 3.2 and Theorem 2.1. We only need to prove that $K_{\hexagon}(\operatorname{Ad}u|_{\mathfrak{C}})=$

$\mathrm{id}_{K_{\hexagon}(\mathfrak{E})}$ for every extension $e\colon \mathfrak{B} \hookrightarrow \mathfrak{E} \twoheadrightarrow \mathfrak{A}$ in $\mathscr{E}_K$ and every unitary $u \in \mathcal{M}(\mathfrak{E})$. Since $\mathfrak{B}$ is an essential ideal in $\mathcal{M}(\mathfrak{E})$, we have that $\mathcal{M}(\mathfrak{E}) \subseteq \mathcal{M}(\mathfrak{B})$ (and, just as in Remark 3.1, this embedding is unital). The quotient map from $\mathfrak{E}$ to $\mathfrak{A}$ can be extended to a surjective *-homomorphism $\mathcal{M}(\mathfrak{E}) \to \mathcal{M}(\mathfrak{A})$ by [11, Proposition 3.12.10], which of course is unital. Now Lemma 3.2 directly implies that $K_{\hexagon}(\mathrm{Ad}\, u|_{\mathfrak{E}}) = \mathrm{id}_{K_{\hexagon}(\mathfrak{E})}$ (in the above settings).

5. The range results

In this section we prove the (slight) improvement of Rørdam's range result in the stable case, Theorem 2.3. Using this result, we prove the range results in the two other cases, Theorems 2.5 and 2.7.

Proof of extra assertion in Theorem 2.3. In [15, Proposition 5.4] Rørdam proves everything except that the extension can be chosen to be essential. Since the functors K_0 and K_1 are split exact, $K_{\hexagon}(e)$ degenerates into two split exact sequences (with zero index and exponential map) for every trivial extension $e\colon \mathfrak{B} \hookrightarrow \mathfrak{E} \twoheadrightarrow \mathfrak{A}$ in $\mathscr{E}_K$. So if the given sequence in the theorem does not consist of two split exact sequences $0 \to M_{0+i} \to M_{1+i} \to M_{2+i} \to 0$, for $i = 0, 3$, then the extension constructed by Rørdam is necessarily essential.

So assume that $0 \to M_{0+i} \to M_{1+i} \to M_{2+i} \to 0$, for $i = 0, 3$, are split exact sequences. Then there exist stable Kirchberg algebras $\mathfrak{A}$ and $\mathfrak{B}$ in the UCT class with $G_0 \cong K_0(\mathfrak{B})$, $G_2 \cong K_0(\mathfrak{A})$, $G_3 \cong K_1(\mathfrak{B})$, $G_5 \cong K_1(\mathfrak{A})$. Because $\mathfrak{B}$ is stable and $\mathfrak{A}$ is separable, there exists an essential trivial extension $e\colon \mathfrak{B} \hookrightarrow \mathfrak{E} \twoheadrightarrow \mathfrak{A}$ (represent $\mathfrak{A}$ on $\mathbb{B}$, use $\mathfrak{B} \cong \mathfrak{B} \otimes \mathbb{K}$ and $\mathfrak{B} \otimes \mathbb{K} \subseteq \mathcal{M}(\mathfrak{B}) \otimes_* \mathbb{B} \subseteq \mathcal{M}(\mathfrak{B} \otimes \mathbb{K})$ to create an injective *-homomorphism $\tau\colon \mathfrak{A} \to \mathcal{M}(\mathfrak{B})$ with the range not intersecting $\mathfrak{B}$). Because this extension is trivial, it realizes the cyclic six term exact sequence.

