NON-STATIONARY AND NO MOMENTS ASYMPTOTICS FOR THE ARCH MODEL

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Abstract: *We establish consistency and asymptotic normality of the quasi-maximum likelihood estimator in the linear ARCH model. Contrary to existing literature we allow the parameters to be in the region where no stationary version of the process exists.*

1. Introduction

Consider the ARCH (autoregressive conditional heteroscedastic) model by Engle (1982), as given by

$$\begin{aligned} y_t &= \sigma_t^2 z_t \\ \sigma_t^2 &= \omega + \alpha y_{t-1}^2 \end{aligned} \tag{1}$$

for $t = 1, ..., T$, $\alpha \geq 0$, $\omega > 0$ and with z_t an i.i.d$(0, 1)$ process. Asymptotic inference for the ARCH(1) and more general ARCH models, including GARCH models, has been studied in e.g. Lee and Hansen (1994), Lumsdaine (1991), Weiss (1986) and recently Kristensen and Rahbek (2002). These as well as existing literature all assume as a minimal requirement that the ARCH process y_t is suitably ergodic or stationary such that laws of large numbers apply. Moreover the generic assumption for asymptotic normality is that the squared error process, z_t^2 has a finite variance, $V(z_t^2) = \zeta < \infty$.

We relax the condition about stability of the y_t process and allow it to be non-stationary and in particular not to have any moments. Surprisingly we find that the derivations in this the non-stationary case are more easy and straightforward when compared to the stationary case. We are convinced that the analogue results applies to the non-stationary version of the GARCH(1,1) process.

For exposition and without loss of generality we henceforth set $\omega = 1$.

2. Inference

By Nelson (1990) and Bougerol and Picard (1992), y_t is stationary (a stationary version exists) and ergodic if and only if $E\log\left(\alpha z_t^2\right) < 0$. Equivalently, if z_t is Gaussian, $\alpha < \frac{1}{2}\exp\left(-\Psi\left(\frac{1}{2}\right)\right) \simeq 3.56$ where $\Psi(\cdot)$ is the Euler psi function, see Nelson (1990).

As mentioned our analysis is under the assumption that y_t does not have a stationary version or equivalently,

$$E\log\left(\alpha z_t^2\right) \geq 0. \tag{2}$$

Consider the likelihood estimator based on maximization of the quasi-likelihood

$$\ell_T(\alpha,\beta) = -\tfrac{1}{2}\sum_{t=1}^{T}\left[\log\sigma_t^2 + \frac{y_t^2}{\sigma_t^2}\right] \tag{3}$$

from which the QMLE (quasi maximum likelihood estimator) $\hat{\alpha}$ is found by maximization. Note that this the true likelihood if z_t is Gaussian. Our main result is the following:

Theorem 1. *Assume that the ARCH process y_t in (1) does not allow a stationary version or equivalently (2) holds. Assume further that the i.i.d.(0,1) process z_t is such that $V(z_t^2) = \zeta$ is finite. Then as $T \to \infty$ the sequence of QMLE $\hat{\alpha}$ is consistent, and asymptotically normal,*

$$\sqrt{T}\left(\hat{\alpha} - \alpha\right) \xrightarrow{D} N(0,\sigma^2)$$

where

$$\sigma^2 = \zeta\alpha^2 > 0.$$

Remark 2. *Note that if z_t is Gaussian then $\sigma^2 = 2\alpha^2$ in Theorem 1.*

Proof: Together Lemma 5, Lemma 6 and Lemma 7 in the next Section establish the classical Cramér type conditions, see e.g. Lehmann (1999). □

3. Derivation

With the likelihood function given by (3), the score, information and the third derivative of the log-likelihood with respect to α are easily found to be given by

$$\frac{\partial}{\partial\alpha}\ell_T(\alpha) = -\tfrac{1}{2}\sum_{t=1}^{T}\left[1-\frac{y_t^2}{\sigma_t^2}\right]\frac{y_{t-1}^2}{\sigma_t^2} \tag{4}$$

$$\frac{\partial^2}{\partial\alpha^2}\ell_T(\alpha) = \tfrac{1}{2}\sum_{t=1}^{T}\left[1-2\frac{y_t^2}{\sigma_t^2}\right]\frac{y_{t-1}^4}{\sigma_t^4} \tag{5}$$

$$\frac{\partial^3}{\partial\alpha^3}\ell_T(\alpha) = -\sum_{t=1}^{T}\left[1-3\frac{y_t^2}{\sigma_t^2}\right]\frac{y_{t-1}^6}{\sigma_t^6} \tag{6}$$

