Classification of Cuntz-Krieger algebras up to stable isomorphism

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Abstract

In this paper we classify all Cuntz-Krieger algebras whose adjacency matrices satisfy condition (II) of Cuntz. The invariant arises naturally from the ideal lattice and the six-term exact sequences from K-theory, while the proof of this invariant being complete depends on recent results on flow equivalence of shifts of finite type by Mike Boyle and Danrun Huang.

Shortly after Franks had made a successful classification of irreducible shifts of finite type up to flow equivalence ([7]), Cuntz raised the question of whether this invariant or the K_0-group alone classifies simple Cuntz-Krieger algebras up to stable isomorphism. He sketched in [5] that it was enough to answer whether $\mathcal{O}_2$ and $\mathcal{O}_{2_-}$ are isomorphic, where $\mathcal{O}_2$ resp. $\mathcal{O}_{2_-}$ are the Cuntz-Krieger algebras associated to the matrices

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{resp.} \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

This question remained open until Rørdam in [16] showed that $\mathcal{O}_2$ and $\mathcal{O}_{2_-}$ are isomorphic and elaborated the argument of Cuntz to show that the K_0-group is a complete invariant of simple Cuntz-Krieger algebras up to stable isomorphism. Nowadays we also know this from the work of Kirchberg and Phillips (the references are [11], Thm. C, and [14]; see also [18], Thm. 8.4.1).

Danrun Huang found a complete stable isomorphism invariant of Cuntz-Krieger algebras with exactly one non-trivial ideal ([9]) and a complete invariant of all Cuntz-Krieger algebras with vanishing K_1-group ([8]). Later, Rørdam showed that the six-term exact sequence from K-theory is a complete stable isomorphism invariant of Cuntz-Krieger algebras with exactly one non-trivial ideal ([17]). The result is actually more general than this, and the invariant fits well with those usually considered in the classification theory of C^*-algebras.

Since Huang's and Rørdam's works in the mid-nineties it has been an open problem to classify general (non-simple) Cuntz-Krieger algebras up to stable isomorphism. In this article we classify all Cuntz-Krieger algebras (whose adjacency matrices satisfy condition (II) of Cuntz) up to stable isomorphism using recent results of Boyle and Huang concerning classification of general (reducible) shifts of finite type up to flow equivalence.

In the first three sections we will repeat some definitions and results about shifts of finite type and Cuntz-Krieger algebras – this material is mainly from [6] and [4]. In Section 4 the main result will be phrased. In Section 5 and 7 are definitions and results originating from two new articles of Boyle and Huang, [2] and [3]. In Section 6 we show an isomorphism theorem which is the key ingredient in the proof in Section 8 of the main theorem. In Section 9 is given a list of open questions naturally raised by this article. If one is only interested in the results and not the proofs, one can skip Section 5–8.

Parts of this paper have appeared in the author's Master's thesis, [15].

1 Preliminaries

Let A be a $n \times n$ matrix with entries in $\mathbb{Z}_+$, the non-negative integers, and let $A(i,j)$ denote the (i,j)'th entry of A. Then A is called *irreducible* iff (if and only if) for every $i, j \in \{1, \dots, n\}$ there exists $k \in \mathbb{N} = \{1, 2, 3, \dots\}$ such that $A^k(i,j) \neq 0$. We say that A is *non-degenerate* iff all its rows and columns are non-zero. We say that A is *essentially irreducible* iff the maximal non-degenerate principal submatrix of A is irreducible.

For every non-degenerate matrix $A \in \mathrm{Mat}_n(\{0,1\})$ we define the *Cuntz-Krieger algebra*, $\mathcal{O}_A$, associated to A to be the universal C^*-algebra generated by n partial isometries $s_1, \dots, s_n$ satisfying the relations

$$\mathbb{1} = s_1 s_1^* + \cdots + s_n s_n^*$$

$$s_i^* s_i = \sum_{j=1}^{n} A(i,j) s_j s_j^* \quad \text{for all } i = 1, \dots, n.$$

Let $A \in \mathrm{Mat}_n(\{0,1\})$ be a non-degenerate matrix and let $\Sigma = \{1, \dots, n\}$. Let $\Sigma^{\mathbb{Z}}$ be the set of bi-infinite sequences of elements in Σ equipped with the product topology induced of the discrete topology on Σ. Let

$$X_A := \{(x_i)_{i\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}} \mid A(x_i, x_{i+1}) = 1 \text{ for all } i \in \mathbb{Z}\}$$

and let $\sigma_A \colon X_A \to X_A$ be the shift map; i.e. $\sigma((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}$. Then X_A is a compact space and σ_A is a homeomorphism; the topological dynamical system (X_A, σ_A) is called the *shift of finite type* (abbreviated SFT) associated to A. It can be useful to think of X_A as the set of bi-infinite paths in the directed graph G_A with adjacency matrix A.

Two SFTs X_A and $X_{A'}$ are called *topologically conjugate* iff there exists a homeomorphism $\phi \colon X_A \to X_{A'}$, such that $\phi \circ \sigma_A = \sigma_{A'} \circ \phi$ (this means exactly that the two topological dynamical systems are isomorphic).

Let $A \in \mathrm{Mat}_n(\{0,1\})$ be a non-degenerate matrix and let $\Sigma = \{1, \dots, n\}$. For $i, j \in \Sigma$ write $i \succeq j$ iff the transition from i to j is possible, i.e. iff there is a path from vertex i to vertex j in the graph G_A. We call i and j equivalent iff $i \succeq j \wedge j \succeq i$ and write Γ_A for the poset (partially ordered set) of equivalence classes of elements $i \in \Sigma$ for which $i \succeq i$ – the partial order $\succeq$ lifts to a partial order on Γ_A, which we also will denote $\succeq$. For each subset γ of Σ denote by A_γ the restriction $(A(i,j))_{i,j\in\gamma}$ of A to γ. By definition of Γ_A the matrices A_γ, $\gamma \in \Gamma_A$ are irreducible. We say that the matrix A satisfies *condition (I)* iff for every minimal $\gamma \in \Gamma_A$, the matrix A_γ is not a permutation matrix. We say that the matrix A satisfies *condition (II)* iff for every $\gamma \in \Gamma_A$, the matrix A_γ is not a permutation matrix – clearly (II) implies (I).

Let (X, f) be a topological dynamical system; i.e. X is a compact topological Hausdorff space and $f \colon X \to X$ a homeomorphism. Let $\sim$ be the equivalence relation on $X \times \mathbb{R}$ generated by identifying $(x, s+1)$ and $(f(x), s)$ for all $x \in X$ and $s \in \mathbb{R}$. The *standard suspension space* of (X, f) is defined as the identification space

$$Y = X \times \mathbb{R} / \sim .$$

We define the *suspension flow* over (X, f) as $\Phi_t([(x,s)]) = [(x, s+t)]$ for all $x \in X$ and $s, t \in \mathbb{R}$.

Two SFTs are *flow equivalent* (abbreviated FE) iff there exists a homeomorphism between their standard suspension spaces carrying flow lines onto flow lines and preserving orientation. Also, we say that two matrices A and A' are FE iff the corresponding SFTs are FE – and in this case, we write $A \sim_{FE} A'$. Parry and Sullivan have made a useful matrix characterization of FE of SFTs ([13]).

In this article we will focus on Cuntz-Krieger algebras associated to matrices satisfying (II). Cuntz and Krieger have shown in [6] that if A satisfies (I), then the condition, that $\mathcal{O}_A$ is universal, is superfluous: all C^*-algebras generated by non-zero partial isometries satisfying the relations above are canonically isomorphic. Cuntz has shown in [4] that if we assume that all the matrices satisfy (II), then the stabilized Cuntz-Krieger algebras are an invariant of SFTs up to FE.

In the mid-nineties, Rørdam showed in [16] that for simple Cuntz-Krieger algebras, $\mathcal{O}_A$ and $\mathcal{O}_{A'}$, $\mathcal{O}_A \otimes \mathbb{K} \cong \mathcal{O}_{A'} \otimes \mathbb{K}$ if and only if $K_0(\mathcal{O}_A) \cong K_0(\mathcal{O}_{A'})$, thus solving a 12 years old open problem.

2 Ideal structure

Cuntz gives in [4] a very satisfactory description of the ideal structure of Cuntz-Krieger algebras. For a C^*-algebra $\mathfrak{A}$ let $\mathfrak{Ideal}(\mathfrak{A})$ denote the lattice of closed (two-sided) ideals in $\mathfrak{A}$.

To be able to describe the ideal structure of $\mathcal{O}_A$ we need a little more terminology. We assume throughout this section that A is a $\{0,1\}$-matrix satisfying (II).

Definition 2.1 We call a subset H of Γ_A *hereditary*, iff for all $\gamma_1, \gamma_2 \in \Gamma_A$

$$\gamma_1 \in H \wedge \gamma_1 \succeq \gamma_2 \Rightarrow \gamma_2 \in H.$$

Definition 2.2 For each $\gamma_0 \in \Gamma_A$ we let

$$\begin{aligned} H(\gamma_0) &= \{\gamma \in \Gamma_A \mid \gamma_0 \succeq \gamma\}, \\ H_-(\gamma_0) &= \{\gamma \in \Gamma_A \mid \gamma_0 \succeq \gamma \wedge \gamma_0 \neq \gamma\} = H(\gamma_0) \setminus \{\gamma_0\}. \end{aligned}$$

So $H(\gamma_0)$ is the hereditary subset of Γ_A generated by γ_0.

Definition 2.3 For every subset $N \subseteq \Gamma_A$ we define

$$\Sigma(N) = \{i \in \Sigma \mid \exists i_1, i_2 \in \bigcup_{\gamma \in N} \gamma : i_1 \succeq i \succeq i_2\}.$$

Definition 2.4 We call a subset $\gamma \subseteq \Sigma$ *saturated* (with resp. to A), iff

$$\{i \in \Sigma \mid A(i,j) = 1 \Rightarrow j \in \gamma\} \subseteq \gamma.$$

Remark that $\emptyset$ and Σ are saturated, and that every intersection of saturated subsets is saturated. Thus, for every subset $\gamma \subseteq \Sigma$ we can define the *saturation*, $\overline{\gamma}$, of γ (with resp. to A) to be the smallest saturated subset of Σ containing γ; i.e.

$$\overline{\gamma} = \bigcap_{\substack{\gamma \subseteq \gamma' \\ \gamma' \text{ is saturated}}} \gamma'.$$

As pointed out in [4], one easily shows:

Lemma 2.5 *Let H be a hereditary subset of Γ_A, then*

$$\overline{\Sigma(H)} = \{i \in \Sigma \mid \forall j \in \bigcup_{\gamma \in \Gamma_A \setminus H} \gamma : i \not\succeq j\} = \Sigma \setminus \Sigma(\Gamma_A \setminus H).$$

Definition 2.6 For each hereditary subset H of Γ_A, we let $\mathcal{J}_H$ denote the closed (two-sided) ideal in $\mathcal{O}_A$ generated by the set $\{s_i \mid i \in \Sigma(H)\}$.

From Theorem 2.5 in [4] (and its proof) we have:

Theorem 2.7 *The map $H \mapsto \mathcal{J}_H$ is a lattice isomorphism of the set of hereditary subsets of Γ_A onto the set of closed ideals in $\mathcal{O}_A$. Let H be a hereditary subset of Γ_A, then*

(a) $\overline{\Sigma(H)} = \{i \in \Sigma \mid s_i \in \mathcal{J}_H\}$.

