Microlocal methods in symplectic geometry

1. Microlocal methods: wave front sets, singular supports. Sheaves, $\mathcal{D}$-modules, pseudo-differential operators. Distributions.

2. Symplectic geometry: Floer cohomology, Fukaya category.

3. Deformation quantization.

4. Sheaf-theoretical microlocal methods in symplectic geometry. The works of Nadler-Zaslow and Tamarkin.

5. Oscillatory modules.

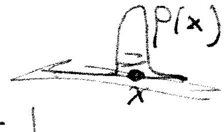
1. Microlocal methods.

i) Wave front set of a distribution.

$u: C_c^\infty(M) \to \mathbb{C}$ $\qquad$ M – manifold

$\cancel{M} = \mathbb{R}^n$: if $\hat{u}$ is rapidly decaying in ξ, then u is <u>smooth</u>

a) localize u near $x \in M$: $\quad u \rightsquigarrow p(x) \cdot u$



$$\bigcap_{\substack{p \\ p(x) \neq 0}} \left\{ \xi \overset{\neq 0}{:} \ \widehat{pu} \text{ is } \underline{\text{not}} \text{ rapidly decaying in direction } \xi \right\}$$

$\cancel{WF(x)}$ ξ – Singular direction at x

$WF(u) = \{(x, \xi) : \xi \neq 0 \text{ sing at } x\}$ $\subset \mathbb{R}^{2n}$

<u>Ex</u> $\cdot$ u smooth: $\quad WF(u) = \emptyset$.

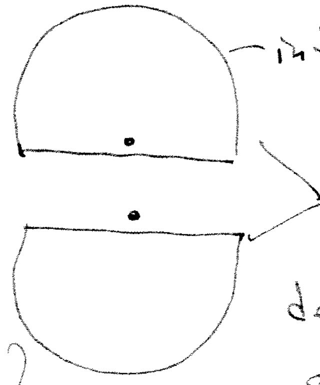
$\cdot$ $u = \delta_0$; $\quad WF(u) = \{(0, \xi)\}$

$pu = \text{const} \cdot \delta_0$; $\quad \widehat{pu} = c$;

$M = \mathbb{R}$ $\cdot$ $u(x) = \dfrac{1}{x + i0}$ $\qquad \langle u, f \rangle = \lim_{\varepsilon \to 0} \displaystyle\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x + i\varepsilon}$

Fourier transform by contour integration

$$\lim_{\varepsilon \to 0} \int \frac{e^{i\xi x}}{x + i\varepsilon}\,dx$$



integrand small

or

$$\hat{u} = c \cdot \begin{cases} 1, & \xi > 0 \\ 0, & \xi < 0 \end{cases}$$

depends on $\operatorname{sgn}(\xi)$

$$\left(WF\left(\frac{1}{x + i0}\right) = \{(0, \xi) : \xi > 0\} \right)$$

$\widehat{pu}$: Convolution with a rapidly decaying function $\hat{p}(\xi)$.

b) Check behavior of $WF(u)$ under diffeomorphisms

$$\varphi : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n \qquad u \mapsto u(\varphi(x))$$

<u>Fact</u>: For a manifold M, $WF(u)$ is a well-defined subset of T^*M.

It is
- <u>closed</u>
- <u>conical</u>, $(x,\xi) \in WF(u) \mapsto (x, t\xi) \in WF(u)$, $t>0$

<u>What is a "coordinate-free" dfn?</u>

distributions on $M \times M$: gen. fns $K(x,y)$

Those define operators $C_c^\infty(M) \to C_c^\infty(M)'$

$$\text{"} f(y) \longmapsto \int K(x,y) f(y)\, dy \text{"}$$

<u>Among them</u>: <u>pseudodifferential</u> operators.

<u>What are they</u>? quite a special class; in particular, $WF(K) \subset \mathrm{diag} = \{ \cancel{(x,\xi); (x,-\xi} \; (x, x; \xi, -\xi) \} \subset T^*(M \times M)$

<u>Or</u>: ~~after being localized!~~

$$A : C^\infty(M) \to C^\infty(M)$$

$$\psi \circ A \circ \rho : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$$

must be of the form

$$f(y) \longmapsto \int K(x,y) f(y)\, dy$$

where

$$K(x,y) = \int e^{i(x-y)\xi} a(x,\xi)\, d\xi$$

<u>and</u> $a(x,\xi)$ ~~is of s~~ has special <u>growth condition</u> for these integrals to make sense.

Ex. $a(x,\xi)$ polynomial in ξ

$\rightsquigarrow K(x,y) = \sum a_i(x) . \delta^{(n)}(x-y)$

A - differential operator.

- A ΨDO has a principal symbol:
  $\sigma(x,\xi)$ function on the cotangent bundle $\{\xi \neq 0\}$

A is elliptic: $\sigma(x,\xi)$ invertible

A is elliptic at (x,ξ): ——— " ——— near x,ξ.

Another definition of WF(u):

(x,ξ): $\xi \neq 0$ $\exists$ A - elliptic near (x,ξ) s.t.
$\not\in$
WF(u) $A \cdot u$ is smooth.

We also know:

A is elliptic iff it is invertible modulo smoothing operators $f(y) \mapsto \int K(x,y) f(y)\, dy$
K - smooth kernel

Suddenly we land in the realm of standard algebra:

$\mathcal{A} = \{\Psi DO\}$ modulo $\{$Smoothing $\Psi DO\}$: algebra

$\mathcal{M} = \{$ distributions $\}$ modulo $\{$ Smooth functions $\}$: module

$\mathcal{A}$ - Sheaf of algebras on $S^*M = T^*M - \{(x,0)\} / \mathbb{R}_+^*$

$(x,\xi) \mapsto (x, t\xi)$

$\mathcal{M}$ - sheaf of modules

WF(u) - SUPPORT of this sheaf of modules

$\mathrm{Supp}(m) = \{ y : \nexists\, a \in \mathcal{A}$ inv. at y; $a \cdot m \neq 0 \}$

So: these are MICRO local considerations;
local in S^*M, not M.

~~An~~ Other versions : local in T^*M, <u>not</u> S^*M.

a) <u>$\mathcal{D}$ - modules.</u>

In coords: $A = \sum_{|n| \leq N} a_n(x) \left(i\frac{\partial}{\partial x}\right)^n$

$\mathcal{D}_M$ - <u>sheaf</u> of operators on M

<u>Filtration</u> : $F_N \mathcal{D}_M = \{$ operators of order $N \}$

$gr_N^F = F_N / F_{N-1}$; $gr^F \mathcal{D}_M = \left\{ \begin{array}{l} \text{Fns on } T^*M \\ \text{polyn in } \xi \end{array} \right\}$

$\mathcal{M}$ - sheaf of $\mathcal{D}_M$ - modules <u>with a compatible filtration</u> (+ one cond...)