In the following we study the asymptotic behaviour of these in order to establish consistency and asymptotic normality of the QMLE. First, consider the asymptotic behaviour of y_t:

Lemma 3. *Assume that (2) holds, then*

$$y_t^2 \overset{\text{a.s.}}{\to} \infty$$

as $t \to \infty$.

Proof: This follows by Theorem 2, Nelson (1990). □

Next, consider the asymptotic behaviour of the following type of averages:

Lemma 4. *Assume that (2) holds, then with $m \leq n$ positive integers,*

$$\frac{y_{t-1}^{2m}}{\left(1+\alpha y_{t-1}^2\right)^k} \overset{\text{a.s.}}{\to} \begin{cases} \frac{1}{\alpha^m} \text{ if } m = k \\ 0 \text{ if } m < k \end{cases} \tag{7}$$

and likewise,

$$\tfrac{1}{T}\textstyle\sum_1^T \frac{y_{t-1}^{2m}}{\left[1+\alpha y_{t-1}^2\right]^k} \overset{\text{a.s.}}{\to} \begin{cases} \frac{1}{\alpha^m} \text{ if } m = k \\ 0 \text{ if } m < k \end{cases} \tag{8}$$

Proof: The results follow by Lemma 3. □

Turn to asymptotic normality of the score:

Lemma 5. *Under the assumptions of Theorem 1, then with* $\frac{\partial}{\partial\alpha}\ell_T(\alpha)$ *given by (4),*

$$\sqrt{T}\frac{\partial}{\partial\alpha}\ell_T(\alpha) \overset{D}{\to} N(0, \tfrac{\zeta}{4\alpha^2})$$

as $T\to\infty$.

Proof: By definition $\sqrt{T}\frac{\partial}{\partial\alpha}\ell_T(\alpha) = \frac{1}{\sqrt{T}}\sum_{t=1}^{T} s_t$ where

$$s_t = -\tfrac{1}{2}\left[1-\frac{y_t^2}{\sigma_t^2}\right]\frac{y_{t-1}^2}{\sigma_t^2}.$$

The process s_t is a Martingale difference sequence with respect to $\mathcal{F}_t = \sigma\{y_t, y_{t-1}, ..., y_0\}$ as $E|s_t| \leq E|1-z_t^2| < \infty$ and $E(s_t|\mathcal{F}_{t-1}) = -\frac{1}{2}E(1-z_t^2)\frac{y_{t-1}^2}{\sigma_t^2} = 0$. Next, using (8)

$$\frac{1}{T}\sum_{t=1}^{T} E\left(s_t^2\middle|\mathcal{F}_{t-1}\right) = \frac{1}{4}\frac{1}{T}\sum_{t=1}^{T}\left(\frac{y_{t-1}^2}{1+\alpha y_{t-1}^2}\right)^2\zeta \overset{a.s.}{\to} \tfrac{\zeta}{4\alpha^2} > 0,$$

where $\zeta = E(1-z_t^2)^2 = V(z_t^2)$. Furthermore, as s_t^2 is bounded by $\mu_t^2 = [1-z_t^2]^2$ we derive the Lindeberg type condition,

$$\frac{1}{T}\sum_{t=1}^{T} E\left(s_t^2 1\left(|s_t| > \sqrt{T}\delta\right)\right) \leq E\left(\mu_t^2 1\left(|\mu_t| > \sqrt{T}\delta\right)\right) \to 0,$$

for some $\delta > 0$ and as T tends to ∞ using $V(z_t^2) = \zeta < \infty$. By the central limit theorem in Brown (1971) the desired result follows. □

Lemma 6. *Under the assumptions of Theorem 1, then with the observed information* $\frac{\partial^2}{\partial\alpha^2}\ell_T(\alpha)$ *given by (5),*

$$\tfrac{1}{T}\left[-\frac{\partial^2}{\partial\alpha^2}\ell_T(\alpha)\right] \overset{a.s.}{\to} \tfrac{1}{2\alpha^2} > 0$$

as $T\to\infty$.