(b) The matrices $A_{\Sigma(H)}$ and $A_{\Sigma(\Gamma_A \setminus H)}$ satisfy both (II), and $\mathcal{J}_H \otimes \mathbb{K} \cong \mathcal{O}_{A_{\Sigma(H)}} \otimes \mathbb{K}$ and $\mathcal{O}_A / \mathcal{J}_H \cong \mathcal{O}_{A_{\Sigma(\Gamma_A \setminus H)}}$.

We recall that for every C^*-algebra $\mathfrak{A}$, the map $\mathcal{I} \mapsto \mathcal{I} \otimes \mathbb{K}$ is a lattice isomorphism between the ideal lattices of $\mathfrak{A}$ and $\mathfrak{A} \otimes \mathbb{K}$.

3 K-theory

For every $n \times n$ integer matrix A, we let $\ker A = \{x \in \mathbb{Z}^n \mid Ax = 0\}$ and $\operatorname{cok} A = \mathbb{Z}^n / A\mathbb{Z}^n$ (all vectors are regarded as column-vectors). In [4] Cuntz has computed the K-theory of Cuntz-Krieger algebras as follows:

Theorem 3.1 (Prop. 3.1 in [4]) *Let $A \in \operatorname{Mat}_n(\{0,1\})$ satisfy (I). Then*

$$\begin{aligned} K_0(\mathcal{O}_A) &\cong \operatorname{cok}(I - A^{\mathsf{T}}), \\ K_1(\mathcal{O}_A) &\cong \ker(I - A^{\mathsf{T}}) \ \textit{(on } \mathbb{Z}^n\textit{)}. \end{aligned}$$

More precisely, let $p_i = s_i s_i^*$ be the range projections of s_i and $[p_i]$, $i = 1, \ldots, n$, their classes in $K_0(\mathcal{O}_A)$, let $e_1, \ldots, e_n$ be the canonical basis of $\mathbb{Z}^n$ and $[e_1], \ldots, [e_n]$ its image in $\operatorname{cok}(I - A^{\mathsf{T}})$. Then the map $[p_i] \mapsto [e_i]$ extends to an isomorphism of $K_0(\mathcal{O}_A)$ onto $\operatorname{cok}(I - A^{\mathsf{T}})$.

Furthermore Rørdam has in [16] a concrete description of the isomorphism of $K_1(\mathcal{O}_A)$ onto $\ker(I - A^{\mathsf{T}})$.

Lemma 3.2 *Let B be an integer matrix, which is written as a block-matrix $B = \begin{pmatrix} B_1 & X \\ 0 & B_2 \end{pmatrix}$. Then we have an induced six-term exact sequence*

$$\begin{tikzcd}
\operatorname{cok} B_1 \arrow[r, "\lambda\colon {[x]} \mapsto {\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right]}"] & \operatorname{cok} B \arrow[r, "\mu\colon {\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]} \mapsto {[y]}"] & \operatorname{cok} B_2 \arrow[d, "0 \quad \textit{(the zero map)}"] \\
\ker B_2 \arrow[u, "\Delta\colon y \mapsto {[Xy]}"] & \ker B \arrow[l, "\beta\colon y \mapsfrom \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)"] & \ker B_1 . \arrow[l, "\alpha\colon \left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) \mapsfrom x"]
\end{tikzcd}$$

Proof. It is straightforward – but boring – to show this directly (see the proof of [10], Thm. 2.1, for some of the cases). But this Lemma is actually just a special case of the Snake Lemma used on the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}^m \arrow[r] \arrow[d, "B_1"] & \mathbb{Z}^m \oplus \mathbb{Z}^n \arrow[r] \arrow[d, "B"] & \mathbb{Z}^n \arrow[r] \arrow[d, "B_2"] & 0 \\
0 \arrow[r] & \mathbb{Z}^m \arrow[r] & \mathbb{Z}^m \oplus \mathbb{Z}^n \arrow[r] & \mathbb{Z}^n \arrow[r] & 0.
\end{tikzcd}$$

The only thing we have to show is that the connecting homomorphism is given by $y \mapsto [Xy]$. But this is easily seen from the definition of the connecting homomorphism by diagram-chase: Given $y \in \ker B_2$, then

$$\begin{tikzcd}
 & (0,y) \arrow[r, mapsto, "2"] \arrow[d, mapsto, "3"] & y \arrow[d, mapsto, "1"] \\
Xy \arrow[r, mapsto, "5"] & (Xy,0) \arrow[r, mapsto, "4"] & 0.
\end{tikzcd}$$

□

Proposition 3.3 *Given a matrix $A \in \operatorname{Mat}_n(\{0,1\})$ satisfying (II). Suppose that $H \subsetneq \Gamma_A$ is a non-empty hereditary subset. We can – modulo conjugation by a permutation matrix – write A in the following form*

$$\begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix},$$

where $A_{11} = A_{\overline{\Sigma(H)}}$ *and* $A_{22} = A_{\Sigma(\Gamma_A \setminus H)}$. *Suppose that* A *is written in this form and let* k_1 *and* k_2 *denote the size of* A_{11} *and* A_{22}, *resp. The short-exact sequence*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{J}_H \arrow[r, hook, "\iota"] & \mathcal{O}_A \arrow[r, two heads, "\pi"] & \mathcal{O}_A/\mathcal{J}_H \arrow[r] & 0
\end{tikzcd}$$

induces a cyclic six-term exact sequence from K*-theory, which is isomorphic to the sequence obtained from Lemma 3.2 with the actual block-structure of* A:

$$\begin{tikzcd}
0 \arrow[r] & K_1(\mathcal{J}_H) \arrow[r, "K_1(\iota)"] \arrow[d, "\cong"', "\xi_1"] & K_1(\mathcal{O}_A) \arrow[r, "K_1(\pi)"] \arrow[d, "\cong"', "\xi"] & K_1(\mathcal{O}_A/\mathcal{J}_H) \arrow[r, "\Delta_1"] \arrow[d, "\cong"', "\xi_2"] & K_0(\mathcal{J}_H) \arrow[r, "K_0(\iota)"] \arrow[d, "\cong"', "\delta_1"] & K_0(\mathcal{O}_A) \arrow[r, "K_0(\pi)"] \arrow[d, "\cong"', "\delta"] & K_0(\mathcal{O}_A/\mathcal{J}_H) \arrow[r] \arrow[d, "\cong"', "\delta_2"] & 0 \\
0 \arrow[r] & \ker(I - A_{11}^{\mathsf{T}}) \arrow[r, "\alpha"] & \ker(I - A^{\mathsf{T}}) \arrow[r, "\beta"] & \ker(I - A_{22}^{\mathsf{T}}) \arrow[r, "\Delta"] & \operatorname{cok}(I - A_{11}^{\mathsf{T}}) \arrow[r, "\lambda"] & \operatorname{cok}(I - A^{\mathsf{T}}) \arrow[r, "\mu"] & \operatorname{cok}(I - A_{22}^{\mathsf{T}}) \arrow[r] & 0,
\end{tikzcd}$$

where ξ *and* ξ_2 *is defined in a obvious way from Rørdam's concrete isomorphism in [16],* $\delta_1 \colon [p_i]_0 \mapsto [(\delta_{1,i}, \ldots, \delta_{k_1,i})^{\mathsf{T}}]$, $\delta \colon [p_i]_0 \mapsto [(\delta_{1,i}, \ldots, \delta_{n,i})^{\mathsf{T}}]$ *and* $\delta_2 \colon [\pi(p_i)]_0 \mapsto [(\delta_{k_1+1,i}, \ldots, \delta_{n,i})^{\mathsf{T}}]$.

Proof. From Theorem 2.7 and 3.1 we clearly have the isomorphisms $\xi, \xi_2, \delta, \delta_2$. These theorems also give the isomorphism from $K_0(\mathcal{J}_H)$ onto $\operatorname{cok}(I - A_{\Sigma(H)}^{\mathsf{T}})$; and an easy argument shows that $\operatorname{cok}(I - A_{\Sigma(H)}^{\mathsf{T}})$ is canonically isomorphic to $\operatorname{cok}(I - A_{\overline{\Sigma(H)}}^{\mathsf{T}})$.

It is not hard to show that the second, fourth and fifth squares in the diagram commute. Thus, we have from K-theory that the first row is an exact sequence. Therefore there exists an isomorphism ξ_1 such that the first square commutes.

Now, it only remains to show the commutativity of the middle square. By use of Rørdam's description of ξ_2 and the definition of the index map Δ_1 (see e.g. [19], Prop. 9.2.4), a long and boring computation shows that $\delta_1 \circ \Delta_1 = \Delta \circ \xi_2$ (see pp. 81–83 in the author's Master's thesis, [15] (in Danish), for the computation). □

4 Main result

We want to study Cuntz-Krieger algebras with a fixed ideal structure.

Let $\mathfrak{A} \neq \{0\}$ be a C^*-algebra with finitely many closed ideals. Let $\mathfrak{I}_1(\mathfrak{A})$ be the set of closed ideals $\mathcal{I} \neq \{0\}$ in $\mathfrak{A}$, such that for all ideals $\mathcal{I}_1, \mathcal{I}_2$ in $\mathfrak{A}$

$$\mathcal{I}_1 + \mathcal{I}_2 = \mathcal{I} \Rightarrow \mathcal{I}_1 = \mathcal{I} \vee \mathcal{I}_2 = \mathcal{I}.$$

It is easy to show that for every ideal $\mathcal{I} \in \mathfrak{I}_1(\mathfrak{A})$ there exists a greatest ideal $\mathcal{I}^- \subsetneq \mathcal{I}$. Let

$$\begin{aligned}
\mathfrak{I}_2(\mathfrak{A}) &= \{\mathcal{I}^- \,|\, \mathcal{I} \in \mathfrak{I}_1(\mathfrak{A})\}, \\
C(\mathfrak{A}) &= \{(\mathcal{I}, \mathcal{J}) \in \mathfrak{I}_1(\mathfrak{A}) \times \mathfrak{I}_2(\mathfrak{A}) \,|\, \mathcal{I} \subsetneq \mathcal{J} \wedge \forall \mathcal{I}' \in \mathfrak{I}_1(\mathfrak{A}) : \mathcal{I} \subseteq \mathcal{I}' \subseteq \mathcal{J} \Rightarrow \mathcal{I} = \mathcal{I}'\}.
\end{aligned}$$

For all closed ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ with $\mathcal{I}_1 \subseteq \mathcal{I}_2$ we have a short-exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_1 \arrow[r, hook, "\iota"] & \mathcal{I}_2 \arrow[r, two heads, "\pi"] & \mathcal{I}_2/\mathcal{I}_1 \arrow[r] & 0.
\end{tikzcd}$$

This sequence induces a cyclic six-term exact sequence from K-theory:

$(\star)$

$$\begin{tikzcd}
K_0(\mathcal{I}_1) \arrow[r] & K_0(\mathcal{I}_2) \arrow[r] & K_0(\mathcal{I}_2/\mathcal{I}_1) \arrow[d] \\
K_1(\mathcal{I}_2/\mathcal{I}_1) \arrow[u] & K_1(\mathcal{I}_2) \arrow[l] & K_1(\mathcal{I}_1). \arrow[l]
\end{tikzcd}$$

Definition 4.1 Let $\mathfrak{I}(\mathfrak{A}) = \mathfrak{I}_1(\mathfrak{A}) \cup \mathfrak{I}_2(\mathfrak{A})$. By the *reduced filtrated K-theory*, $\mathfrak{K}_r^{\mathrm{filt}}(\mathfrak{A})$, we mean the families

$$(K_0(\mathcal{I}))_{\mathcal{I}\in\mathfrak{I}(\mathfrak{A})}, \quad (K_0(\mathcal{I}/\mathcal{I}^-))_{\mathcal{I}\in\mathfrak{I}_1(\mathfrak{A})}, \quad (K_1(\mathcal{I}/\mathcal{I}^-))_{\mathcal{I}\in\mathfrak{I}_1(\mathfrak{A})}$$

together with the sequences

$$K_1(\mathcal{I}/\mathcal{I}^-) \longrightarrow K_0(\mathcal{I}^-) \longrightarrow K_0(\mathcal{I}) \longrightarrow K_0(\mathcal{I}/\mathcal{I}^-) \qquad \text{for } \mathcal{I} \in \mathfrak{I}_1(\mathfrak{A}),$$

$$K_0(\mathcal{I}) \longrightarrow K_0(\mathcal{J}) \qquad \text{for } (\mathcal{I},\mathcal{J}) \in C(\mathfrak{A})$$

originating from ($\star$).