$\mathbb{R}^x$ - nw... Information partly lost; works better/$\mathbb{C}$	$gr^F \mathcal{M}$ - sheaf of $gr^F \mathcal{D}_M$ - modules Its <u>support</u> $\subseteq T^*M$: $SS(\mathcal{M})$ SINGULAR SUPPORT of $\mathcal{M}$; Closed conical subset of T^*M.

Roughly, from a distribution to a $\mathcal{D}$-module :

$$u \longrightarrow \mathcal{D}_M \cdot u$$

(or $A \cdot u$, $A = \frac{\Psi DO}{\text{Smoothing}}$)

b) <u>Asymptotic</u> distributions/operators.

$$a(x, \xi, \hbar) \rightsquigarrow Op_\hbar(a) : C^\infty(M) \circlearrowleft$$

$\hbar$ - dependent function on T^*M

~~Pro~~

Kernel of $Op_\hbar(a)$:

$$K(x,y) = \frac{1}{(2\pi\hbar)^{n/2}} \int e^{\frac{1}{i\hbar}(x-y)\xi} a(x, \xi, \hbar) \, d\xi$$

<u>Roughly</u>:

$$a(x, \xi, \hbar) \longmapsto a\left(x, i\hbar\frac{\partial}{\partial x}, \hbar\right)$$

For <u>polynomials</u> in ξ: literally so.

Before: to localize wrt S^*X, factor out smooth functions/kernels

Now: to localize wrt T^*X, factor out those that decrease faster than any $\hbar^N$ (i.e. $O(\hbar^\infty)$).

Fact: $\forall N$
$$Op_\hbar(a) \circ Op_\hbar(b) = Op_\hbar\left(a \cdot b + \sum_{k=1}^{N} (i\hbar)^k P_k(a,b)\right) + O(\hbar$$

$P_k(a,b)$ local in T^*M. Actually, in coords:
$$P_k(a,b) = \sum_{|\alpha|, |\beta| \leq M} A^{(k)}_{\alpha\beta}(x)\, \partial^\alpha a \cdot \partial^\beta b \quad \text{(bidifferential)}$$

Ex. $X = \mathbb{R}^n$: ...

What are asymptotic distributions?

GS: (locally of the form)
$$u(x,\hbar) = \frac{1}{(2\pi\hbar)^{n/2}} \int e^{\frac{1}{i\hbar}\varphi(x,\theta)}\, a(x,\theta,\hbar)\, d\theta$$

θ: additional parameter; some requirements on φ and on $a(x,\theta,\hbar)$. (No problem if a smooth, comp. supp.)

- at fixed $\hbar = 1$: The classical theory of Lagrangian distributions (Hörmander; Maslov; ...) $\varphi(x,\theta)$ required to be homogeneous of order one in θ; growth condition on $a(x,\theta)$ similar to above.

Fact: $WF(u) = \{(x,\xi) : \xi = \varphi_x(x,\theta) \text{ where } \varphi_\theta(x,\theta) = 0\}$ ☆

(Automatically conical because φ homog in θ)

- As an asymptotic distribution:
  No problem if $a(x,\theta)$ comp. supp.
  (In general: GS?)

$WF_{as}(u) = \{(x,\xi) \in T^*M \mid Op_\hbar(a)\cdot u \neq 0 \text{ if } a(x,\xi,\hbar) \neq 0\}$

<u>Example</u> $\{\theta\} = \{0\}$

$$u(x,\hbar) = e^{\frac{p(x)}{i\hbar}} \underbrace{a(x,\hbar)}_{\text{smooth}}$$

$WF_{as}(u) = \{\xi = \varphi'(x)\}$; but at any given $\hbar$

$WF(u) = \emptyset$ since u <u>smooth</u>

<u>Algebra</u>: $C^\infty(M)[[\hbar]]$ $\quad a * b = \sum_{k=0}^{\infty} (i\hbar)^k P_k(a,b)$

$\mathcal{A}$ $\quad M = T^*X$ $\quad P_0(a,b) = ab$; P_k <u>bidifferential</u>

In fact $P_1(a,b) = \frac{1}{2}\{a,b\} = \frac{1}{2}(\partial_\xi a \cdot \partial_x b - \partial_x a \cdot \partial_\xi b)$

$\uparrow$ the Poisson bracket

(<u>Deformation quantization</u> of T^*X)

<u>Module</u>: $(X = \mathbb{R}^n)$

<u>Motivation</u>: $\quad u(x,\hbar) = \frac{1}{(2\pi\hbar)^{n/2}} \int e^{\frac{1}{i\hbar}\varphi(x,\theta)} a(x,\theta,\hbar)\, d\theta$

$\mathcal{M}$ <u>consists of</u>: $a(x,\theta,\hbar)$ formal series in $\hbar$

$$\left[i\hbar \frac{\partial}{\partial\theta} + \varphi_\theta(x,\theta)\right] \cdot a(x,\theta) \sim 0$$

Action of $\mathcal{A}$:

$$\xi \text{ by } i\hbar\frac{\partial}{\partial x} + \varphi_x(x,\theta)$$

$$x \text{ by } x.$$

$\mathcal{A}$ - sheaf of algebras on T^*X

$\mathcal{M}$ - sheaf of modules. Its support:

$\star$ $\{(x,\xi) \mid \xi = \varphi_x(x,\theta) \text{ where } \varphi_\theta(x,\theta) = 0\}$

Examples $\{\theta\} = \{0\}$ R. Nest-B.T.

$u(x,\hbar) = e^{\varphi/i\hbar} a(x,\hbar)$

$\mathcal{M} = \{a(x,\hbar)\}$ $\xi \sim i\hbar \frac{\partial}{\partial x} + \varphi'(x)$

$x \sim x$ $[\xi, x] = i\hbar$

$u(x,\hbar) \; \underline{A} \; \int e^{\frac{1}{i\hbar}(x\theta - \psi(\theta))} a(x,\theta,\hbar)\, d\theta$

$\mathcal{M} = \{a(x,\theta,\hbar)\}/\sim$ $i\hbar \partial_\theta a + x a \sim 0$

Whenever you see $x.a$, can replace...

$\mathcal{M} \sim C^\infty(\{\theta\})[[\hbar]]$ $\xi \sim \theta.$

$x \sim -i\hbar \frac{\partial}{\partial \theta} + \psi'(\theta)$

So! one algebra $\mathcal{A}$; many modules $\mathcal{M}$

$M = T^*X$:

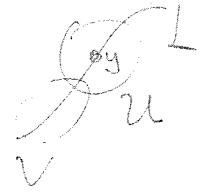
- Symplectic structure $\omega = d\xi\, dx = d\alpha$
  $\alpha = \xi\, dx$
- $L \subset M$ Lagrangian: $\omega|T_yL = 0$; $\dim L = n$.
  (or: T_yL is maximal isotropic for ω).