Proof of Theorems 2.5 and 2.7. Let $(M_n, \partial_n)_{n \in \mathbb{Z}}$ be an exact sequence in $\mathscr{H}$. From Rørdam's range result, Theorem 2.3, we know that there exists an essential stable extension $e\colon \mathfrak{B} \hookrightarrow \mathfrak{E} \twoheadrightarrow \mathfrak{A}$ in $\mathscr{E}_K$, such that we have an isomorphism $\alpha = (\alpha_n)_{n \in \mathbb{Z}}\colon K_{\hexagon}(e) \to (M_n, \partial_n)_{n \in \mathbb{Z}}$. For notational convenience, we may assume that $\mathfrak{B}$ is an ideal in $\mathfrak{E}$, and that $\mathfrak{A}$ is the quotient $\mathfrak{E}/\mathfrak{B}$.

If a C^*-algebra has a full, properly infinite projection, then every element of K_0 is of the form $[p]_0$ for a full, properly infinite projection p in the algebra. Since $\mathfrak{A}$ is purely infinite, there exists a full, properly infinite projection q in $\mathfrak{A}$ such that $[q]_0 = 0$ in $K_0(\mathfrak{A})$. Because of [15, Proposition 4.1] we can lift it to a projection $q_0 \in \mathfrak{E}$. By [15, Proposition 4.5] q_0 is a full, properly infinite projection in $\mathfrak{E}$.

Now we prove Theorem 2.5. Let $m_1 \in M_1$ be a given element. Then there exists a full, properly infinite projection p in $\mathfrak{E}$ such that $\alpha_1([p]_0) = m_1$. Then $p\mathfrak{E}p$ is a full, hereditary sub-C^*-algebra of $\mathfrak{E}$ —

actually a full corner — and $p\mathfrak{E}p \cap \mathfrak{B} = p\mathfrak{B}p$ (note that this does not imply that $p\mathfrak{B}p$ is unital, because $p \notin \mathfrak{B}$). By Brown's theorem ([3, Theorem 2.8]) we have $p\mathfrak{E}p \otimes \mathbb{K} \cong \mathfrak{A} \otimes \mathbb{K}$. The functor $-\otimes \mathbb{K}$ preserves the ideal lattice, therefore $p\mathfrak{E}p$ has exactly one non-trivial ideal, say $\mathfrak{J}$. The ideal $\mathfrak{B}$ is the only non-trivial ideal in $\mathfrak{E}$, so by ([10, Theorem 3.2.7]) $\mathfrak{J} = p\mathfrak{E}p \cap \mathfrak{B}$. Therefore $p\mathfrak{B}p$ is the only non-trivial ideal in $p\mathfrak{E}p$. Moreover, $p\mathfrak{B}p$ and $p\mathfrak{E}p/p\mathfrak{B}p$ are simple (being hereditary sub-C^*-algebras of simple C^*-algebras), so we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p\mathfrak{B}p & \hookrightarrow & p\mathfrak{E}p & \twoheadrightarrow & p\mathfrak{E}p/p\mathfrak{B}p & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \iota|_{p\mathfrak{B}p}} & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \bar{\iota}} & & \\ 0 & \longrightarrow & \mathfrak{B} & \hookrightarrow & \mathfrak{E} & \twoheadrightarrow & \mathfrak{A} & \longrightarrow & 0 \end{array}$$

from Lemma 3.5 where all the vertical injections are embeddings as full, hereditary sub-C^*-algebras. Now we will let e' denote the extension $e' \colon p\mathfrak{B}p \hookrightarrow p\mathfrak{E}p \twoheadrightarrow p\mathfrak{E}p/p\mathfrak{B}p$.

From Brown's Theorem for K-theory (Corollary 3.3) we get isomorphisms in the following diagram:

$$\begin{array}{ccccccc} K_i(p\mathfrak{B}p) & \longrightarrow & K_i(p\mathfrak{E}p) & \longrightarrow & K_i(p\mathfrak{E}p/p\mathfrak{B}p) & \longrightarrow & K_{i+1}(p\mathfrak{B}p) \\ \cong\downarrow{\scriptstyle K_i(\iota|_{p\mathfrak{B}p})} & & \cong\downarrow{\scriptstyle K_i(\iota)} & & \cong\downarrow{\scriptstyle K_i(\bar{\iota})} & & \cong\downarrow{\scriptstyle K_{i+1}(\iota|_{p\mathfrak{B}p})} \\ K_i(\mathfrak{B}) & \longrightarrow & K_i(\mathfrak{E}) & \longrightarrow & K_i(\mathfrak{A}) & \longrightarrow & K_{i+1}(\mathfrak{B}) \end{array}$$