Let $\mathfrak{B} \neq \{0\}$ be an other C^*-algebra with finitely many closed ideals. By a *reduced filtrated K-theory isomorphism* $(\kappa,\rho)\colon \mathfrak{K}_r^{\mathrm{filt}}(\mathfrak{A}) \to \mathfrak{K}_r^{\mathrm{filt}}(\mathfrak{B})$, we understand isomorphisms

$$\rho\colon \mathfrak{Ideal}(\mathfrak{A}) \to \mathfrak{Ideal}(\mathfrak{B}) \quad \text{(lattice isomorphism)},$$

$$\kappa_{\mathcal{I}}^{0,i}\colon K_0(\mathcal{I}) \to K_0(\rho(\mathcal{I})), \quad \mathcal{I} \in \mathfrak{I}(\mathfrak{A}),$$

$$\kappa_{\mathcal{I}}^{0,q}\colon K_0(\mathcal{I}/\mathcal{I}^-) \to K_0(\rho(\mathcal{I})/\rho(\mathcal{I}^-)), \quad \mathcal{I} \in \mathfrak{I}_1(\mathfrak{A}),$$

$$\kappa_{\mathcal{I}}^{1,q}\colon K_1(\mathcal{I}/\mathcal{I}^-) \to K_1(\rho(\mathcal{I})/\rho(\mathcal{I}^-)), \quad \mathcal{I} \in \mathfrak{I}_1(\mathfrak{A})$$

such that all the diagrams coming from sequences in $\mathfrak{K}_r^{\mathrm{filt}}(\mathfrak{A})$ and $\mathfrak{K}_r^{\mathrm{filt}}(\mathfrak{B})$ commute; i.e.

$$\begin{tikzcd}
K_1(\mathcal{I}/\mathcal{I}^-) \arrow[r] \arrow[d, "\kappa_{\mathcal{I}}^{1,q}"] & K_0(\mathcal{I}^-) \arrow[r] \arrow[d, "\kappa_{\mathcal{I}^-}^{0,i}"] & K_0(\mathcal{I}) \arrow[r] \arrow[d, "\kappa_{\mathcal{I}}^{0,i}"] & K_0(\mathcal{I}/\mathcal{I}^-) \arrow[d, "\kappa_{\mathcal{I}}^{0,q}"] \\
K_1(\rho(\mathcal{I})/\rho(\mathcal{I}^-)) \arrow[r] & K_0(\rho(\mathcal{I}^-)) \arrow[r] & K_0(\rho(\mathcal{I})) \arrow[r] & K_0(\rho(\mathcal{I})/\rho(\mathcal{I}^-))
\end{tikzcd}$$

commutes for all $\mathcal{I} \in \mathfrak{I}_1(\mathfrak{A})$ and

$$\begin{tikzcd}
K_0(\mathcal{I}) \arrow[r] \arrow[d, "\kappa_{\mathcal{I}}^{0,i}"] & K_0(\mathcal{J}) \arrow[d, "\kappa_{\mathcal{J}}^{0,i}"] \\
K_0(\rho(\mathcal{I})) \arrow[r] & K_0(\rho(\mathcal{J}))
\end{tikzcd}$$

commutes for all $(\mathcal{I},\mathcal{J}) \in C(\mathfrak{A})$.

We may now state our main result. Proof in Section 8.

Theorem 4.2 (Main theorem) *Let A and A' be $\{0,1\}$-matrices satisfying (II). Then the following are equivalent*

(1) $\mathcal{O}_A \otimes \mathbb{K} \cong \mathcal{O}_{A'} \otimes \mathbb{K}$

(2) there exists a reduced filtrated K-theory isomorphism from $\mathfrak{K}_r^{\mathrm{filt}}(\mathcal{O}_A)$ to $\mathfrak{K}_r^{\mathrm{filt}}(\mathcal{O}_{A'})$.

Remark 4.3 From the definition of the reduced filtrated K-theory, it might seem very complicated to compute the reduced filtrated K-theory – and in fact this is the case in general. Furtunately, for Cuntz-Krieger algebras this invariant is very nice and computable. For Cuntz-Krieger algebras we have a very satisfactory description of the ideal structure (Theorem 2.7), of the K-theory (Theorem 3.1), and of the six-term exact sequences in K-theory (Proposition 3.3); we could in fact use a computer to compare the invariants for given matrices A and A' (satisfying (II)).

Definition 4.4 We define the *full filtrated K-theory*, $\mathfrak{K}^{\mathrm{filt}}(\mathfrak{A})$, of a C^*-algebra $\mathfrak{A} \neq \{0\}$ to be the families

$$(K_0(\mathcal{I}))_{\mathcal{I}\in\mathfrak{Ideal}(\mathfrak{A})}, \quad (K_1(\mathcal{I}))_{\mathcal{I}\in\mathfrak{Ideal}(\mathfrak{A})}, \quad (K_0(\mathcal{I}_2/\mathcal{I}_1))_{\mathcal{I}_1,\mathcal{I}_2\in\mathfrak{Ideal}(\mathfrak{A}),\mathcal{I}_1\subseteq\mathcal{I}_2}, \quad (K_1(\mathcal{I}_2/\mathcal{I}_1))_{\mathcal{I}_1,\mathcal{I}_2\in\mathfrak{Ideal}(\mathfrak{A}),\mathcal{I}_1\subseteq\mathcal{I}_2}$$

together with the six-term exact sequences

$$\begin{tikzcd}
K_0(\mathcal{I}_1) \arrow[r] & K_0(\mathcal{I}_2) \arrow[r] & K_0(\mathcal{I}_2/\mathcal{I}_1) \arrow[d] \\
K_1(\mathcal{I}_2/\mathcal{I}_1) \arrow[u] & K_1(\mathcal{I}_2) \arrow[l] & K_1(\mathcal{I}_1) \arrow[l]
\end{tikzcd}$$

for every $\mathcal{I}_1, \mathcal{I}_2 \in \mathfrak{Ideal}(\mathfrak{A})$ with $\mathcal{I}_1 \subseteq \mathcal{I}_2$.

Let $\mathfrak{B} \neq \{0\}$ be a C^*-algebra. By a *full filtrated K-theory isomorphism* $(\kappa, \rho)\colon \mathfrak{K}^{\mathrm{filt}}(\mathfrak{A}) \to \mathfrak{K}^{\mathrm{filt}}(\mathfrak{B})$, we understand isomorphisms

$$\begin{gathered}
\rho\colon \mathfrak{Ideal}(\mathfrak{A}) \to \mathfrak{Ideal}(\mathfrak{B}) \quad \text{(lattice isomorphism)}, \\
\kappa_{\mathcal{I}}^{0,i}\colon K_0(\mathcal{I}) \to K_0(\rho(\mathcal{I})), \quad \mathcal{I} \in \mathfrak{Ideal}(\mathfrak{A}), \\
\kappa_{\mathcal{I}}^{1,i}\colon K_1(\mathcal{I}) \to K_1(\rho(\mathcal{I})), \quad \mathcal{I} \in \mathfrak{Ideal}(\mathfrak{A}), \\
\kappa_{\mathcal{I}_1,\mathcal{I}_2}^{0,q}\colon K_0(\mathcal{I}_2/\mathcal{I}_1) \to K_0(\rho(\mathcal{I}_2)/\rho(\mathcal{I}_1)), \quad \mathcal{I}_1,\mathcal{I}_2 \in \mathfrak{Ideal}(\mathfrak{A}), \mathcal{I}_1 \subseteq \mathcal{I}_2, \\
\kappa_{\mathcal{I}_1,\mathcal{I}_2}^{1,q}\colon K_1(\mathcal{I}_2/\mathcal{I}_1) \to K_1(\rho(\mathcal{I}_2)/\rho(\mathcal{I}_1)), \quad \mathcal{I}_1,\mathcal{I}_2 \in \mathfrak{Ideal}(\mathfrak{A}), \mathcal{I}_1 \subseteq \mathcal{I}_2
\end{gathered}$$

such that all the diagrams coming from the six-term exact sequences commute.

5 $GL_{\mathcal{P}}$- and $SL_{\mathcal{P}}$-equivalence

To be able to prove the main theorem in the previous section, we use results of Danrun Huang and Mike Boyle – especially the articles [2] and [3]. This section contains a (alternative, but equivalent) definition of the objects $\mathfrak{Mat}_{\mathcal{P}}(\mathbb{Z})$, $GL_{\mathcal{P}}(\mathbb{Z})$ and $SL_{\mathcal{P}}(\mathbb{Z})$; and of the relations $GL_{\mathcal{P}}$-equivalence and $SL_{\mathcal{P}}$-equivalence (all from these two articles). Remark that here we denote the order on $\mathcal{P}$ opposite of that in [2] and [3].

Definition 5.1 Let $N \in \mathbb{N}$ be given. Let $\mathbf{n} = (n_1, \dots, n_N) \in \mathbb{N}^N$ and set $k = n_1 + \cdots + n_N$. Then every matrix $B \in \mathrm{Mat}_k(\mathbb{Z})$ can be written in the following block-form

$$B = \begin{pmatrix} B_{11} & \dots & B_{1N} \\ \vdots & \ddots & \vdots \\ B_{N1} & \dots & B_{NN} \end{pmatrix},$$

where $B_{ij} \in \mathrm{Mat}_{n_i \times n_j}(\mathbb{Z})$ for all $i, j = 1, \dots, N$. We let $\mathfrak{Mat}(\mathbf{n}, \mathbb{Z})$ denote the set $\mathrm{Mat}_k(\mathbb{Z})$, where we also remember this block-structure. Given a matrix $B \in \mathfrak{Mat}(\mathbf{n}, \mathbb{Z})$, we let B_{ij}, $i, j = 1, \dots, N$, denote these block-matrices. If we let $B(l, m)$ denote the *number* in the l^{th} row and m^{th} column of B, then this should not lead to any confusion.

It is an important observation that addition and multiplication of the elements in $\mathfrak{Mat}(\mathbf{n}, \mathbb{Z})$ correspond to do the corresponding block-wise operations.