Locally (in T^*X), near $y = (x,\xi) \in L$:
L is of the form $(\star)$ i.e. local phase fns exist

So: local modules $\mathcal{M}_U$ over $\mathcal{A}|_U$ $U \subset T^*X$

Can be glued into a sheaf of modules over the sheaf of algebras $\mathcal{A}[e^{a/i\hbar}]_{a\in\mathbb{R}}$

Thm: $\mathcal{M}_U \simeq \left(C^\infty(L|U)\otimes|\Lambda|^{1/2}\right)[e^{\cdot/i\hbar}]$



<u>Transition automorphisms:</u>

$$e^{\alpha_{UV}/i\hbar}\left[g_{UV}(1+i\hbar\ldots)\right]$$

g_{UV} - transition isoms of the <u>Maslov local system</u> on L.

<u>Another microlocal theory: sheaves on mflds</u>

<u>Why sheaves?</u>

$\mathcal{D}_X$-modules $\rightsquigarrow$ (complexes of) Sheaves on X

$\mathcal{M}$ a $\mathcal{D}_X$-module: a generalization of a vector bundle $\mathcal{E}$ with a flat connection ∇ $\longrightarrow$ De Rham complex $(\Omega^\cdot(X,\mathcal{E});\nabla)$

$\Omega^\cdot(X,\mathcal{M}):\ \Omega^0\otimes_{\mathcal{O}_X}\mathcal{M}\to\Omega^1\otimes_{\mathcal{O}_X}\mathcal{M}\ldots$

MORE homologically:

$$\mathbb{R}\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{O}_X,\mathcal{M})$$

[Up to shift + tensoring by a local syst...]

<u>Example</u>: $\mathcal{M}=\mathcal{O}_X$ $\quad\Omega^\cdot$ - the usual DR complex (sheaf of); same as the constant sheaf $\mathbb{C}_X$

$X=\mathbb{R}$ $\quad \mathcal{M}=\bigoplus_{k\geq 0}\mathbb{C}\cdot\delta_0^{(k)}$; $\quad \mathcal{M}\xrightarrow{\partial/\partial x}\mathcal{M}$

Skyscraper sheaf $\mathbb{C}_0$ in degree one

So:

no direct link, but at least strong analogies:

Distribution on X	$\mathcal{D}_X$-module	Sheaf on X (complex of)

$$u \rightsquigarrow \mathcal{D}_X \cdot u \rightsquigarrow \text{De Rham complex of } \mathcal{D}_X \cdot u$$

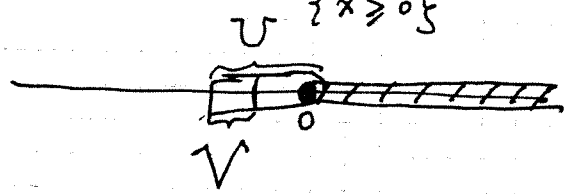
Singular support of a sheaf

(Kashiwara-Schapira)

Idea: $\mathcal{F}$ - a sheaf of vector spaces on X

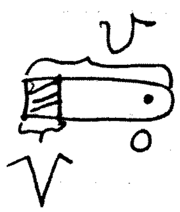
Examples: $\mathcal{F} = \mathbb{C}_{\{x \geq 0\}}$ on $X = \mathbb{R}$

NEAR $x=0$:



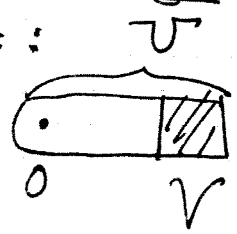
Sheaf changes if one moves to the left:

1. • a section on U
 • no sections on V



Sheaf does not change if one moves to the right:

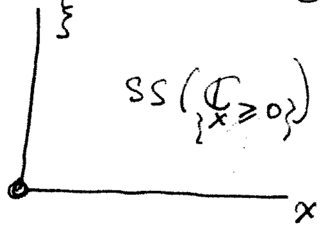
2. • a section on U
 • a section on V



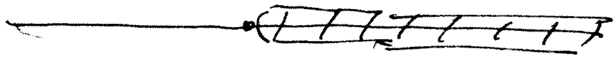
OR: the restriction map

$$\Gamma(U, \mathcal{F}) \to \Gamma(V, \mathcal{F})$$ ① not an isom
②

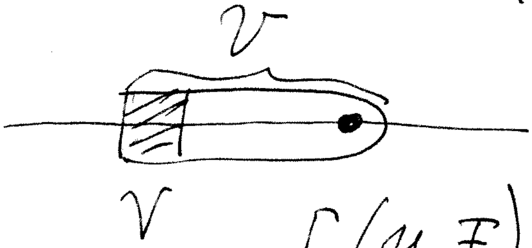
a) $SS(\mathcal{F}) = \{(x,0) : x \geq 0\} \cup \{(0,\xi) : \xi \geq 0\}$



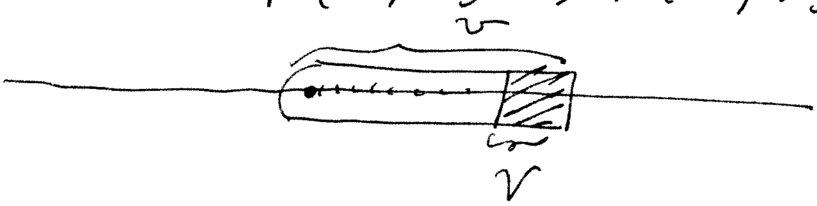
b) $SS(\mathbb{C}_{\{x>0\}}) = \{(x,0) : x \geq 0\} \cup \{(0,\xi) : \xi \leq 0\}$



