On block identities and block inclusions

JØRN B. OLSSON

(joint work with D. Stanton, C. Bessenrodt, G. Navarro, P.H. Tiep)

1. The Navarro-Willems conjecture

Let G be a finite group. We consider for a prime p and a p-block B_p of G the set $\mathrm{Irr}(B_p)$ of irreducible complex characters of B_p. It was conjectured by Navarro and Willems [4], that if for different primes p, q we have a *block equality* $\mathrm{Irr}(B_p) = \mathrm{Irr}(B_q)$ then $|\mathrm{Irr}(B_p)| = 1$. Thus both blocks should be of defect 0. We call such an equality *trivial*.

The Navarro-Willems conjecture holds for all blocks in solvable groups [4], for all blocks blocks in the symmetric groups [5] and their covering groups [3]. Also in [2] the conjecture was verified for *principal blocks* in all finite groups by reducing the question to simple groups.

It has however been noticed by C. Bessenrodt that the extension group $6.A_7$ of the alternating group A_7 provides a counterexample to the conjecture for non-principal blocks $(p, q = 5, 7)$. Such counterexamples are expected to be rare. It would be interesting to have more.

2. Block identities

The fact that all block identities in the symmetric groups and their covering groups are trivial follows from the classification of the nontrivial block inclusions in these groups, as described below. There is an interesting explicit description of all block identities for different primes p and q ([5], [3]).

In the case of blocks of the symmetric groups, the number of such block identities is finite and in fact equal to $\frac{1}{p+q}\binom{p+q}{q}$. The maximum n for which an equality occurs is $n = \frac{(p^2-1)(q^2-1)}{24}$. ([1],[5]).

The above results of course involve the classification of partitions which are cores for two primes simultaneously, due to the Nakayama conjecture. Actually for the classification p and q need not be prime numbers, only relatively prime positive integers. There is a unique partition κ_{pq} of the maximum number $\frac{(p^2-1)(q^2-1)}{24}$, which is a p-core and a q-core. It is still an open question whether the Young diagram of any other partition which is a p-core and a q-core is contained in that of κ_{pq}. See [5] for details.

In the case of spin blocks of the covering groups the number of block identities is again finite. In this case we have to classify partitions into distinct parts which are bar cores for p and q simultaneously. Again p and q need only be relatively prime *odd* positive integers, not necessarily primes. The total number of spin block identities is $\binom{s+t}{t}$ where $s = \frac{p-1}{2},\ t = \frac{q-1}{2}$. In this case it can be shown that there is a maximal partition $\hat{\kappa}_{pq}$, whose Young diagram contains those of all the others.

In both cases the possibilites are descibed by paths in certain diagrams of integers. As an example the shown diagram is the so-called (7,17)-*Yin-Yang diagram.* It has been taken from [3] and shows the possible parts of all partitions into distinct

10	3	4	11	18	25	32	39
27	20	13	6	1	8	15	22
44	37	30	23	16	9	2	5

parts which are bar cores for the primes 7 and 17. (In this diagram the numbers 10,3,27,20,13... below the dividing line are exactly the parts in the partition $\hat{\kappa}_{7\ 17}$.)

Let us finally mention that if $p_1, p_2, ..., p_k$ are distinct odd primes then there exists another prime number q such that $p_1 p_2 ... p_k \mid q+1$. This shows that the group $\mathrm{GL}(3,q)$ contains a unipotent irreducible character which is of defect 0 for all the primes $p_1, p_2, ..., p_k$. Thus we may have simultaneous (trivial) block identites for arbitrarily many primes.

3. Block inclusions

More generally nontrivial *block inclusions* $\mathrm{Irr}(B_p) \subseteq \mathrm{Irr}(B_q)$ in a finite group G may be studied. We call the inclusion *trivial* if $|\mathrm{Irr}(B_p)| = 1$, i.e., if the smaller block has defect 0. Nontrivial block inclusions occur frequently, for instance if G has a selfcentralizing normal q-subgroup; then G has only one q-block and for any p-block of G of positive defect we get a nontrivial inclusion.

It is possible to classify all nontrivial block inclusions $\mathrm{Irr}(B_p) \subseteq \mathrm{Irr}(B_q)$ in the the symmetric groups and their covering groups. It can only happen when the "small" block B_p has defect 1 and the core of the block has a very special property. However the number of occurrencies is infinite.

This classification easily implies that all block identities in these groups are trivial, as mentioned above.

In the case of blocks of symmetric groups the partition κ_{pq} plays a special rôle as a kind of treshold: It is the smallest p-core which can occur as the core for a block B_p in a non-trivial block inclusion $\mathrm{Irr}(B_p) \subseteq \mathrm{Irr}(B_q)$. Thus when the block identities stop, then the non-trivial block inclusions start! A similar statement holds for spin blocks when κ_{pq} is replaced by $\hat{\kappa}_{pq}$.

References

[1] J. Anderson, *Partitions which are simultaneously t_1- and t_2-core* Discrete Math. **248** (2002), 237–243.

[2] C. Bessenrodt, G. Navarro, J.B. Olsson and P.H. Tiep, *On the Navarro-Willems conjecture for blocks of finite groups.* Preprint 2005. To appear in J. Pure Applied Algebra.

[3] C. Bessenrodt, J.B. Olsson, *Spin block inclusions.* Preprint 2005. To appear in J. Algebra.

[4] G. Navarro, W. Willems, *When is a p-block a q-block?* Proc. Amer. Math. Soc. **125** (1997), 1589-1591

[5] J.B. Olsson, D. Stanton, *Block inclusions and cores of partitions*, Preprint 2005. To appear in Aequationes Math.