Assumption 5.2 In what follows, we will let $\succeq$ be an (partial) order on $\mathcal{P} := \{1, \dots, N\}$, where $N \in \mathbb{N}$, which satisfies

$$i \succeq j \Rightarrow i \leq j$$

for all $i, j \in \mathcal{P}$, where $\leq$ denotes the usual order on $\mathbb{Z}$.

Definition 5.3 For each $\mathbf{n} \in \mathbb{N}^N$, we let $\mathfrak{Mat}_{\mathcal{P}}(\mathbf{n}, \mathbb{Z})$ denote the set of matrices $B \in \mathfrak{Mat}(\mathbf{n}, \mathbb{Z})$, satisfying

$$\forall i, j \in \mathcal{P}(B_{ij} \neq 0 \Rightarrow i \succeq j).$$

So if $B \in \mathfrak{Mat}_{\mathcal{P}}(\mathbf{n}, \mathbb{Z})$ then $B_{ij} = 0$ for all $i > j$ – in other words: B is a block-upper triangular-matrix.

We define the subsets $GL_{\mathcal{P}}(\mathbf{n}, \mathbb{Z})$ and $SL_{\mathcal{P}}(\mathbf{n}, \mathbb{Z})$ of $\mathfrak{Mat}_{\mathcal{P}}(\mathbf{n}, \mathbb{Z})$ by

$$B \in GL_{\mathcal{P}}(\mathbf{n}, \mathbb{Z}) \Leftrightarrow \det B_{11}, \dots, \det B_{NN} \in \{-1, 1\},$$

$$B \in SL_{\mathcal{P}}(\mathbf{n}, \mathbb{Z}) \Leftrightarrow \det B_{11} = \cdots = \det B_{NN} = 1$$

for $B \in \mathfrak{Mat}_{\mathcal{P}}(\mathbf{n}, \mathbb{Z})$.

Definition 5.4 Let $\mathbf{n} = (n_1, \dots, n_N), \mathbf{r} = (r_1, \dots, r_N) \in \mathbb{N}^N$. We write $\mathbf{n} \leq \mathbf{r}$, iff $n_i \leq r_i$ for all $i = 1, \dots, N$. For each $\mathbf{n} \leq \mathbf{r}$, we define an embedding $\iota_{\mathbf{r},\mathbf{n}} \colon \mathfrak{Mat}_{\mathcal{P}}(\mathbf{n}, \mathbb{Z}) \to \mathfrak{Mat}_{\mathcal{P}}(\mathbf{r}, \mathbb{Z})$ as follows. Let $B \in \mathfrak{Mat}_{\mathcal{P}}(\mathbf{n}, \mathbb{Z})$. Then the ij^{th} block in $\iota_{\mathbf{r},\mathbf{n}}(B)$ has B_{ij} as its upper left corner. Outside this corner, the ij^{th} block in $\iota_{\mathbf{r},\mathbf{n}}(B)$ equals zero if $i \neq j$, and equals the corresponding entries of the identity-matrix if $i = j$.

Remark 5.5 It is clear that $\leq$ is an upwards directed order on $\mathbb{N}^N$. Because $\succeq$ is transitive, $\mathfrak{Mat}_{\mathcal{P}}(\mathbf{n}, \mathbb{Z})$ is a monoid – under matrix-multiplication – for each $\mathbf{n} \in \mathbb{N}^N$ (i.e. a semigroup with an identity).

Given $U \in GL_{\mathcal{P}}(\mathbf{n}, \mathbb{Z})$ and $U' \in \mathfrak{Mat}(\mathbf{n}, \mathbb{Z})$ such that $UU' = U'U = I$. By elementary matrix theory U' is known to be a block-upper triangular-matrix. Suppose $U'_{ij} \neq 0$ for some $i < j$. The matrix U_{ii} is invertible thus $U_{ii}U'_{ij} \neq 0$. By matrix multiplication it is seen that

$$(UU')_{ij} = \sum_{k=i}^{j} U_{ik}U'_{kj} = 0,$$

thus there exists a $k \in \mathcal{P}$ such that $i < k \leq j$, $i \succeq k$, and $U'_{kj} \neq 0$. By induction is therefore $i \succeq j$. So $U^{-1} \in GL_{\mathcal{P}}(\mathbf{n}, \mathbb{Z})$ for all $U \in GL_{\mathcal{P}}(\mathbf{n}, \mathbb{Z})$. Thus we have seen that $GL_{\mathcal{P}}(\mathbf{n}, \mathbb{Z})$ and $SL_{\mathcal{P}}(\mathbf{n}, \mathbb{Z})$ are groups for every $\mathbf{n} \in \mathbb{N}^N$.

It is easy to see that $\iota_{\mathbf{r},\mathbf{n}}$ is an injective homomorphism, that $\iota_{\mathbf{r},\mathbf{n}}\left(GL_{\mathcal{P}}(\mathbf{n}, \mathbb{Z})\right) \subseteq GL_{\mathcal{P}}(\mathbf{r}, \mathbb{Z})$, and that $\iota_{\mathbf{r},\mathbf{n}}\left(SL_{\mathcal{P}}(\mathbf{n}, \mathbb{Z})\right) \subseteq SL_{\mathcal{P}}(\mathbf{r}, \mathbb{Z})$ for all $\mathbf{n}, \mathbf{r} \in \mathbb{N}^N$ with $\mathbf{n} \leq \mathbf{r}$. It is clear that $\iota_{\mathbf{r},\mathbf{m}} = \iota_{\mathbf{r},\mathbf{n}} \circ \iota_{\mathbf{n},\mathbf{m}}$ for all $\mathbf{m}, \mathbf{n}, \mathbf{r} \in \mathbb{N}^N$ with $\mathbf{m} \leq \mathbf{n} \leq \mathbf{r}$.

Definition 5.6 From preceding remark it follows that $(\mathfrak{Mat}_{\mathcal{P}}(\mathbf{n}, \mathbb{Z}), \iota_{\mathbf{r},\mathbf{n}})$, $\mathbf{n}, \mathbf{r} \in \mathbb{N}^N$, $\mathbf{n} \leq \mathbf{r}$ is an inductive system in the category of monoids while $(GL_{\mathcal{P}}(\mathbf{n}, \mathbb{Z}), \iota_{\mathbf{r},\mathbf{n}})$, $\mathbf{n}, \mathbf{r} \in \mathbb{N}^N$, $\mathbf{n} \leq \mathbf{r}$ and $(SL_{\mathcal{P}}(\mathbf{n}, \mathbb{Z}), \iota_{\mathbf{r},\mathbf{n}})$, $\mathbf{n}, \mathbf{r} \in \mathbb{N}^N$, $\mathbf{n} \leq \mathbf{r}$ are inductive systems in the category of groups. We let $\mathfrak{Mat}_{\mathcal{P}}(\mathbb{Z})$, $GL_{\mathcal{P}}(\mathbb{Z})$ resp. $SL_{\mathcal{P}}(\mathbb{Z})$ denote the inductive limits of these three systems.

With a slight abuse of notation, we let $(\cdot)_\infty$ denote all the embeddings. It is a very important observation that we can get every element in $\mathfrak{Mat}_{\mathcal{P}}(\mathbb{Z})$ (resp. $GL_{\mathcal{P}}(\mathbb{Z})$ and $SL_{\mathcal{P}}(\mathbb{Z})$) as B_∞, where $B \in \mathfrak{Mat}_{\mathcal{P}}(\mathbf{n}, \mathbb{Z})$ (resp. $GL_{\mathcal{P}}(\mathbf{n}, \mathbb{Z})$ and $SL_{\mathcal{P}}(\mathbf{n}, \mathbb{Z})$) for an $\mathbf{n} \in \mathbb{N}^N$. Also, we can canonically consider $SL_{\mathcal{P}}(\mathbb{Z})$ as a subgroup of $GL_{\mathcal{P}}(\mathbb{Z})$, and we can also canonically consider $GL_{\mathcal{P}}(\mathbb{Z})$ as a sub-monoid of $\mathfrak{Mat}_{\mathcal{P}}(\mathbb{Z})$.

Remark 5.7 One could also get quite a nice description of these inductive limits by having the matrices being represented as $N \times N$-block-matrices – where each block is a $\aleph_0 \times \aleph_0$-matrix (a countably infinite matrix). This is done in [2]. The description offered here is particularly comfortable to anyone familiar with K-theory.

Definition 5.8 Given $\underline{B} \in \mathfrak{Mat}_{\mathcal{P}}(\mathbb{Z})$ and $\underline{B}' \in \mathfrak{Mat}_{\mathcal{P}}(\mathbb{Z})$. Then we say that $\underline{B}$ and $\underline{B}'$ are *$GL_{\mathcal{P}}$-equivalent* (resp. *$SL_{\mathcal{P}}$-equivalent*) iff there exist $\underline{U}, \underline{V} \in GL_{\mathcal{P}}(\mathbb{Z})$ (resp. $SL_{\mathcal{P}}(\mathbb{Z})$) such that $\underline{U}\,\underline{B}\,\underline{V} = \underline{B}'$. It is easy to verify that these are equivalence relations.

Now let $B \in \mathfrak{Mat}_{\mathcal{P}}(\mathbf{n}, \mathbb{Z})$ and $B' \in \mathfrak{Mat}_{\mathcal{P}}(\mathbf{n}', \mathbb{Z})$. Then it is easy to see that B_∞ and B'_∞ are $GL_{\mathcal{P}}$-equivalent (resp. $SL_{\mathcal{P}}$-equivalent) if and only if there exist $\mathbf{r} \geq \mathbf{n}, \mathbf{n}'$ and $U, V \in GL_{\mathcal{P}}(\mathbf{r}, \mathbb{Z})$ (resp. $SL_{\mathcal{P}}(\mathbf{r}, \mathbb{Z})$) such that $U\iota_{\mathbf{r},\mathbf{n}}(B)V = \iota_{\mathbf{r},\mathbf{n}'}(B')$. We will also say that the matrices B and B' are $GL_{\mathcal{P}}$-equivalent (resp. $SL_{\mathcal{P}}$-equivalent) in the sense of B_∞ and B'_∞ being $GL_{\mathcal{P}}$-equivalent (resp. $SL_{\mathcal{P}}$-equivalent).

6 An isomorphism theorem for $\mathcal{O}_A$

We let $\succeq$ be an order on $\mathcal{P} := \{1,\dots,N\}$ satisfying the conditions of Assumption 5.2.

Remark 6.1 First assume $N=1$. Let $B \in \mathfrak{Mat}_{\mathcal{P}}(n,\mathbb{Z})$ and $B' \in \mathfrak{Mat}_{\mathcal{P}}(n',\mathbb{Z})$. By use of the existence of the Smith normal form, it is easily seen that B is $GL_{\mathcal{P}}$-equivalent to B' iff $\operatorname{cok} B \cong \operatorname{cok} B'$, and that B is $SL_{\mathcal{P}}$-equivalent to B' iff $\operatorname{cok} B \cong \operatorname{cok} B'$ and $\det B = \det B'$.

Now let $N \in \mathbb{N}$ be arbitrary and let furthermore $\mathbf{n},\mathbf{n}' \in \mathbb{N}^N$, $B \in \mathfrak{Mat}_{\mathcal{P}}(\mathbf{n},\mathbb{Z})$ and $B' \in \mathfrak{Mat}_{\mathcal{P}}(\mathbf{n}',\mathbb{Z})$ be given. Then it follows from the above observation that a necessary condition for B being $GL_{\mathcal{P}}$-equivalent (resp. $SL_{\mathcal{P}}$-equivalent) to B' is that $\operatorname{cok} B_{ii} \cong \operatorname{cok} B'_{ii}$, $i \in \mathcal{P}$ (resp. $\det B_{ii} = \det B'_{ii}$ and $\operatorname{cok} B_{ii} \cong \operatorname{cok} B'_{ii}$ for $i \in \mathcal{P}$).