Proof: Rewrite minus the observed information as

$$-\frac{\partial^2}{\partial\alpha^2}\ell_T(\alpha) = \frac{1}{2T}\sum_{t=1}^{T}\kappa_t\gamma_t$$

with

$$\kappa_t = 2z_t^2 - 1 \text{ and } \gamma_t = \frac{y_{t-1}^4}{\sigma_t^4} = \frac{y_{t-1}^4}{\left(1+\alpha y_{t-1}^2\right)^2}.$$

The strong law of large numbers imply

$$\frac{1}{T}\sum_{t=1}^{T}\kappa_t \overset{a.s.}{\to} 1$$

$$\frac{1}{T}\sum_{t=1}^{T}|\kappa_t| \overset{a.s.}{\to} \kappa < \infty$$

while (7) implies $\gamma_t \overset{a.s.}{\to} \frac{1}{\alpha^2}$ and hence the desired result follows. □

Lemma 7. *Assume that the assumptions of Theorem 1 hold. Denote by $N(\alpha,\delta)$ the interval $[\alpha \pm \delta]$, $0 < \delta < \alpha$. Then with $\frac{\partial^3}{\partial\alpha^3}\ell_T(\alpha)$ given by (6), it holds that*

$$\sup_{\tilde{\alpha}\in N(\alpha,\delta)}\left|\tfrac{1}{T}\frac{\partial^3}{\partial\alpha^3}\ell_T(\tilde{\alpha})\right| \leq g(\alpha,\delta,T) \overset{P}{\to} \beta < \infty$$

as $T \to \infty$.

Proof: With $\alpha_l = \alpha - \delta$,

$$\begin{aligned}\left|\tfrac{1}{T}\frac{\partial^3}{\partial\alpha^3}\ell_T(\tilde{\alpha})\right| &= \left|\frac{1}{T}\sum_{t=1}^{T}\left[3\frac{y_t^2}{\sigma_t^2}-1\right]\frac{y_{t-1}^6}{\sigma_t^6}\right| \leq \frac{1}{T}\sum_{t=1}^{T}\left|\left[3\frac{y_t^2}{\sigma_t^2}-1\right]\right|\frac{1}{\alpha_l^3} \\ &= \frac{1}{T}\sum_{t=1}^{T}\left|\left[3\frac{\left(1+\alpha y_{t-1}^2\right)}{\left(1+\tilde{\alpha}y_{t-1}^2\right)}z_t^2-1\right]\right|\frac{1}{\alpha_l^3} \\ &\leq \frac{1}{T}\sum_{t=1}^{T}\left(3\left[1+\tfrac{\alpha}{\alpha_l}\right]z_t^2+1\right)\frac{1}{\alpha_l^3} := g(\alpha,\delta,T)\end{aligned}$$

and the results follows by the law of large numbers. □

Remark 8. *The classical sufficient condition is that*

$$E\sup_{\tilde{\alpha}\in N(\alpha,\delta)}\left|\tfrac{1}{T}\frac{\partial^3}{\partial\alpha^3}\ell_T(\tilde{\alpha})\right| < \infty.$$

In Basawa, Feigin and Heyde (1976, condition (B.7)) this is incorrectly stated as

$$\sup_{\tilde{\alpha}\in N(\alpha,\delta)}E\left|\tfrac{1}{T}\frac{\partial^3}{\partial\alpha^3}\ell_T(\tilde{\alpha})\right| < \infty.$$

The mistake is reproduced in Weiss (1986) and next in Lumsdaine (1991), and their proofs may therefore not be complete.

Remark 9. *We note that in the stationary and ergodic case Lemma 4 and Lemma 6 are established using the ergodic theorem. Based on these the proofs of Lemma 5 and Lemma 7 are unchanged when compared to the stationary and ergodic case.*

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