for $i = 0, 1$. Clearly $\alpha \circ K_{\bigcirc}(\iota)$ is an isomorphism from $K_{\bigcirc}(e')$ to $(M_n, \partial_n)_{n\in\mathsf{Z}}$ mapping $[p]_0 \in K_0(p\mathfrak{E}p)$ to $m_1 \in M_1$. Clearly $p\mathfrak{E}p$ is a unital C^*-algebra with unit p. The C^*-algebras $p\mathfrak{B}p$ and $p\mathfrak{E}p/p\mathfrak{B}p$ are separable, nuclear, and belong to the UCT class (by Brown's Theorem ([3, Theorem 2.8]) and [1, 22.3.5(a)]). Also they are purely infinite ([9, Proposition 4.17]). We have already seen that $p\mathfrak{B}p$ is the only non-trivial ideal in $p\mathfrak{E}p$, so e' is a unital extension in $\mathscr{E}_K$.

Now we consider the non-stable, non-unital case (Theorem 2.7). So let $m_2 \in M_2$ be a given element. Then there exists a full, properly infinite projection p in $\mathfrak{A}$ such that $\alpha_2([p]_0) = m_2$. We can lift p to a positive element $x \in \mathfrak{E}$ (e.g. [14, 2.2.10]). It is easy to see that $\overline{x\mathfrak{E}x}/(\overline{x\mathfrak{E}x} \cap \mathfrak{B}) \cong (\overline{x\mathfrak{E}x} + \mathfrak{B})/\mathfrak{B} = p\mathfrak{A}p$.

Clearly $\overline{x\mathfrak{E}x}$ is a full hereditary sub-C^*-algebra of $\mathfrak{E}$ ($x \in \overline{x\mathfrak{E}x}$ and $x \notin \mathfrak{B}$ since $p \neq 0$). Since $\mathfrak{B}$ is essential in $\mathfrak{E}$ and $x \neq 0$, there exists a $b \in \mathfrak{B}$ such that $bx \neq 0$. Therefore $xb^*bx = (bx)^*bx \neq 0$, and hence $\overline{x\mathfrak{E}x} \cap \mathfrak{B} \neq \{0\}$. Being hereditary subalgebras of simple C^*-algebras, $\overline{x\mathfrak{E}x} \cap \mathfrak{B}$ and $p\mathfrak{A}p$ are simple. Again by [10, Theorem 3.2.7] we see that $\overline{x\mathfrak{E}x} \cap \mathfrak{B}$ is the only non-trivial ideal in $\overline{x\mathfrak{E}x}$. So we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{x\mathfrak{E}x} \cap \mathfrak{B} & \hookrightarrow & \overline{x\mathfrak{E}x} & \twoheadrightarrow & p\mathfrak{A}p & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \iota|_{\overline{x\mathfrak{E}x}\cap\mathfrak{B}}} & & \Big\downarrow{\scriptstyle \iota} & & \Big\downarrow{\scriptstyle \overline{\iota}} & & \\ 0 & \longrightarrow & \mathfrak{B} & \hookrightarrow & \mathfrak{E} & \twoheadrightarrow & \mathfrak{A} & \longrightarrow & 0 \end{array}$$

from Lemma 3.5 where all the vertical injections are embeddings as full, hereditary sub-C^*-algebras. Let $\mathfrak{B}' = \overline{x\mathfrak{E}x} \cap \mathfrak{B}$, $\mathfrak{E}' = \overline{x\mathfrak{E}x}$, and $\mathfrak{A}' = p\mathfrak{A}p$, and let e' denote the extension $e' \colon \mathfrak{B}' \hookrightarrow \mathfrak{E}' \twoheadrightarrow \mathfrak{A}'$.