Let us now consider the case where $A \in \mathrm{Mat}_n(\{0,1\})$ can be written in the following block-form

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1N} \\ 0 & A_{22} & A_{23} & \cdots & A_{2N} \\ 0 & 0 & A_{33} & \cdots & A_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{NN} \end{pmatrix},$$

where each of the diagonal-blocks $A_{11},\dots,A_{NN}$ are irreducible non-permutation matrices and $\Gamma_A \cong \mathcal{P}$ via the identity. Then the above is a good motivation to examine, whether $SL_{\mathcal{P}}$-equivalence between $(I-A)$ and $(I-A')$ classifies these matrices up to FE. And it is also natural to ask, whether $GL_{\mathcal{P}}$-equivalence between $(I-A)$ and $(I-A')$ classifies the associated Cuntz-Krieger algebras up to stable isomorphism. And actually, this seems to be the case (modulo a permutation of Γ_A). This leads to the following definition.

Definition 6.2 Given $\mathbf{n} \in \mathbb{N}^N$. We let $\mathfrak{Mat}^{(\mathrm{II})}_{\mathcal{P},+}(\mathbf{n},\mathbb{Z})$ denote the set of matrices $A \in \mathfrak{Mat}_{\mathcal{P}}(\mathbf{n},\mathbb{Z})$, which satisfy

- All entries in A are non-negative,
- Each diagonal-block, $A_{ii}, i \in \mathcal{P}$, is essentially irreducible,
- A satisfies the condition (II) (so for each $i \in \mathcal{P}$, the maximal irreducible principal sub-matrix of A_{ii} is not a permutation matrix),
- For all $i,j \in \mathcal{P}$ with $i \succeq j$ there exist an index i', appearing on a cycle of A_{ii}, an index j', appearing on a cycle of A_{jj}, and a $k \in \mathbb{N}$, such that $A^k(i',j') > 0$ (so Γ_A is order-isomorphic to $\mathcal{P}$ via the identity).

Remark that $\mathfrak{Mat}^{(\mathrm{II})}_{\mathcal{P},+}(\mathbf{n},\mathbb{Z})$ is a subset of the set $\mathfrak{M}^{\circ}_{\mathcal{P},+}(\mathbf{n},\mathbb{Z})$ defined in [2], Def. 2.2.

Remark 6.3 The definition of $\mathfrak{Mat}^{(\mathrm{II})}_{\mathcal{P},+}(\mathbf{n},\mathbb{Z})$ is a bit technical. Requiring a matrix to belong to $\mathfrak{Mat}^{(\mathrm{II})}_{\mathcal{P},+}(\mathbf{n},\mathbb{Z})$ seems very restrictive. But actually in dynamic settings – as we will see later – this is nothing but to demand condition (II) (modulo topological conjugation). So the main idea of $\mathfrak{Mat}^{(\mathrm{II})}_{\mathcal{P},+}(\mathbf{n},\mathbb{Z})$ is to write the matrices in a nice way to make the notation much easier.

In what follows, we will restrict our interest to $\{0,1\}$-matrices satisfying condition (II). More specific, we will look at matrices $A \in \mathfrak{Mat}^{(\mathrm{II})}_{\mathcal{P},+}(\mathbf{n},\mathbb{Z})$ and $A' \in \mathfrak{Mat}^{(\mathrm{II})}_{\mathcal{P},+}(\mathbf{n}',\mathbb{Z})$ with entries from $\{0,1\}$. So we have that Γ_A, $\Gamma_{A'}$ and $\mathcal{P}$ are order isomorphic via the identity.

Assuming that the adjacency matrices of the irreducible components are non-permutation matrices, it follows from [2], Thm. 3.1, that

Theorem 6.4 (Boyle) *Given* $A \in \mathfrak{Mat}^{(\mathrm{II})}_{\mathcal{P},+}(\mathbf{n},\mathbb{Z})$ *and* $A' \in \mathfrak{Mat}^{(\mathrm{II})}_{\mathcal{P},+}(\mathbf{n}',\mathbb{Z})$ *with entries from* $\{0,1\}$*. Then we have that* $A \sim_{FE} A'$ *iff there modulo a permutation of* $\Gamma_{A'}$ *exists a* $SL_{\mathcal{P}}$*-equivalence from* $B = I - A$ *to* $B' = I - A'$*. In other words:* $A \sim_{FE} A'$ *iff there exist an order isomorphism* $\rho\colon \Gamma_A \to \Gamma_{A'}$ *and a* $SL_{\mathcal{P}}$*-equivalence from* $(I - A)$ *to* $(I - P_\rho^{-1} A' P_\rho)$*, where* P_ρ *is the permutation matrix induced by* ρ*.*

From this we immediately get

Corollary 6.5 (Boyle) *Given* $A \in \mathfrak{Mat}^{(\mathrm{II})}_{\mathcal{P},+}(\mathbf{n},\mathbb{Z})$ *and* $A' \in \mathfrak{Mat}^{(\mathrm{II})}_{\mathcal{P},+}(\mathbf{n}',\mathbb{Z})$ *with entries from* $\{0,1\}$*. If there exists a* $SL_{\mathcal{P}}$*-equivalence from* $(I - A)$ *to* $(I - A')$*, then* $A \sim_{FE} A'$*.*

Definition 6.6 For every matrix $A \in \mathrm{Mat}_n(\{0,1\})$ we define two matrices $A_- \in \mathrm{Mat}_{n+2}(\{0,1\})$ and $A_\sim \in \mathrm{Mat}_{n+3}(\{0,1\})$ by

$$A_- = \begin{pmatrix} A & \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{pmatrix}, \quad A_\sim = \begin{pmatrix} A & \begin{pmatrix} 0 & 0 & 1 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \end{pmatrix}.$$

Remark that $A_\sim$ is always irreducible, and that A_- is irreducible if A is irreducible.

The proofs of the following two results are very much inspired by [8], Lem. 4.3, and [9], Thm. 2.6, resp. These are again inspired by Rørdams elaboration in the appendix of [16] of an idea pointed out by Cuntz in [5]. These are the key ingredients in proving the main theorem.

Lemma 6.7 *Given* $A \in \mathfrak{Mat}^{(\mathrm{II})}_{\mathcal{P},+}(\mathbf{n},\mathbb{Z})$ *with entries in* $\{0,1\}$*. Write* A *in the block-form*

$$A = \begin{pmatrix} M & X & Y \\ 0 & Q & Z \\ 0 & 0 & N \end{pmatrix},$$

where Q *is the diagonal-block* $A_{i_0 i_0}$*,* $i_0 \in \mathcal{P}$ *– and so it is essentially irreducible. Set*

$$A_* = \begin{pmatrix} M & X_* & Y \\ 0 & Q_* & Z_* \\ 0 & 0 & N \end{pmatrix} \text{ and } A_{**} = \begin{pmatrix} M & X_{**} & Y \\ 0 & Q_{**} & Z_{**} \\ 0 & 0 & N \end{pmatrix},$$

where $Q_* = Q_\sim$*,* $X_* = \begin{pmatrix} X & 0 \end{pmatrix}$*,* $Z_* = \begin{pmatrix} Z \\ 0 \end{pmatrix}$*,* $Q_{**} = (Q_*)_- = (Q_\sim)_-$*,* $X_{**} = \begin{pmatrix} X_* & 0 \end{pmatrix} = \begin{pmatrix} X & 0 \end{pmatrix}$*, and* $Z_{**} = \begin{pmatrix} Z_* \\ 0 \end{pmatrix} = \begin{pmatrix} Z \\ 0 \end{pmatrix}$ *(we add a "suitable number of zeros"). In other words*

$$A_* = \begin{pmatrix} M & X & 0 & Y \\ 0 & Q & \begin{pmatrix} 0 & 0 & 1 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1 \end{pmatrix} & Z \\ 0 & \begin{pmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} & 0 \\ 0 & 0 & 0 & N \end{pmatrix}, \quad A_{**} = \begin{pmatrix} M & X & 0 & Y \\ 0 & Q & \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} & Z \\ 0 & \begin{pmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} & 0 \\ 0 & 0 & 0 & N \end{pmatrix}.$$

Then we have:

(a) $A_* \in \mathfrak{Mat}^{(\mathrm{II})}_{\mathcal{P},+}(\mathbf{n}_*,\mathbb{Z})$ *and* $A_{**} \in \mathfrak{Mat}^{(\mathrm{II})}_{\mathcal{P},+}(\mathbf{n}_{**},\mathbb{Z})$*, where* $\mathbf{n}_* = (n_1, \dots, n_{i_0-1}, n_{i_0} + 3, n_{i_0+1}, \dots, n_N)$ *and* $\mathbf{n}_{**} = (n_1, \dots, n_{i_0-1}, n_{i_0} + 5, n_{i_0+1}, \dots, n_N)$*.*

(b) $\mathcal{O}_{A_*} \cong \mathcal{O}_{A_{**}}$.

(c) $(I - A)$ *is* $SL_{\mathcal{P}}$*-equivalent to* $(I - A_{**})$.

(d) There exist invertible matrices $U_1, V_1 \in \operatorname{Mat}_{n_{i_0}+3}(\mathbb{Z})$ *with* $\det U_1 = 1$ *and* $\det V_1 = -1$*, such that*

$$\begin{pmatrix} I & 0 & 0 \\ 0 & U_1 & 0 \\ 0 & 0 & I \end{pmatrix} (I - A_*) \begin{pmatrix} I & 0 & 0 \\ 0 & V_1 & 0 \\ 0 & 0 & I \end{pmatrix} = \begin{pmatrix} I - M & -X & 0 & -Y \\ 0 & I - Q & 0 & -Z \\ 0 & 0 & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & 0 \\ 0 & 0 & 0 & I - N \end{pmatrix} = \iota_{\mathbf{n}_*,\mathbf{n}}(I - A).$$

So $(I - A)$ *is* $GL_{\mathcal{P}}$*-equivalent to* $(I - A_*)$ *with these special kinds of matrices inducing this equivalence.*

(e) $\mathcal{O}_A \otimes \mathbb{K} \cong \mathcal{O}_{A_*} \otimes \mathbb{K}$.

The corresponding results hold if M *or* N *is "empty".*

Proof. (a): This is easily seen.

(b): This part of the lemma – that $\mathcal{O}_{A_*} \cong \mathcal{O}_{A_{**}}$ – is proven in [8], Lem. 4.3.

(c): Define A_0 such that

$$I - A_0 = \iota_{\mathbf{n}_{**},\mathbf{n}}(I - A) = \begin{pmatrix} I - M & -X & 0 & -Y \\ 0 & I - Q & 0 & -Z \\ 0 & 0 & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} & 0 \\ 0 & 0 & 0 & I - N \end{pmatrix},$$

and let

$$U_0 = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} & 0 \\ 0 & 0 & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \quad \text{and} \quad V_0 = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & \begin{pmatrix} -1 & \cdots & -1 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} & 0 \\ 0 & 0 & 0 & I \end{pmatrix}.$$

Then

$$U_0(I - A_0)V_0 = \begin{pmatrix} I - M & -X & 0 & -Y \\ 0 & I - Q & \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix} & -Z \\ 0 & \begin{pmatrix} -1 & \cdots & -1 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} & 0 \\ 0 & 0 & 0 & I - N \end{pmatrix} = I - A_{**}.$$

It is clear that $\det U_0 = 1$ and $\det V_0 = 1$.