$\mathbb{C}_{\{x>0\}}$



$\Gamma(U,\mathcal{F}) \xrightarrow{\sim} \Gamma(V,\mathcal{F}) = 0$

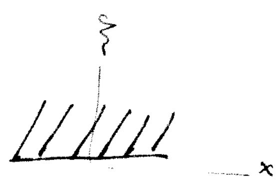


$0 = \Gamma(U,\mathcal{F}) \longrightarrow \Gamma(V,\mathcal{F}) \neq 0$

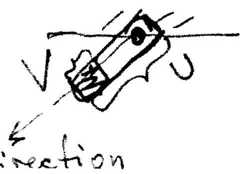
c) $SS(\mathbb{C}) = \{(x,0)\}$ 

d) $SS(\mathbb{C}_{\{0\}}) = \{(0,\xi)\}$ 

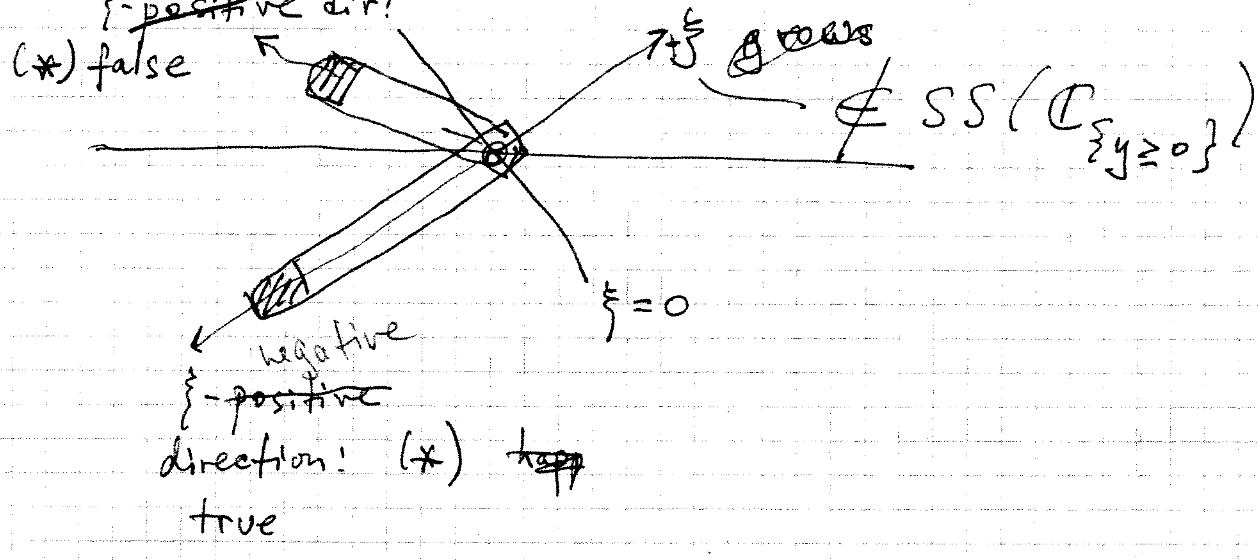
e) $X = \mathbb{R}^2$; $\mathcal{F} = \mathbb{C}_{\{y\geq 0\}}$

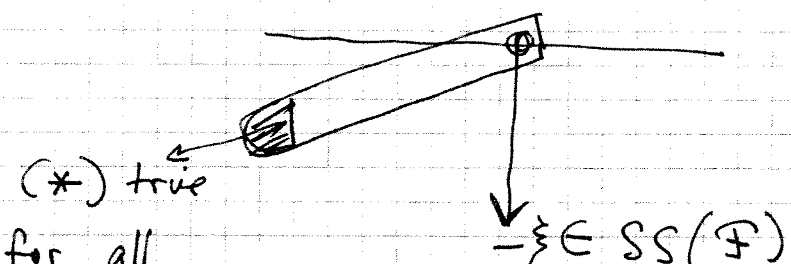


$\underset{\overset{\shortparallel}{0}}{\Gamma(U,\mathcal{F})} \overset{\sim}{\not\to} \underset{\overset{\shortparallel}{0}}{\Gamma(V,\mathcal{F})}$

(*) 

If (*) happens for all ~~ξ-positive~~ negative directions from x: then $(x, \xi) \in SS(\mathcal{F})$.





for all ~~positive~~ negative ξ directions.

$SS(\mathbb{C}_{\{y \geq 0\}}) = \{(x, y; 0, 0) : y > 0\} \cup \{(x, 0; 0, -\eta) : \eta \geq 0\}$

This is the idea of the definition.

$SS(\mathcal{F})$: a closed conic subset of T^*X.

Replace Γ by the complex $R\Gamma$.

Floer cohomology and the Fukaya category

Symplectic manifold (M, ω)

Assumption: $c_1(TM) = 0$

~~CM~~ almost complex structure compatible with ω: $J: TM \circlearrowleft$ $J^2 = -1$

$\omega(x, Jy)$ positive definite symmetric ~~quadra~~ bilinear form

(i.e. $g(x,x) = \omega(x, Jx) > 0$ $x \neq 0$)

(TM viewed as a complex v.bdle)

· ~~Graded~~ Rough idea:

Objects: Lagrangian submanifolds $L \subset M$

· Two objects $L_1, L_2 \rightsquigarrow$ Complex $C^\bullet(L_1, L_2)$

$C^\bullet(L_1, L_2) \xrightarrow{d} C^{\bullet+1}(L_1, L_2)$

· Three objects $L_1, L_2, L_3 \to$ composition: morphism of complexes

$$C^\bullet(L_1, L_2) \otimes C^\bullet(L_2, L_3) \to C^\bullet(L_1, L_3)$$

Not associative, but up to homotopy.

Higher compositions:

$$C^\bullet(L_1, L_2) \otimes C^\bullet(L_2, L_3) \otimes C^\bullet(L_3, L_4) \to C^\bullet(L_1, L_4)$$

of degree -1 ; ...

$$C^\bullet(L_1, L_2) \otimes \dots \otimes C^\bullet(L_n, L_{n+1}) \to C^\bullet(L_1, L_{n+1})$$

of degree n

Condition:

$$\sum \pm m_r(\ldots m_{\ldots}(\_\_)\_\_) = 0$$

(A_∞ category)

a DG category: A_∞ cat with $m_k = 0$, $k \geq 3$.

Rectification: Any A_∞ category $C^\cdot$

$\wr$

DG category $\mathcal{C}^\cdot$ with same objs;

quasi-isoms of complexes $\mathcal{C}^\cdot(L_1, L_2) \xrightarrow{\sim} C^\cdot(L_1, L_2)$

$\forall$ pair of objects L_1, L_2.

Transfer of structure:

Given: a DG category $\mathcal{C}^\cdot$

complexes $C^\cdot(L_1, L_2)$ $\forall L_1, L_2$ - objs

qvis of cplexes $\mathcal{C}^\cdot(L_1, L_2) \xrightarrow{\sim} C^\cdot(L_1, L_2)$

This defines an A_∞ structure on $C^\cdot(-,-)$

So, essentially: to choose an A_∞ category IS to choose a DG category with same objects

OR: An A_∞ category is another, ~~tor~~ usually smaller, model of a dg category.

How to construct the Fukaya category?