From Brown's Theorem for K-theory (Corollary 3.3) we get isomorphisms in the following diagram:

$$\begin{array}{ccccccc} K_i(\mathfrak{B}') & \longrightarrow & K_i(\mathfrak{E}') & \longrightarrow & K_i(\mathfrak{A}') & \longrightarrow & K_{i+1}(\mathfrak{B}') \\ \cong\Big\downarrow{\scriptstyle K_i(\iota|_{\mathfrak{B}'})} & & \cong\Big\downarrow{\scriptstyle K_i(\iota)} & & \cong\Big\downarrow{\scriptstyle K_i(\overline{\iota})} & & \cong\Big\downarrow{\scriptstyle K_{i+1}(\iota|_{\mathfrak{B}'})} \\ K_i(\mathfrak{B}) & \longrightarrow & K_i(\mathfrak{E}) & \longrightarrow & K_i(\mathfrak{A}) & \longrightarrow & K_{i+1}(\mathfrak{B}) \end{array}$$

for $i = 0, 1$. Clearly $\alpha \circ K_{\hexagon}(\iota)$ is an isomorphism from $K_{\hexagon}(e')$ to $(M_n, \partial_n)_{n\in\mathbb{Z}}$ mapping $[p]_0 \in K_0(\mathfrak{A}')$ to $m_2 \in M_2$. Clearly $\mathfrak{A}'$ is a unital C^*-algebra with unit p. The C^*-algebras $\mathfrak{A}'$ and $\mathfrak{B}'$ are separable, nuclear, and belong to the UCT class (by Brown's Theorem ([3, Theorem 2.8]) and [1, 22.3.5(a)]). Also they are purely infinite ([9, Proposition 4.17]). We have already seen that $\mathfrak{B}'$ is the only non-trivial ideal in $\mathfrak{E}'$, so e' is an extension in $\mathscr{E}_K$. We just do not know whether it is unital or not.

If e' is non-unital, then we are done. Assume that e' is unital. Let e_0 denote the direct sum extension $e_0 \colon \mathfrak{B}' \hookrightarrow \mathfrak{B}' \oplus \mathfrak{A}' \twoheadrightarrow \mathfrak{A}'$. Let τ' and τ_0 be the Busby invariant of e' and e_0, resp. Then τ' is unital, but τ_0 is non-unital. Let $\tau'' = \tau' \oplus \tau_0$. Then τ'' is an essential, non-unital extension, and $[\tau''] = [\tau' \oplus \tau_0] = [\tau']$ in $\operatorname{Ext}(\mathfrak{A}', \mathfrak{B}')$. Hence by [15, Proposition 2.1] the cyclic six term exact sequences corresponding to τ'' and τ' are congruent, i.e. there exists an isomorphism between $K_{\hexagon}(e'')$ and $K_{\hexagon}(e')$ which is the identity on $K_0(\mathfrak{B}')$, $K_1(\mathfrak{B}')$, $K_0(\mathfrak{A}')$, and $K_1(\mathfrak{A}')$, where e'' is the extension corresponding to τ''. This assures us that — when composed with the isomorphism from $K_{\hexagon}(e')$ to $(M_n, \partial_n)_{n\in\mathbb{Z}}$ — the class of the unit in $p\mathfrak{A}p$, $[p]_0 \in K_0(p\mathfrak{A}p)$, will be mapped onto m_2. By the above remarks, e'' is a non-stable, non-unital extension in $\mathscr{E}_K$.

6. The classification theorem — the non-stable, non-unital case

In this section we prove the classification theorem in the non-stable, non-unital case, Theorem 2.6. First we need to recall some notation and some facts from [15].