(d): Define $U_1, V_1 \in \operatorname{Mat}_{n_{i_0}+3}(\mathbb{Z})$ by

$$U_1 = \begin{pmatrix} I & \begin{pmatrix} 0 & -1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & -1 & 0 \end{pmatrix} \\ 0 & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix} \text{ and } V_1 = \begin{pmatrix} I & 0 \\ \begin{pmatrix} 1 & \cdots & 1 \\ -1 & \cdots & -1 \\ 0 & \cdots & 0 \end{pmatrix} & \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix} \end{pmatrix}.$$

Then

$$U_1(I-Q_*)V_1 = \begin{pmatrix} I-Q & 0 \\ 0 & \left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}\right)\end{pmatrix}, \quad U_1 Z_* = Z_*, \quad X_* V_1 = X_*.$$

Part (d) is now seen by multiplying the matrices together.

(e): In part (c) we have shown that $(I-A)$ is $SL_{\mathcal{P}}$-equivalent to $(I-A_{**})$. By Corollary 6.5 is therefore $A \sim_{FE} A_{**}$. Because the stabilized Cuntz-Krieger algebra is a FE-invariant of SFTs, we get that $\mathcal{O}_A \otimes \mathbb{K} \cong \mathcal{O}_{A_{**}} \otimes \mathbb{K}$. Now the assertion follows from part (b).

The last assertion is easily seen by going once again through the proof. $\square$

Theorem 6.8 *Given* $A \in \mathfrak{Mat}_{\mathcal{P},+}^{(\mathrm{II})}(\mathbf{n},\mathbb{Z})$ *and* $A' \in \mathfrak{Mat}_{\mathcal{P},+}^{(\mathrm{II})}(\mathbf{n}',\mathbb{Z})$ *with entries from* $\{0,1\}$. *If* $B = I - A$ *is* $GL_{\mathcal{P}}$*-equivalent to* $B' = I - A'$, *then* $\mathcal{O}_A \otimes \mathbb{K} \cong \mathcal{O}_{A'} \otimes \mathbb{K}$.

Proof. Suppose that $B = I - A$ is $GL_{\mathcal{P}}$-equivalent to $B' = I - A'$. Therefore there exist $\mathbf{r} \geq \mathbf{n}, \mathbf{n}'$ and $U, V \in GL_{\mathcal{P}}(\mathbf{r},\mathbb{Z})$, such that $U\iota_{\mathbf{r},\mathbf{n}}(B)V = \iota_{\mathbf{r},\mathbf{n}'}(B')$. Set $B_0 = \iota_{\mathbf{r},\mathbf{n}}(B)$ and $B_0' = \iota_{\mathbf{r},\mathbf{n}'}(B')$. Then $UB_0V = B_0'$. We want to show that we w.l.o.g. can assume that all diagonal-blocks in U and V have determinant 1. From Corollary 6.5 we then get that $A \sim_{FE} A'$. This will be done by induction. Given $i \in \{1, \dots, N\}$ and suppose that $\det(U_{jj}) = \det(V_{jj}) = 1$ for all $j \in \mathbb{N}$ with $j < i$. Then we want to show that there exists a matrix $A'' \in \mathfrak{Mat}_{\mathcal{P},+}^{(\mathrm{II})}(\mathbf{n}'',\mathbb{Z})$ such that $\mathcal{O}_A \otimes \mathbb{K} \cong \mathcal{O}_{A''} \otimes \mathbb{K}$ and such that there is a $GL_{\mathcal{P}}$-equivalence from $(I - A'')$ to $(I - A')$, where the determinant of the first i diagonal-blocks in each of the $GL_{\mathcal{P}}$-matrices inducing the equivalence is 1. We consider diagonal-block i and divide into 3 cases:

Case 1: Suppose $\det U_{ii} = \det V_{ii} = 1$. This is what we want.

Case 2: Suppose $\det U_{ii} = 1$ and $\det V_{ii} = -1$. Set $\mathbf{m} := (n_1, \dots, n_{i-1}, n_i + 3, n_{i+1}, \dots, n_N)$. We know from Lemma 6.7(d) that there exist $U' \in SL_{\mathcal{P}}(\mathbf{m},\mathbb{Z})$ and $V' \in GL_{\mathcal{P}}(\mathbf{m},\mathbb{Z})$, such that $\det V_{ii}' = -1$, $\det V_{jj}' = 1$ for all $j \neq i$ and $U'\iota_{\mathbf{m},\mathbf{n}}(I-A)V' = I - A_*$. Find (by the upward directedness) an $\mathbf{r}_0 \geq \mathbf{r}, \mathbf{m}$. Then

$$\iota_{\mathbf{r}_0,\mathbf{m}}(U')\iota_{\mathbf{r}_0,\mathbf{n}}(I-A)\iota_{\mathbf{r}_0,\mathbf{m}}(V') = \iota_{\mathbf{r}_0,\mathbf{m}}(I-A_*) \text{ and } \iota_{\mathbf{r}_0,\mathbf{r}}(U^{-1})\iota_{\mathbf{r}_0,\mathbf{n}'}(I-A')\iota_{\mathbf{r}_0,\mathbf{r}}(V^{-1}) = \iota_{\mathbf{r}_0,\mathbf{n}}(I-A).$$

So we have

$$\iota_{\mathbf{r}_0,\mathbf{m}}(U')\iota_{\mathbf{r}_0,\mathbf{r}}(U^{-1})\iota_{\mathbf{r}_0,\mathbf{n}'}(I-A')\iota_{\mathbf{r}_0,\mathbf{r}}(V^{-1})\iota_{\mathbf{r}_0,\mathbf{m}}(V') = \iota_{\mathbf{r}_0,\mathbf{m}}(I-A_*)$$

and $\det\left(\iota_{\mathbf{r}_0,\mathbf{m}}(U')\iota_{\mathbf{r}_0,\mathbf{r}}(U^{-1})\right)_{jj} = \det U_{jj}$ for all $j = 1, \dots, N$, $\det\left(\iota_{\mathbf{r}_0,\mathbf{r}}(V^{-1})\iota_{\mathbf{r}_0,\mathbf{m}}(V')\right)_{jj} = \det V_{jj}$ for all $j \neq i$ and $\det\left(\iota_{\mathbf{r}_0,\mathbf{r}}(V^{-1})\iota_{\mathbf{r}_0,\mathbf{m}}(V')\right)_{ii} = 1$. In this manner, we get a $GL_{\mathcal{P}}$-equivalence between $(I - A_*)$ and $(I - A')$, which satisfies that the determinants of the diagonal-blocks in the $GL_{\mathcal{P}}$-matrices inducing the equivalence are the same as of U resp. V – except that the determinant of the i^{th} diagonal-blocks are 1. From Lemma 6.7(e) we have that $\mathcal{O}_A \otimes \mathbb{K} \cong \mathcal{O}_{A_*} \otimes \mathbb{K}$. So we can w.l.o.g. assume that $\det U_{ii} = \det V_{ii} = 1$.

Case 3: Suppose $\det U_{ii} = -1$. Roughly speaking, this case is dealt with by going "far out in U_{ii} and V_{ii}" and interchange two neighbour columns in both U_{ii} and V_{ii}. A bit more precise: Let $\mathbf{r}' := (r_1, \dots, r_{i-1}, r_i + 2, r_{i+1}, \dots, r_N)$, and consider the matrices $\iota_{\mathbf{r}',\mathbf{r}}(U)$, $\iota_{\mathbf{r}',\mathbf{r}}(V)$, $\iota_{\mathbf{r}',\mathbf{r}}(B_0)$ and $\iota_{\mathbf{r}',\mathbf{r}}(B_0')$. Let U_0 (resp. V_0) be equal to $\iota_{\mathbf{r}',\mathbf{r}}(U)$ (resp. $\iota_{\mathbf{r}',\mathbf{r}}(V)$) except from that the $(r_1+\cdots+r_i+1)^{\text{th}}$ and the $(r_1+\cdots+r_i+2)^{\text{th}}$ column are interchanged. Then $U_0\iota_{\mathbf{r}',\mathbf{r}}(B_0)V_0 = \iota_{\mathbf{r}',\mathbf{r}}(B_0')$, $\det(U_0)_{ii} = 1$ and $\det(V_0)_{ii} = -\det V_{ii}$; now we proceed to case 1 or 2, depending on the determinant of $(V_0)_{ii}$.

By induction of i, it is now proven that we w.l.o.g. can assume that $B = I - A$ is $SL_{\mathcal{P}}$-equivalent to $B' = I - A'$. But from Corollary 6.5 we see that $A \sim_{FE} A'$. We know that the stabilized Cuntz-Krieger algebra is a FE-invariant for SFTs, hence $\mathcal{O}_A \otimes \mathbb{K} \cong \mathcal{O}_{A'} \otimes \mathbb{K}$. $\square$

7 K-web

Boyle has shown in [2] that SFTs coming from matrices in e.g. $\mathfrak{Mat}_{\mathcal{P},+}^{(\mathrm{II})}(\mathbf{n},\mathbb{Z})$ can be classified up to FE by $SL_{\mathcal{P}}$-equivalences. Boyle and Huang have in [3] tried to make a more "K-theoretic" complete invariant, but this is not fully satisfactory, because of complications related to signs.

However, dealing with Cuntz-Krieger algebras up to stable isomorphism, these signs are irrelevant – as well as we in the preceding section saw that $SL_{\mathcal{P}}$-equivalence has to be replaced by $GL_{\mathcal{P}}$-equivalence. Fortunately Boyle and Huang have in [3] succeeded in making a "K-theoretical" characterization (the so-called K-web) of $GL_{\mathcal{P}}$-equivalence. Here we denote the order on $\mathcal{P}$ opposite of that in [3] and we use $\breve{H}$ instead of the use of H in [3], to avoid confusion with the H's in Definition 2.2. Otherwise, the rest of this section is just a reformulation of definitions and results from [3].

We let $\succeq$ be an order on $\mathcal{P} := \{1, \dots, N\}$ satisfying the conditions of Assumption 5.2.

Definition 7.1 We say that a subset S of $\mathcal{P}$ is *convex* iff $S \neq \emptyset$ and

$$\forall i, j, k \in \mathcal{P}\left((i, j \in S \wedge i \succeq k \succeq j) \Rightarrow k \in S\right).$$

For each $i \in \mathcal{P}$ we let

$$\begin{aligned}\breve{H}(i) &:= \{j \in \mathcal{P} \mid j \succeq i\},\\ \breve{H}_-(i) &:= \{j \in \mathcal{P} \mid j \succeq i \wedge i \neq j\},\\ \operatorname{Imm}(i) &:= \{j \in \mathcal{P} \mid j \succeq i \wedge i \neq j \wedge \forall k \in \mathcal{P}\,(j \succeq k \succeq i \Rightarrow k = i \vee k = j)\}.\end{aligned}$$

Given $B \in \mathfrak{Mat}_{\mathcal{P}}(\mathbf{n}, \mathbb{Z})$. For a non-empty subset S of $\mathcal{P}$ let B_S denote the principal sub-matrix of B we get by deleting all rows and columns, whose number belongs to $\mathcal{P} \setminus S$. For each convex set $S \subseteq \mathcal{P}$ we let

$$C_S(B) := \operatorname{cok} B_S$$

and for each $i \in \mathcal{P}$ we let

$$K_i(B) := \ker B_{ii}.$$

Remark that these groups are the same for $\iota_{\mathbf{r},\mathbf{n}}(B)$ for all $\mathbf{r} \geq \mathbf{n}$.