Physical Motivation

Recall: a symplectic manifold such as $M = T^*X$: Phase space of a QM system.

a particle moving on X

$(x, \xi) \in T^*X$: x - position

ξ - momentum

$H(x, \xi)$ - Hamiltonian function

$\hat{H} = Op_H : L_2(X) \circlearrowleft$

$L_2(X) = \mathcal{H}$

Schrödinger equation of evolution:

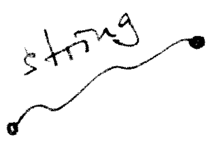
$$\dot{\psi} = \frac{1}{i\hbar}\hat{H}\psi$$

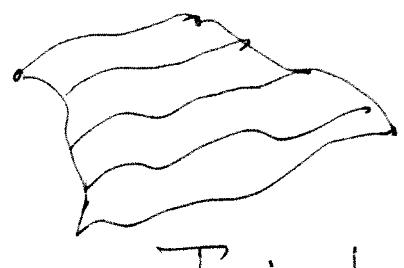
$$\psi(t) = \exp\left(\frac{1}{i\hbar}\hat{H}\right)\psi(0)$$

etc.

<u>Now</u>: string theory.

~~Particle~~ String moves on ~~X~~ M (ev dim; say, complex projective, whence <u>symplectic</u>, space).





Trajectory of a string: <u>surface</u> in M

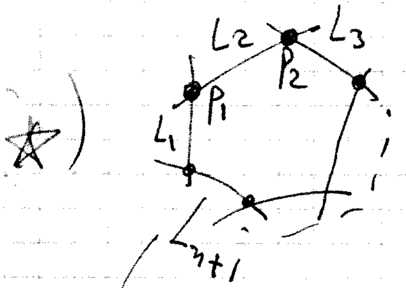
~~Action & trajectory ↔ number~~

Points of new space: curves in M

Function on that: $\int p\,dq - H\,dt$

<u>Critical points</u>: trajectories of the Hamilt. H

<u>Trajectories of grad flow</u>: (pseudo) holomorphic curves...

(☆) 

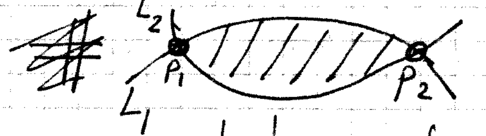
Say, all L_is transverse

$$C^\bullet(L_1, L_2) = \bigoplus_{p \in L_1 \cap L_2} \mathbb{K} \cdot p$$

<u>differential</u>:

$$C^\bullet(L_1, L_2) \to C^{\bullet+1}(L_1, L_2)$$

$$p_1 \longmapsto \sum \pm \, e^{\frac{1}{i\hbar} \text{Area}(\;)}$$



<u>holomorphic</u> films

...

$$m_n(p_1, \ldots, p_n) = \sum_{\text{holom. films } (\star)} \pm \, e^{\frac{1}{i\hbar} \text{Area}(\;)}$$

=

- <u>What</u> is $\sum$ over holom maps $\to M$ with certain boundary conditions?
- degree of m_n must be $2-n$. ~~tdd~~

<u>How</u> are these degrees defined?
How can one choose the signs $\pm$ to achieve $d^2 = 0$ and, more generally, the A_∞ equations?

<u>Answers</u>: a) Lagrangian submanifolds L must be GRADED: the Maslov loc syst triv.

b) Transverse inters pts of graded Lagrs have <u>Maslov index</u> $\in \mathbb{Z}$ $\mu(p)$

c) Provided $\mu(p_{n+1}) = \sum^n \mu(p_i) + 2 - n$, the set of holom films is DISCRETE. Compactness

d) Choice of $\pm$! Lagrs L must be Pin

Deformation quantization of symplectic manifolds

(M, ω)

Local version at a point:

$\hat{\mathbb{A}} = \mathbb{C}[[\hat{x}_1, \ldots, \hat{x}_n, \hat{\xi}_1, \ldots, \hat{\xi}_n, \hbar]] := \mathbb{C}[[\hat{x}, \hat{\xi}, \hbar]]$

$Sp(2n)$-invariant product $*$ (Weyl prodt)

$[\hat{\xi}_k, \hat{x}_\ell] = i\hbar\, \delta_{k\ell}$; $[\hat{\xi}_k, \hat{\xi}_\ell] = [\hat{x}_k, \hat{x}_\ell] = 0.$

$G := \mathrm{Aut}(\hat{\mathbb{A}})$; $\mathfrak{g} := \mathrm{Der}(\hat{\mathbb{A}})$

Fact: $\mathfrak{g} \simeq \frac{1}{i\hbar}\hat{\mathbb{A}} \Big/ \frac{1}{i\hbar}\mathbb{C}[[\hbar]]$ with $a*b - b*a$

(all derivations are of the form $[\frac{1}{i\hbar}F, ?]$ $F \in \hat{\mathbb{A}}$)

Grading: $|\hat{x}_k| = |\hat{\xi}_\ell| = 1$; $|\hbar| = 2$

$$\mathfrak{g} = \prod_{k \geq -1} \mathfrak{g}_k \qquad [\mathfrak{g}_k, \mathfrak{g}_\ell] \subset \mathfrak{g}_{k+\ell}$$

$$\tilde{\mathfrak{g}} := \frac{1}{i\hbar}\hat{\mathbb{A}} \qquad \tilde{\mathfrak{g}} = \prod_{k \geq -2} \tilde{\mathfrak{g}}_k$$

$$0 \to \frac{1}{i\hbar}\mathbb{C}[[\hbar]] \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$$

$$\tilde{\mathfrak{g}}_{\geq 1} := \prod_{k=1}^{\infty} \tilde{\mathfrak{g}}_k , \quad \mathfrak{g}_{\geq 1} = \ldots$$

Fact:

$$G = Sp(2n) \ltimes \exp(\mathfrak{g}_{\geq 1})$$

$$\exp\left(\tfrac{1}{i\hbar}\mathbb{C} \oplus \mathbb{C}\right) \times \widetilde{Sp}(2n) \ltimes \exp(\tilde{\mathfrak{g}}_{\geq 1}) =: \tilde{G}$$

$\mathcal{G}$ is the group of expressions

$$\exp\left(\frac{1}{i\hbar}\left[\underbrace{q(\hat{x},\hat{\xi})}_{\text{quadratic expressions}} + \text{cube}(\hat{x},\hat{\xi}) + \dots\right]\right)$$

$\tilde{\mathcal{G}}$ is the group of expressions

$$\cancel{\exp\left(\frac{1}{i\hbar}a + b + \frac{1}{i\hbar}\right.}$$

$$\exp\left(\frac{1}{i\hbar}\left[(a + i\hbar b + (i\hbar)^2 c + \dots) + q(\hat{x},\hat{\xi}) + \text{cube}(\ ) + \dots\right)\right.$$

in other words, of expressions

$$\exp\left(\frac{1}{i\hbar}\underbrace{\varphi(\hat{x},\hat{\xi},\hbar)}\right)$$

power series in $\hat{x}, \hat{\xi}, \hbar$
with no linear term ~~in~~
$a\hat{x} + b\hat{\xi}$ $\quad a,b \in \mathbb{C}^n$

The Fedosov construction

On a manifold (M,ω):