If $\mathfrak{B}$ is a stable C^*-algebra and $\mathfrak{A}$ is a C^*-algebra, then we will denote the set of essential extensions $e \colon \mathfrak{B} \hookrightarrow \mathfrak{E} \twoheadrightarrow \mathfrak{A}$ by $\mathcal{E}\mathrm{xt}\,(\mathfrak{A}, \mathfrak{B})$. For each injective *-homomorphism $\varphi \colon \mathfrak{A}_1 \to \mathfrak{A}_2$ and for each (essential) extension $e \colon \mathfrak{B} \hookrightarrow \mathfrak{E} \twoheadrightarrow \mathfrak{A}_2$ in $\mathcal{E}\mathrm{xt}\,(\mathfrak{A}_2, \mathfrak{B})$, there exists a unique extension

$\varphi \cdot e \colon \mathfrak{B} \hookrightarrow \mathfrak{E}' \twoheadrightarrow \mathfrak{A}_1$ in $\mathcal{E}\mathrm{xt}\,(\mathfrak{A}_1, \mathfrak{B})$, where $\mathfrak{E}'$ is a sub-C^*-algebra of $\mathfrak{E}$, making the following diagram commute:

$$\begin{array}{llcccc}
\varphi \cdot e\colon & \mathfrak{B} & \hookrightarrow & \mathfrak{E}' & \twoheadrightarrow & \mathfrak{A}_1 \\
& \Big\| {\scriptstyle \mathrm{id}_{\mathfrak{B}}} & & \big\downarrow & & \big\downarrow {\scriptstyle \varphi} \\
e\colon & \mathfrak{B} & \hookrightarrow & \mathfrak{E} & \twoheadrightarrow & \mathfrak{A}_2.
\end{array}$$

For each isomorphism $\psi \colon \mathfrak{B}_1 \to \mathfrak{B}_2$ and for each $e \colon \mathfrak{B}_1 \hookrightarrow \mathfrak{E} \twoheadrightarrow \mathfrak{A}$ in $\mathcal{E}\mathrm{xt}\,(\mathfrak{A}, \mathfrak{B}_1)$ there exists a unique extension $e \cdot \psi \colon \mathfrak{B}_2 \hookrightarrow \mathfrak{E} \twoheadrightarrow \mathfrak{A}$ in $\mathcal{E}\mathrm{xt}\,(\mathfrak{A}, \mathfrak{B}_2)$ making the diagram commute:

$$\begin{array}{llcccc}
e\colon & \mathfrak{B}_1 & \hookrightarrow & \mathfrak{E} & \twoheadrightarrow & \mathfrak{A} \\
& \big\downarrow {\scriptstyle \psi} & & \Big\| {\scriptstyle \mathrm{id}_{\mathfrak{E}}} & & \Big\| {\scriptstyle \mathrm{id}_{\mathfrak{A}}} \\
e \cdot \psi\colon & \mathfrak{B}_2 & \hookrightarrow & \mathfrak{E} & \longrightarrow & \mathfrak{A}.
\end{array}$$

Let $x_{\mathfrak{A},\mathfrak{B}} \colon \mathcal{E}\mathrm{xt}\,(\mathfrak{A}, \mathfrak{B}) \to \mathrm{Ext}(\mathfrak{A}, \mathfrak{B})$ be the natural map. The following propositions are Proposition 1.1 and Proposition 1.2 in [15].

PROPOSITION 6.1. *Suppose* $\varphi \colon \mathfrak{A}_1 \to \mathfrak{A}_2$ *is an injective* **-homomorphism and* $\psi \colon \mathfrak{B}_1 \to \mathfrak{B}_2$ *is a* **-isomorphism. If* $e_1 \in \mathcal{E}\mathrm{xt}\,(\mathfrak{A}, \mathfrak{B}_1)$ *and* $e_2 \in \mathcal{E}\mathrm{xt}\,(\mathfrak{A}_2, \mathfrak{B})$*, then*

$$x_{\mathfrak{A},\mathfrak{B}_2}(e_1 \cdot \psi) = x_{\mathfrak{A},\mathfrak{B}_1}(e_1) \cdot KK(\psi)$$
$$x_{\mathfrak{A}_1,\mathfrak{B}}(\varphi \cdot e_2) = KK(\varphi) \cdot x_{\mathfrak{A}_2,\mathfrak{B}}(e_2).$$

PROPOSITION 6.2. *Let* $e_j \colon \mathfrak{B}_j \hookrightarrow \mathfrak{E}_j \twoheadrightarrow \mathfrak{A}_j \in \mathcal{E}\mathrm{xt}\,(\mathfrak{A}_j, \mathfrak{B}_j)$ *be given, for* $j = 1, 2$.