For each $i \in \mathcal{P}$ with $\breve{H}_-(i) \neq \emptyset$ we get an exact sequence from Lemma 3.2

$$K_i(B) \to C_{\breve{H}_-(i)}(B) \to C_{\breve{H}(i)}(B) \to C_{\{i\}}(B) \to 0. \tag{1}$$

And for every pair $(i, j) \in \mathcal{P} \times \mathcal{P}$ satisfying $j \in \operatorname{Imm}(i) \wedge \operatorname{Imm}(i) \setminus \{j\} \neq \emptyset$ is $\breve{H}(j) \subsetneq \breve{H}_-(i)$ – and so we have a homomorphism

$$C_{\breve{H}(j)}(B) \to C_{\breve{H}_-(i)}(B) \tag{2}$$

originating from the exact sequence (cf. Lemma 3.2)

$$\ker B_{\breve{H}_-(i)\setminus\breve{H}(j)} \to C_{\breve{H}(j)}(B) \to C_{\breve{H}_-(i)}(B) \to C_{\breve{H}_-(i)\setminus\breve{H}(j)}(B) \to 0.$$

Set

$$J_{\mathcal{P}} := \{\{i\} \mid i \in \mathcal{P}\} \cup \{\breve{H}(i) \mid i \in \mathcal{P}\} \cup \{\breve{H}_-(i) \mid i \in \mathcal{P} \wedge \breve{H}_-(i) \neq \emptyset\}.$$

By the *K-web* of B, $K(B)$, we mean the families $\left(C_S(B)\right)_{S \in J_{\mathcal{P}}}$ and $\left(K_i(B)\right)_{i \in \mathcal{P}}$ together with all the homomorphisms from the sequences (1) and (2). Let now also $B' \in \mathfrak{Mat}_{\mathcal{P}}(\mathbf{n}', \mathbb{Z})$. By a *$K$-web isomorphism*, $\kappa\colon K(B) \to K(B')$, we mean families $(\phi_S\colon C_S(B) \to C_S(B'))_{S \in J_{\mathcal{P}}}$ and $(\psi_i\colon K_i(B) \to K_i(B'))_{i \in \mathcal{P}}$ of isomorphisms, satisfying that the ladders coming from the sequences in $K(B)$ and $K(B')$ commute.

Given $B \in \mathfrak{Mat}_{\mathcal{P}}(\mathbf{n}, \mathbb{Z})$. It is clear that $K(B)$ and $K(\iota_{\mathbf{r},\mathbf{n}}(B))$ are naturally isomorphic for every $\mathbf{r} \geq \mathbf{n}$. Therefore we can define the K-web for elements in $\mathfrak{Mat}_{\mathcal{P}}(\mathbb{Z})$.

Remark 7.2 Remark that K-web isomorphism is stronger than demanding all the exact sequences to be isomorphic (as exact sequences). The different sequences can overlap in such a way that groups in different sequences can originate from the same convex subset of $\mathcal{P}$; for groups in different sequences originating from the same convex subset of $\mathcal{P}$, the isomorphisms have to be the same in all these sequences. If one visualizes the K-web as a diagram (as e.g. in [3], Ex. 3.6), one gets a notion of why it is called the *K-web*.

Proposition 7.3 (Prop. 3.2 from [3]) *Let* $B = \left(\begin{smallmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{smallmatrix}\right)$, $B' = \left(\begin{smallmatrix} B'_{11} & B'_{12} \\ 0 & B'_{22} \end{smallmatrix}\right)$, $U = \left(\begin{smallmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{smallmatrix}\right)$, *and* $V = \left(\begin{smallmatrix} V_{11} & V_{12} \\ 0 & V_{22} \end{smallmatrix}\right)$ *be matrices with matching block-structures such that* $UBV = B'$. *Then we have an induced isomorphism of exact sequences given by*

$$\begin{tikzcd}
\ker B_{22} \arrow[r] \arrow[d, "a"'] & \operatorname{cok} B_{11} \arrow[r] \arrow[d, "b"'] & \operatorname{cok} B \arrow[r] \arrow[d, "c"'] & \operatorname{cok} B_{22} \arrow[r] \arrow[d, "d"'] & 0 \\
\ker B'_{22} \arrow[r] & \operatorname{cok} B'_{11} \arrow[r] & \operatorname{cok} B' \arrow[r] & \operatorname{cok} B'_{22} \arrow[r] & 0,
\end{tikzcd}$$

where $a\colon x \mapsto V_{22}^{-1}x$, $b\colon [x] \mapsto [U_{11}x]$, $c\colon [x] \mapsto [Ux]$ *and* $d\colon [x] \mapsto [U_{22}x]$.

Proof. An easy exercise. ☐

Definition 7.4 Given $B \in \mathfrak{Mat}_{\mathcal{P}}(\mathbf{n}, \mathbb{Z})$ and $B' \in \mathfrak{Mat}_{\mathcal{P}}(\mathbf{n}', \mathbb{Z})$. Suppose that B and B' are $GL_{\mathcal{P}}$-equivalent. Thus, there exist $\mathbf{r} \geq \mathbf{n}, \mathbf{n}'$ and $U, V \in GL_{\mathcal{P}}(\mathbf{r}, \mathbb{Z})$, such that $UB_0V = B_0'$ for $B_0 = \iota_{\mathbf{r},\mathbf{n}}(B)$ and $B_0' = \iota_{\mathbf{r},\mathbf{n}'}(B')$.

It is seen from the preceding proposition that the pair (U, V) induces a K-web isomorphism between $K(B_0) = K(B)$ and $K(B_0') = K(B')$. We denote this *induced* K*-web isomorphism* $\kappa_{(U,V)}$.

Because we restrict ourselves to $\{0, 1\}$-matrices, A, to generate SFTs and Cuntz-Krieger algebras, we can assume that the matrices $(I - A)$ have entries from $\{-1, 0, 1\}$. Thus the greatest common divisor of the elements of every diagonal-block is 1. From [3], Cor. 4.7, we thus have the following

Theorem 7.5 (Boyle and Huang) *Let* $B, B' \in \mathfrak{Mat}_{\mathcal{P}}(\mathbb{Z})$ *be given matrices with* $\{-1, 0, 1\}$*-entries. Then every* K*-web isomorphism from* $K(B)$ *to* $K(B')$ *is induced by a* $GL_{\mathcal{P}}$*-equivalence from* B *to* B'. *The matrices* B *and* B' *are* $GL_{\mathcal{P}}$*-equivalent iff the* K*-webs* $K(B)$ *and* $K(B')$ *are* K*-web isomorphic.*

8 $\mathfrak{K}_r^{\mathsf{filt}}(\mathcal{O}_A)$ is a complete invariant of $\mathcal{O}_A$ up to stable isomorphism

Assumption 8.1 In what follows, we let $\succeq$ be an order on $\mathcal{P} := \{1, \ldots, N\}$, where $N \in \mathbb{N}$, which satisfies

$$i \succeq j \Rightarrow i \leq j$$

for all $i, j \in \mathcal{P}$. Set

$$J = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & ⋰ & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Then $J^2 = I$. Let $\mathcal{P}_J^{\mathsf{T}}$ denote the set $\mathcal{P}$ equipped with the order $\succeq_J^{\mathsf{T}}$ given by $i \succeq_J^{\mathsf{T}} j$, iff $N - j + 1 \succeq N - i + 1$. Clearly $\mathcal{P}_J^{\mathsf{T}}$ also satisfies

$$i \succeq_J^{\mathsf{T}} j \Rightarrow i \leq j$$

for all $i, j \in \mathcal{P}_J^{\mathsf{T}}$. We can think of $\mathcal{P}_J^{\mathsf{T}}$ as $\mathcal{P}$ equipped with the opposite order, and afterwards we have made the permutation $i \mapsto N - i + 1$ of the elements in $\mathcal{P}$ to ensure that the opposite order satisfies the corresponding requirement that the block-matrices have to be block-upper triangular-matrices.

Remark 8.2 Let there be given $B, B' \in \mathfrak{Mat}_{\mathcal{P}}(\mathbf{n}, \mathbb{Z})$. Let $\mathbf{n}_J^{\mathsf{T}} = (n_N, n_{N-1}, \ldots, n_1) \in \mathbb{N}^N$ and remark that $(JBJ)^{\mathsf{T}}, (JB'J)^{\mathsf{T}} \in \mathfrak{Mat}_{\mathcal{P}_J^{\mathsf{T}}}(\mathbf{n}_J^{\mathsf{T}}, \mathbb{Z})$ with the block-structure preserved. Then B and B' are $GL_{\mathcal{P}}$-equivalent if and only if $(JBJ)^{\mathsf{T}}$ and $(JB'J)^{\mathsf{T}}$ are $GL_{\mathcal{P}_J^{\mathsf{T}}}$-equivalent. This, because of $UBV = B'$ if and only if

$$(JVJ)^{\mathsf{T}}(JBJ)^{\mathsf{T}}(JUJ)^{\mathsf{T}} = (JB'J)^{\mathsf{T}}.$$

We now want to prove the main theorem, Theorem 4.2. It is clear that (1) implies (2) in Theorem 4.2. The opposite implication follows from

Proposition 8.3 *Let A and A' be $\{0,1\}$-matrices satisfying (II) and suppose that $\mathfrak{K}_r^{\mathrm{filt}}(\mathcal{O}_A) \cong \mathfrak{K}_r^{\mathrm{filt}}(\mathcal{O}_{A'})$. Then $\mathcal{O}_A \otimes \mathbb{K} \cong \mathcal{O}_{A'} \otimes \mathbb{K}$.*

Proof. By state splitting (see e.g. [12], §2.4), there exist $\{0,1\}$-matrices A_0 and A_0' satisfying (II) such that every transition state has exactly one out-edge and X_A and X_{A_0} (resp. $X_{A'}$ and $X_{A_0'}$) are topologically conjugate. So we can w.l.o.g. assume that every transition state has exactly one out-edge.

Given a reduced filtrated K-theory isomorphism $(\kappa,\rho)\colon \mathfrak{K}_r^{\mathrm{filt}}(\mathcal{O}_A) \to \mathfrak{K}_r^{\mathrm{filt}}(\mathcal{O}_{A'})$. So by Theorem 2.7 this induces a lattice isomorphism $\eta\colon Her(\Gamma_A) \to Her(\Gamma_{A'})$, where $Her(\Gamma_A)$ (resp. $Her(\Gamma_{A'})$) denotes the hereditary subsets of Γ_A (resp. $\Gamma_{A'}$). It is easy to show that $\mathrm{card}(\eta(H(\gamma)) \setminus \eta(H_-(\gamma))) = 1$ and that the map sending γ into the element in $\eta(H(\gamma)) \setminus \eta(H_-(\gamma))$ is an order isomorphism. Let $\delta\colon \Gamma_A \to \Gamma_{A'}$ denote this isomorphism. Then $\rho(\mathcal{J}_{H(\gamma)}) = \mathcal{J}_{\eta(H(\gamma))} = \mathcal{J}_{H(\delta(\gamma))}$ for all $\gamma \in \Gamma_A$.