- a bundle of algebras $\hat{\mathbb{A}}_M$ with fiber $\hat{\mathbb{A}}$ (associated to the tangent bundle which is given by $g_{UV}: U \cap V \to Sp(2n,\mathbb{R})$)
- a flat connection ∇ in $\hat{\mathbb{A}}$

Locally: $\nabla = d + A \qquad A \in \Omega^1(U, \mathfrak{g})$

lifts to a $\tilde{\mathfrak{g}}$-valued connection $\tilde{\nabla}$

Locally: $\tilde{\nabla} = d + \tilde{A} \qquad \tilde{A} \in \Omega^1(U, \tilde{\mathfrak{g}})$

Can be chosen: $\nabla^2 = \frac{1}{i\hbar}\omega$

The local picture: (x, ξ) - Darboux coords

Local sections of $\hat{\mathbb{A}}_M$: $a(x, \xi; \underbrace{\hat{x}, \hat{\xi}, \hbar}_{\text{formal variables}})$

Product: Weyl product in $\hat{x}, \hat{\xi}, \hbar$

Connection ∇: $\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \hat{x}}\right) dx + \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \hat{\xi}}\right) d\xi$

$$\nabla^2 = 0$$

Lifted connection $\tilde{\nabla}$:

$$\tilde{\nabla} = \left(\frac{\partial}{\partial x} - \frac{1}{i\hbar}\hat{\xi}\right) dx + \left(\frac{\partial}{\partial \xi} + \frac{1}{i\hbar}\hat{x}\right) d\xi$$

$$\tilde{\nabla}^2 = \frac{1}{i\hbar} d\xi\, dx = \frac{1}{i\hbar}\omega$$

Flat sections of ∇: $f(x+\hat{x}, \xi+\hat{\xi}; \hbar)$ where $f(x, \xi)$ is a C^∞ function. More precisely: $\sum \frac{\partial_x^p \partial_\xi^q f(x, \xi)}{p!\, q!} \cdot \hat{x}^p \hat{\xi}^q$

The product on

$$C^\infty(M)^{[[\hbar]]} \simeq \ker\left(\nabla : \Gamma(M, \hat{\mathbb{A}}_M) \to \Omega^1(M, \hat{\mathbb{A}}_M)\right)$$

is DEFORMATION QUANTIZATION of M.

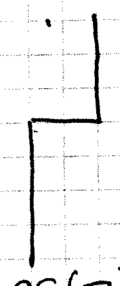
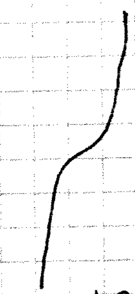
Sheaf-theoretical microlocal methods in symplectic geometry. The works of Nadler-Zaslow and Tamarkin.

Nadler-Zaslow:

[Derived category of constructible sheaves on X] $\xrightarrow{\sim}$ [Derived cat of the Fukaya category of T^*X]

Sheaf $\rightsquigarrow$ Lagrangian submfld:

$\mathcal{F} \rightsquigarrow SS(\mathcal{F})$ (conical!) $\rightsquigarrow$ its smoothing

$\mathcal{F} = \mathbb{C}_{[0,1]}$ $\rightsquigarrow$  $\rightsquigarrow$  [$L = \varphi(SS(\mathcal{F}))$ for some strictly Hamiltonian φ]

$SS(\mathcal{F})$ $\qquad$ L-smoothing of $SS(\mathcal{F})$

General construction of homological algebra: DG (or A_∞) category

$\rightsquigarrow$ DERIVED CATEGORY

Tamarkin: A version of NZ with ideas how to glue a category for M out of categories for Darboux charts

$$U_i \underset{\text{open}}{\hookrightarrow} T^*X_i$$

Objects: (complexes of) sheaves on $\underset{(x,\,t)}{X \times \mathbb{R}}$

with extra condition:

$$SS(\mathcal{F}) \cap \{\tau < 0\} = \emptyset$$

Examples of such:

(Will be associated to a Lagrangian ~~$\xi = d\varphi$~~ $\xi = \varphi'(x)$)

$$\mathbb{C}_{\{t-\varphi(x)\geq 0\}}$$

More generally, to $L = \{(x,\xi) : \xi = \varphi_x(x,\theta);\ \varphi_\theta(x,\theta)=0\}$

$$\mathrm{proj}_* \,\mathbb{C}_{\{t-\varphi(x,\theta)\geq 0\}}$$

where $\mathrm{proj} : (x,\theta) \mapsto x$

Microsupport of such:

$$SS(\mathcal{F}) \cap \{\tau = 1\}$$

~~Thus~~ The DG subcategory with objects $\{\xi = \varphi'(x)\}$ (or graph $(d\varphi)$)

is isomorphic to the following:

objects: Morse functions φ

$\mathrm{Mor}(\varphi, \psi) = (\text{Witten}) - \text{Morse complex of } \varphi - \psi$

~~By A~~

By Morse theory: this computes $H^\bullet(X)$.

$C^\bullet(\varphi,\psi) =$ Morse ~~function~~ complex of $\varphi-\psi$

Kontsevich-Soibelman

Fukaya subcat of objects graph$(d\varphi)$

Morse theory

quis

isom (Tamarkin)

$C^\bullet(\varphi,\psi) \equiv C^\bullet(X)$
composition = cup product

Tamarkin's subcat of objs ~~gr~~ $\{t \geq \varphi(x)\}$

equivalence

partial case of Nadler-Zaslow

Subcategory with one object $\varphi = 0$

$\mathbb{R}\mathrm{Hom}_{\text{sheaves}}(\mathbb{C}_X, \mathbb{C}_X)$
$\cong$
$C^\bullet(X)$

<u>So</u>: Tamarkin's correspondence between microlocal and Fukaya is much more faithful. <u>Also</u>: good prospects to glue and globalize.

<u>For example</u>: $M = \mathbb{T}^2$;

Tamarkin defines $\mathbb{Z}^2$-equivariant objects of his objects on $\mathbb{R}^2$.