(i) If $\varphi \colon \mathfrak{A}_1 \to \mathfrak{A}_2$ *and* $\psi \colon \mathfrak{B}_1 \to \mathfrak{B}_2$ *are* **-isomorphisms, then* e_1 *is isomorphic to* $e_1 \cdot \psi$ *and* e_2 *is isomorphic to* $\varphi \cdot e_2$*.*

(ii) e_1 *is isomorphic to* e_2 *if and only if* $e_1 \cdot \psi$ *is strongly isomorphic to* $\varphi \cdot e_2$ *for some* **-isomorphisms* $\varphi \colon \mathfrak{A}_1 \to \mathfrak{A}_2$ *and* $\psi \colon \mathfrak{B}_1 \to \mathfrak{B}_2$*.*

(iii) If e_1 *is isomorphic to* e_2*, then* $\mathfrak{E}_1$ *is isomorphic to* $\mathfrak{E}_2$*, and if* $\mathfrak{A}_j, \mathfrak{B}_j$ *are simple, then* $\mathfrak{E}_1$ *is isomorphic to* $\mathfrak{E}_2$ *implies that* e_1 *is isomorphic to* e_2*.*

Let $\gamma_0 \colon KK(\mathfrak{A}, \mathfrak{B}) \to \mathrm{Hom}(K_0(\mathfrak{A}), K_0(\mathfrak{B}))$ denote the canonical map from the UCT.

PROOF OF THEOREM 2.6. Assume that we have two non-stable, non-unital extensions $e_1 \colon \mathfrak{B}_1 \hookrightarrow \mathfrak{E}_1 \twoheadrightarrow \mathfrak{A}_1$ and $e_2 \colon \mathfrak{B}_2 \hookrightarrow \mathfrak{E}_2 \twoheadrightarrow \mathfrak{A}_2$ in $\mathscr{E}_K$ and an isomorphism $(\alpha_n)_{n \in \mathbb{Z}} \colon K_{\hexagon}(e_1) \to K_{\hexagon}(e_2)$ satisfying $\alpha_2([\mathbb{1}_{\mathfrak{A}_1}]_0) = [\mathbb{1}_{\mathfrak{A}_2}]_0$.

By the proof of Theorem 3.2 in [15], there exist invertible elements $a \in KK(\mathfrak{A}_1, \mathfrak{A}_2)$ and $b \in KK(\mathfrak{B}_1, \mathfrak{B}_2)$ such that $x_{\mathfrak{A}_1,\mathfrak{B}_1}(e_1) \cdot b = a \cdot x_{\mathfrak{A}_2,\mathfrak{B}_2}(e_2)$ in $\mathrm{Ext}(\mathfrak{A}_1, \mathfrak{B}_2)$ and $\gamma_0(a) = \alpha_2$. By Kirchberg-Phillips' classification theorem for Kirchberg algebras (see e.g. [16, Theorem 8.4.1] for a good exposition) there exist *-isomorphisms $\varphi \colon \mathfrak{A}_1 \to \mathfrak{A}_2$

and $\psi\colon \mathfrak{B}_1 \to \mathfrak{B}_2$ such that $KK(\varphi) = a$ and $KK(\psi) = b$. Hence, by Proposition 6.1, $x_{\mathfrak{A}_1,\mathfrak{B}_2}(e_1 \cdot \psi) = x_{\mathfrak{A}_1,\mathfrak{B}_2}(\varphi \cdot e_2)$.