So Γ_A and $\Gamma_{A'}$ have the same number of elements; let N denote this number. Write $\Gamma_A = \{\gamma_1, \dots, \gamma_N\}$ such that

$$\gamma_i \succeq \gamma_j \Rightarrow j \leq i$$

for $i, j = 1, \dots, N$. And set $\gamma_i' = \delta(\gamma_i)$ for $i = 1, \dots, N$. Find permutation matrices P and P' such that

$$PAP^{-1} = \begin{pmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ A_{21} & A_{22} & 0 & \cdots & 0 \\ A_{31} & A_{32} & A_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & A_{N3} & \cdots & A_{NN} \end{pmatrix}, \quad P'A'P'^{-1} = \begin{pmatrix} A_{11}' & 0 & 0 & \cdots & 0 \\ A_{21}' & A_{22}' & 0 & \cdots & 0 \\ A_{31}' & A_{32}' & A_{33}' & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{N1}' & A_{N2}' & A_{N3}' & \cdots & A_{NN}' \end{pmatrix},$$

where the vertices corresponding to A_{ii} are $\overline{\gamma_i}$, and the vertices corresponding to A_{ii}' are $\overline{\gamma_i'}$ (here we use that every transition state has exactly one out-edge and thus belonging to exactly one of $\overline{\gamma}_1, \dots, \overline{\gamma}_N$). Define $\mathbf{n}, \mathbf{n}' \in \mathbb{N}^N$, so that they reflect these block structures. Let $\mathcal{P} = \{1, \dots, N\}$, where $i \succeq j$ if and only if $\gamma_j \succeq \gamma_i$. Then $(PAP^{-1})^{\mathsf{T}} \in \mathfrak{Mat}_{\mathcal{P},+}^{(\mathrm{II})}(\mathbf{n}, \mathbb{Z})$ and $(P'A'P'^{-1})^{\mathsf{T}} \in \mathfrak{Mat}_{\mathcal{P},+}^{(\mathrm{II})}(\mathbf{n}', \mathbb{Z})$. It is an important observation that given a transition state $i \in \Sigma$ and a hereditary subset H of Γ_A, then $i \in \overline{\Sigma(H)}$ iff $\gamma \in H$, where $\gamma \in \Gamma_A$ is the first irreducible component we meet, when we start a trip from i. We now want to show that $K(I - (PAP^{-1})^{\mathsf{T}})$ is K-web isomorphic to $K(I - (P'A'P'^{-1})^{\mathsf{T}})$.

Assume w.l.o.g. that A and A' are already written in this form (so $P = I$ and $P' = I$) and let $B = I - A^{\mathsf{T}}$ and $B = I - A'^{\mathsf{T}}$. Then $B_{ii} = I - A_{ii}^{\mathsf{T}}$ and $B_{ij} = -A_{ji}^{\mathsf{T}}$ for all $i, j \in \mathcal{P}$ with $i \neq j$ and correspondingly for B'.

Let $\phi_{\breve{H}(i)} = \kappa^{0,i}_{\mathcal{J}_{H(\gamma_i)}}$, $\phi_{\breve{H}_-(i)} = \kappa^{0,i}_{\mathcal{J}_{H_-(\gamma_i)}}$ (if $\breve{H}_-(i) \neq \emptyset$), $\phi_{\{i\}} = \kappa^{0,q}_{\mathcal{J}_{H(\gamma_i)}}$ and $\psi_i = \kappa^{1,q}_{\mathcal{J}_{H(\gamma_i)}}$. We of course have to check that this is well defined! First of all, because of the special structure of A and A', we have that

$$\begin{aligned} C_{\{i\}}(B) &= K_0(\mathcal{J}_{H(\gamma_i)}/\mathcal{J}_{H_-(\gamma_i)}), & C_{\{i\}}(B') &= K_0(\mathcal{J}_{H(\gamma_i')}/\mathcal{J}_{H_-(\gamma_i')}), \\ K_i(B) &= K_1(\mathcal{J}_{H(\gamma_i)}/\mathcal{J}_{H_-(\gamma_i)}), & K_i(B') &= K_1(\mathcal{J}_{H(\gamma_i')}/\mathcal{J}_{H_-(\gamma_i')}), \\ C_{\breve{H}(i)}(B) &= K_0(\mathcal{J}_{H(\gamma_i)}), & C_{\breve{H}(i)}(B') &= K_0(\mathcal{J}_{H(\gamma_i')}), \\ C_{\breve{H}_-(i)}(B) &= K_0(\mathcal{J}_{H_-(\gamma_i)}), & C_{\breve{H}_-(i)}(B') &= K_0(\mathcal{J}_{H_-(\gamma_i')}) \qquad (\text{if } \breve{H}_-(i) \neq \emptyset) \end{aligned}$$

for all $i \in \mathcal{P}$. So the source- and target-sets are the right ones. But we also have to check that the maps are well-defined. Because the ϕ_S's are defined from the *subsets* S of $\mathcal{P}$, the cases $\breve{H}(i) = \breve{H}_-(j)$ and $\breve{H}_-(i) = \breve{H}_-(j)$ will not cause any problems. So the remaining case is $\{i\} = \breve{H}(i)$; and thus $\breve{H}_-(i) = \emptyset$, $H(\gamma_i) = \{\gamma_i\}$ and $H_-(\gamma_i) = \emptyset$. Because we also have included the exact sequence induced by the short-exact sequence

$$0 \to \mathcal{J}_{H_-(\gamma_i)} = \{0\} \to \mathcal{J}_{H(\gamma_i)} \to \mathcal{J}_{H(\gamma_i)}/\mathcal{J}_{H_-(\gamma_i)} \to 0$$

where the “little ideal” is $\{0\}$ in the definition of the reduced filtrated K-theory of $\mathcal{O}_A$ (and correspondingly for $\mathcal{O}_{A'}$), we get the following commutative diagram

$$\begin{tikzcd}
 & & C_{\{i\}}(B) \arrow[r, equal] \arrow[d, "\cong"] & C_{\{i\}}(B) \arrow[d, "\cong"] & \\
K_1(\mathcal{J}_{H(\gamma_i)}/\mathcal{J}_{H_-(\gamma_i)}) \arrow[r] \arrow[d, "\kappa^{1,q}_{\mathcal{J}_{H(\gamma_i)}}"] & K_0(\mathcal{J}_{H_-(\gamma_i)}) = \{0\} \arrow[r] \arrow[d, "\kappa^{0,i}_{\mathcal{J}_{H_-(\gamma_i)}}"] & K_0(\mathcal{J}_{H(\gamma_i)}) \arrow[r, "\cong"] \arrow[d, "\kappa^{0,i}_{\mathcal{J}_{H(\gamma_i)}}"] & K_0(\mathcal{J}_{H(\gamma_i)}/\mathcal{J}_{H_-(\gamma_i)}) \arrow[r] \arrow[d, "\kappa^{0,q}_{\mathcal{J}_{H(\gamma_i)}}"] & 0 \\
K_1(\mathcal{J}_{H(\gamma_i')}/\mathcal{J}_{H_-(\gamma_i')}) \arrow[r] & K_0(\mathcal{J}_{H_-(\gamma_i')}) = \{0\} \arrow[r] & K_0(\mathcal{J}_{H(\gamma_i')}) \arrow[r, "\cong"] & K_0(\mathcal{J}_{H(\gamma_i')}/\mathcal{J}_{H_-(\gamma_i')}) \arrow[r] & 0 \\
 & & C_{\{i\}}(B') \arrow[r, equal] \arrow[u, "\cong"'] & C_{\{i\}}(B'). \arrow[u, "\cong"'] &
\end{tikzcd}$$

Thus the definition is consistent. We now automatically get all the commutativity from the reduced filtrated K-theory. So $K(I - A^{\mathsf{T}})$ is K-web isomorphic to $K(I - A'^{\mathsf{T}})$. By Theorem 7.5 $(I - A^{\mathsf{T}})$ is therefore $GL_{\mathcal{P}}$-equivalent to $(I - A'^{\mathsf{T}})$.

It now follows from Remark 8.2 that $(I - JAJ) \in \mathfrak{Mat}^{(\mathrm{II})}_{\mathcal{P}_J^{\mathsf{T}},+}(\mathbf{n}_J^{\mathsf{T}}, \mathbb{Z})$, $(I - JA'J) \in \mathfrak{Mat}^{(\mathrm{II})}_{\mathcal{P}_J^{\mathsf{T}},+}(\mathbf{n}_J'^{\mathsf{T}}, \mathbb{Z})$ and that $(I - JAJ)$ is $GL_{\mathcal{P}_J^{\mathsf{T}}}$-equivalent to $(I - JA'J)$. From Theorem 6.8 it now follows that $\mathcal{O}_A \otimes \mathbb{K} \cong \mathcal{O}_{A'} \otimes \mathbb{K}$. $\square$

9 Addendum

There are many interesting questions concerning the material in this paper. In [8], Thm. 4.9, Huang has a complete invariant of Cuntz-Krieger algebras with trivial K_1-group up to unital isomorphism (by a kind of “preserving the unit”). It is natural to ask how to extend this result to the general case:

Question 1 *What additional conditions do we need to make a complete invariant of Cuntz-Krieger algebras up to unital isomorphism?*

This is an open question for Cuntz-Krieger algebras with only one non-trivial ideal, so the first step would be to solve this case:

Question 2 *What, in addition to the six-term exact sequence, do we need to include in our invariant to make it a complete invariant up to unital isomorphism of Cuntz-Krieger algebras with exactly one non-trivial ideal?*

Rørdam told the author that an isomorphism theorem of the two-component Cuntz-Krieger algebras was the “missing link” to succeed in proving the classification result (for three-component Cuntz-Krieger algebras) in the mid nineties. This suggests that the isomorphism theorem is correct, but much deeper than the classification problem:

Question 3 *Let A and A' be $\{0,1\}$-matrices satisfying (II). Is every reduced filtrated K-theory isomorphism from $\mathfrak{K}_r^{\mathfrak{filt}}(\mathcal{O}_A)$ to $\mathfrak{K}_r^{\mathfrak{filt}}(\mathcal{O}_{A'})$ induced by an isomorphism from $\mathcal{O}_A \otimes \mathbb{K}$ to $\mathcal{O}_{A'} \otimes \mathbb{K}$?*

Eilers told the author that using the work of Bonkat, [1], this question can be answered in the positive for two-component Cuntz-Krieger algebras.

And, of course, we can ask, how large a class of (infinite, stable, separable, nuclear) C^*-algebras, our invariant classifies:

Question 4 *Which "nice" classes of C^*-algebras, $\mathfrak{A}$, do the full filtrated K-theory, $\mathfrak{K}^{\mathrm{filt}}(\mathfrak{A})$, classify.*

Of course, this includes Cuntz-Krieger algebras (whose adjacency matrices satisfy condition (II) of Cuntz). It is also worth mentioning that all separable, stable, nuclear C^*-algebras absorbing $\mathcal{O}_2$ ($\mathfrak{A}$ absorbs $\mathcal{O}_2$ iff $\mathfrak{A} \cong \mathfrak{A} \otimes \mathcal{O}_2$) are classified up to unital isomorphism by their ideal lattice (this is proved in [11]) – this means that all the "K-theory information" in the full filtrated K-theory is superfluous.

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