Gets the correct answer (i.e. Polishchuk-

Tamarkin's work on microlocal methods

$\mathcal{D}(X)$ - derived category of sheaves on X

- $f : X \to Y$ $\quad f^{-1}, f^{!} : \mathcal{D}(X) \leftarrow \mathcal{D}(Y)$

$f_*, f_! : \mathcal{D}(X) \to \mathcal{D}(Y)$

subject to adjunction formulas

- Functorial properties of $SS(\mathcal{F})$ under $f^{-1}, \ldots$:

cf. Kashiwara-Schapira's book

Objects: $\mathcal{F} \in \mathcal{D}(\underset{x}{X} \times \underset{t}{\mathbb{R}})$ s.t. $\mathcal{F} \in {}^{\perp}\mathcal{C}_{\leq 0}$ (

where:

$$\mathcal{C}_{\leq 0} = \{ \mathcal{G} : SS(\mathcal{G}) \subseteq \{\tau \leq 0\} \}$$

(here $T^*\mathbb{R} \simeq \{(t, \tau)\}$);

$${}^{\perp}\mathcal{C}_{\leq 0} = \{ \mathcal{F} : \mathbb{R}\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0, \ \forall \mathcal{G} \in \mathcal{C}_{\leq 0} \}$$

$\cong$

$a : X \times \mathbb{R} \times \mathbb{R} \longrightarrow X \times \mathbb{R} \qquad (x, t_1, t_2) \mapsto (x, t_1 + t_2)$

Convolution with sheaves on $\mathbb{R}$:

$\mathcal{F} * S = \mathbb{R} a_! (\mathcal{F} \boxtimes S)$; $\mathcal{F} = \mathcal{F} * \mathbb{K}_{\{0\}}$

Decomposition of $\mathcal{F}$ according to $\mathcal{C}_{\leq 0}$:

$$0 \to \underset{\displaystyle \overset{\cap}{\mathcal{C}_{\leq 0}}}{\mathbb{K}_{(0,\infty)}} \longrightarrow \mathbb{K}_{[0,\infty)} \longrightarrow \mathbb{K}_{\{0\}} \longrightarrow 0$$

$$\mathcal{F} * \mathbb{K}_{[0,\infty)} \longrightarrow \mathcal{F} \longrightarrow \underbrace{\mathcal{F} * \mathbb{K}_{(0,\infty)}[1]}_{\in \mathcal{C}_{\leq 0}}$$

Therefore: $\mathcal{F} \in {}^{\perp}\mathcal{C}_{\leq 0}$ iff

$$\mathcal{F} * \mathbb{K}_{[0,\infty)} \xrightarrow{\sim} \mathcal{F}$$

Action of translations:

$$T_c : (x, t) \mapsto (x, t+c)$$

Morphisms

$$\begin{tikzcd}
\tau_c : T_{c*}\mathcal{F} \arrow[r] \arrow[d, phantom, "\wr"] & \mathcal{F} \arrow[d, phantom, "\wr"] \\
\mathcal{F} * \mathbb{K}_{[c,\infty)} & \mathcal{F} * \mathbb{K}_{[0,\infty)}
\end{tikzcd}
\qquad \boxed{\begin{array}{c} c \geq 0 \\ \mathcal{F} \in {}^{\perp}\mathcal{C}_{\leq 0} \end{array}}$$

$\equiv$

Microsupport: $\mathcal{F} \in {}^{\perp}\mathcal{C}_{\leq 0}$

$$\mu S(\mathcal{F}) = \{ (x, \xi) \in T^*X : \exists t,\ (x, \xi, t, 1) \in SS(\mathcal{F}) \}$$

$$\mathcal{D}_A(X) = \{ \mathcal{F} \in {}^{\perp}\mathcal{C}_{\leq 0} : \mu S(\mathcal{F}) \subseteq A \}$$

$\equiv$

Convolutions, Fourier transforms

$\mathcal{F}_1 \in \mathcal{D}(X_1 \times X_2 \times \mathbb{R}) \qquad \mathcal{F}_2 \in \mathcal{D}(X_2 \times X_3 \times \mathbb{R})$

$$\begin{tikzcd}
 & (x_1, x_2, x_3, t_1, t_2) \arrow[dl, "p_{12}"'] \arrow[d, "p_{13}"] \arrow[dr, "p_{23}"] & \\
(x_1, x_2, t_1) & (x_1, x_3, t_1 + t_2) & (x_2, x_3, t_2)
\end{tikzcd}$$

$$\mathcal{F}_1 \underset{X_2}{\bullet} \mathcal{F}_2 = p_{13!}\left(p_{12}^{-1}(\mathcal{F}_1) \otimes p_{23}^{-1}(\mathcal{F}_2) \right)$$

- $(\mathcal{F}_1 \bullet \mathcal{F}_2) \bullet \mathcal{F}_3 \simeq \mathcal{F}_1 \bullet (\mathcal{F}_2 \bullet \mathcal{F}_3)$

or $(\mathbb{K}_{[0,\infty)} * \mathcal{F}_1) \bullet \mathcal{F}_2 \simeq \mathbb{K}_{[0,\infty)} * (\mathcal{F}_1 \bullet \mathcal{F}_2)$

Thus: $\bullet : \mathcal{C}_{\leq 0} \times \mathcal{C}_{\leq 0} \to \mathcal{C}_{\leq 0}$

Is it true that microsupports behave well under general convolution? Yes, if all microsupps are compact. (Prop. 3.11)

Fourier transforms:

$\mathbb{K}_G = \mathbb{K}_{\{t - x\cdot p \geq 0\}}$ on $E \times E^\vee$

$\mathbb{K}_\Gamma = \mathbb{K}_{\{t + p\cdot x \geq 0\}}$ on $E^\vee \times E$

$$\mathcal{D}(E) \underset{\Phi = ? \bullet_{E^\vee} \mathbb{K}_\Gamma}{\overset{F = ? \bullet_E \mathbb{K}_G}{\rightleftarrows}} \mathcal{D}(E^\vee)$$

Thm F and $\Phi[+n]$ are inverse

Pf: $\mathbb{K}_G \bullet \mathbb{K}_\Gamma = \mathbb{K}_{\{(x_1, x_2, t):\ x_1 = x_2;\ t \geq 0\}}[-n]$

$A \underset{E}{\bullet}$ —— " —— $\simeq A * \mathbb{K}_{[0,\infty)}[-n]$

Thm Microsupport behaves under F via $(x, \xi) \mapsto (\xi, -x)$.

Used: functorial properties of SS, including: Lemma 3.3 for $p : X \times E \to X$

$SS(Rp_* \mathcal{F}),\ SS(Rp_! \mathcal{F}) \subseteq \{(x, \xi) : \exists v \in E\ (x, \xi, v, 0) \in SS(\mathcal{F})\}$

(functorial property of $SS(\mathcal{F})$ under $Rp_{!,*}$ with properness condition relaxed).

Thm Let $\mathcal{F}_1 \in D_{A_1}(X)$, $\mathcal{F}_2 \in D_{A_2}(X)$, A_1, A_2 compact. Then, if $A_1 \cap A_2 = \phi$,

$$R\mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2) = 0.$$

Proof 1) For any sheaf S on $\mathbb{R}$:

$$(\star) \qquad \mathbb{R}\mathrm{Hom}_{X\times\mathbb{R}}(\mathcal{F}_1 * S, \mathcal{F}_2) = \mathbb{R}\mathrm{Hom}_{\mathbb{R}}(S, ?(\mathcal{F}_1, \mathcal{F}_2))$$

$?(\mathcal{F}_1, \mathcal{F}_2)$ found using adjunction formulas.