Since φ and ψ are isomorphisms, e_1 is isomorphic to $e_1 \cdot \psi$ and e_2 is isomorphic $\varphi \cdot e_2$ (by Proposition 6.2), i.e. the following diagrams are commutative

$$\begin{array}{lccccc}
e_1\colon & \mathfrak{B}_1 & \hookrightarrow & \mathfrak{E}_1 & \twoheadrightarrow & \mathfrak{A}_1 \\
 & \downarrow{\scriptstyle \psi} & & \Vert{\scriptstyle \mathrm{id}_{\mathfrak{E}_1}} & & \Vert{\scriptstyle \mathrm{id}_{\mathfrak{A}_1}} \\
e_1 \cdot \psi\colon & \mathfrak{B}_2 & \hookrightarrow & \mathfrak{E}_1 & \twoheadrightarrow & \mathfrak{A}_1
\end{array}$$

$$\begin{array}{lccccc}
\varphi \cdot e_2\colon & \mathfrak{B}_2 & \hookrightarrow & \mathfrak{E}_2 & \twoheadrightarrow & \mathfrak{A}_1 \\
 & \Vert{\scriptstyle \mathrm{id}_{\mathfrak{B}_2}} & & \Vert{\scriptstyle \mathrm{id}_{\mathfrak{E}_2}} & & \downarrow{\scriptstyle \varphi} \\
e\colon & \mathfrak{B}_2 & \hookrightarrow & \mathfrak{E}_2 & \twoheadrightarrow & \mathfrak{A}_2
\end{array}$$

and the vertical maps are isomorphisms. Note that $e_1 \cdot \psi$ and $\varphi \cdot e_2$ are non-unital essential extensions of $\mathfrak{A}_1$ by $\mathfrak{B}_2$. Since $\mathfrak{B}_2$ is a stable purely infinite simple C^*-algebra, by [7, Theorem 17], $e_1 \cdot \psi$ and $\varphi \cdot e_2$ are purely large. Let τ_1 and τ_2 be the Busby invariant associated to $e_1 \cdot \psi$ and $\varphi \cdot e_2$, resp. Then $[\tau_1] = [\tau_2]$ in $\mathrm{Ext}(\mathfrak{A}_1, \mathfrak{B}_2)$. Since $e_1 \cdot \psi$ and $\varphi \cdot e_2$ are non-unital, purely large, essential extensions and $[\tau_1] = [\tau_2]$ in $\mathrm{Ext}(\mathfrak{A}_1, \mathfrak{B}_2)$, by [7, Corollary 16] there exists a unitary $u \in \mathcal{M}(\mathfrak{B}_2)$ such that

$$\mathrm{Ad}(\pi(u)) \circ \tau_1 = \tau_2. \tag{1}$$

Let $\mathfrak{E}_1' = \pi^{-1}(\tau_1(\mathfrak{A}_1))$ and $\mathfrak{E}_2' = \pi^{-1}(\tau_2(\mathfrak{A}_1))$, where $\pi\colon \mathcal{M}(\mathfrak{B}_2) \to \mathcal{M}(\mathfrak{B}_2)/\mathfrak{B}_2$ is the quotient map. Then $\mathfrak{E}_1 \cong \mathfrak{E}_1'$ and $\mathfrak{E}_2 \cong \mathfrak{E}_2'$. By Equation (1), $uxu^* \in \mathfrak{E}_2'$ for all $x \in \mathfrak{E}_1'$ and $u^*yu \in \mathfrak{E}_1'$ for all $y \in \mathfrak{E}_2'$. Define $\eta\colon \mathfrak{E}_1' \to \mathfrak{E}_2'$ by $\eta(x) = uxu^*$. Then η is an *-isomorphism with $\eta^{-1}(y) = u^*yu$ for $y \in \mathfrak{E}_2'$. So $\mathfrak{E}_1'$ is isomorphic to $\mathfrak{E}_2'$, and hence, e_1 is isomorphic to e_2.

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