Actually,

$$\begin{tikzcd} & X\times\mathbb{R}\times\mathbb{R} \arrow[dl, "p_1"'] \arrow[dr, "\pi"] & \\ X\times\mathbb{R} & & \mathbb{R} \end{tikzcd}$$

$$\mathbb{R}\mathrm{Hom}(\mathcal{F}_1 * S, \mathcal{F}_2) = \mathbb{R}\mathrm{Hom}(Rq_!(p_1^{-1}\mathcal{F}_1 \otimes \pi^{-1}S), \mathcal{F}_2)$$
$$= \mathbb{R}\mathrm{Hom}(p_1^{-1}\mathcal{F}_1 \otimes \pi^{-1}S, q^!\mathcal{F}_2) = \mathbb{R}\mathrm{Hom}(\pi^{-1}S, \mathbb{R}\underline{\mathrm{Hom}}(p_1^{-1}\mathcal{F}_1, q^!\mathcal{F}_2))$$
$$= \mathbb{R}\mathrm{Hom}(S, R\pi_* \mathbb{R}\underline{\mathrm{Hom}}(p_1^{-1}\mathcal{F}_1, q^!\mathcal{F}_2)), \text{ so}$$

$$?(\mathcal{F}_1, \mathcal{F}_2) = R\pi_* R\underline{\mathrm{Hom}}(p_1^{-1}\mathcal{F}_1, q^!\mathcal{F}_2)$$

OR: $X\times\mathbb{R}\times\mathbb{R} \xrightarrow{p_2} X\times\mathbb{R} \xrightarrow{q} \mathbb{R}$

$(x, t_1, t_2) \longmapsto (x, t_2) \longmapsto t_2$

$$?(\mathcal{F}_1, \mathcal{F}_2) = \mathbb{R}q_* \mathcal{H}; \quad \mathcal{H} = Rp_{2*} R\underline{\mathrm{Hom}}(p_1^{-1}\mathcal{F}_1, q^!\mathcal{F}_2)$$

2) Now: One finds $SS(\mathbb{R}q_* \mathcal{H})$ using functorial properties of SS wrt $Rq_{..}$, p_1^{-1}, Hom

One gets $SS(Rq_* \mathcal{H}) \subset$ (zero section of $T^*\mathbb{R}$). (

For this, one needs

<u>Lemma 3.7.</u> If $S \in \mathcal{D}_A(X)$, A <u>compact</u>, then

$$SS(S) \subset \{\tau > 0\} \cup \{\xi = \tau = 0\}$$

(the Fourier transform is used in the proof).

3). We have : a) $R\mathcal{H}om_{\mathbb{R}}(\mathbb{K}_{\{0\}}, Rq_* \mathcal{H}) = R\mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2)$

b) $R\mathcal{H}om_{\mathbb{R}}(\mathbb{K}_{\mathbb{R}}, Rq_* \mathcal{H}) = 0$

c) $Rq_* \mathcal{H}$ is a constant sheaf

a) follows from 1), (✯); b) from 1), (✶) and from $\mathbb{K}_{[0,\infty)} * \mathbb{K}_{\mathbb{R}} = 0$ (fibers of a are half-lines, and their $H_c^{\bullet}$ is zero); c), from 2). Combining, we get

$R\mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2) = 0$ if $A_1 \cap A_2 = \emptyset$. ∎

≡

Invariance under Hamiltonian flows

Let $\phi : T^*X \xrightarrow{\sim} T^*X$, $\phi = id$ outside a compact, be Hamiltonian.

Let $L \Subset T^*X$.

1) ϕ induces $T_\phi : \mathcal{D}_L(X) \to \mathcal{D}_{\phi(L)}(X)$

2) The functor T_ϕ is "isomorphic to id modulo torsion", i.e.:

$$\begin{tikzcd}
\mathcal{D}_L(X) \arrow[r, bend left=70, "id"] \arrow[r, phantom, bend left=35, "\Uparrow s_1"] \arrow[r, shift left=1.5] \arrow[r, phantom, "\Downarrow s_2"] \arrow[r, shift right=1.5] \arrow[r, phantom, bend right=35, "\vdots"] \arrow[r, bend right=70, "T_\phi"'] & \mathcal{D}(X)
\end{tikzcd}$$

s_i - natural transformations of functors; $\forall \mathcal{F}$, $\mathrm{cone}(\mathcal{F} \to s_i(\mathcal{F}))$ is <u>torsion</u> (i.e. τ_c acts by zero).

<u>Proof</u> a) $\phi = \phi_1 \dots \phi_N$, ϕ_i "small" (= id outside a coord nbhd)

Enough to prove for a small ϕ.

b) Let $\Sigma(x_1, x_2, \theta)$ be a phase function of

$$L_\phi = \text{graph}(\phi) = \{(x, x', p, -p') : (x', p') = \phi(x, p)\}$$

$$\|$$

$$\{(x, x', p, -p') : p = \Sigma_x (x, x', \theta) ; p' = \Sigma_{x'} (x, x', \theta) ;$$

$$\Sigma_\theta (x, x', \theta) = 0\}$$

T_ϕ = integral transform with kernel

$$R q_! \, \mathbb{K}_{\{t + \Sigma(x, x', \theta) \geq 0\}} \qquad (*)$$

where $q : (x, x', \theta, t) \mapsto (x, x', t)$.

It <u>does</u> send $\mathcal{D}_L$ to $\mathcal{D}_{\phi(L)}$.

Actually, Σ is chosen in a special way:

- There is a requirement that, if

$$\phi(x, p) = (x', p'),$$

then (x, p') is a coordinate system (ϕ close to id - no problem).

- $p\,dx + x'\,dp' = d(S(x, p') + xp')$, so:

$$p - S_x = p'$$
$$x + S_{p'} = x'$$

($\exists S$, supported on a nbhd of 0.)

$$\Sigma(x, x', p') = \cancel{(x_1 - x_2)p' - S(x} - (x - x')p' - S(x, p')$$

Notation: $\Lambda_{S,\pi}$ is the kernel $(*)$, p. 7; (π is the origin in the p-space in whose neighborhood we work).

$$S_+(x,p') = \begin{cases} S(x,p') & \text{if } \geq 0 \\ 0 & \text{if } S(x,p') \leq 0 \end{cases}$$

$$L_S \longrightarrow L_{S_+} \longleftarrow L_0$$

Cones of $L_S \to L_{S_+}$, $L_0 \to L_{S_+}$ are torsion objects in $D(X \times X \times \mathbb{R})$. Indeed: they are zero away from $\min S \leq t \leq \max S$. For such $\mathcal{F}$, $\mathbb{R}\mathrm{Hom}(T_{c*}\mathcal{F}, \mathcal{F}) = 0$ $c \gg 0$