Spin block inclusions

Christine Bessenrodt
Fakultät für Mathematik und Physik, Universität Hannover
Welfengarten 1, D-30167 Hannover, Germany

Jørn B. Olsson
Matematisk Institut, Københavns Universitet
Universitetsparken 5,DK-2100 Copenhagen Ø, Denmark

November 16, 2005
Mathematics Subject Classifications: 20C30, 05A17

Abstract

In this paper we classify all block equalities and all nontrivial block inclusions for spin blocks of the double covers of the symmetric groups at different odd primes. More generally, we describe for an odd integer $s > 1$ explicitly when an $\bar{s}$-block of bar partitions is contained in a $\bar{t}$-block of bar partitions, for $t > 1$ an odd integer or $t = 4$. The question of block equality leads to the study of $(\bar{s}, \bar{t})$-cores. In the case of primes they label spin characters which are of defect 0 for different primes and therefore represent a block equality. We enumerate these cores and show that there is a unique maximal one.

1 Introduction

Let G be a finite group. We consider for a prime p a p-block B_p of G simply as a subset (usually denoted $\mathrm{Irr}(B_p)$) of the set $\mathrm{Irr}(G)$ of irreducible complex characters of G. It was conjectured by Navarro and Willems [8], that if for different primes p, q we have a *block equality* $B_p = B_q$ then $|B_p| = 1$. Recently it was noticed by the first author that the extension group $6.A_7$ of the alternating group A_7 provides a counterexample to the conjecture for non-principal blocks ($p, q = 5, 7$). We expect such counterexamples to be rare.

More generally nontrivial *block inclusions* $B_p \subseteq B_q$ in a finite group G may be studied. We call the inclusion *trivial* if $|B_p| = 1$, i.e., if the smaller block has defect 0. Nontrivial block inclusions occur for instance if G has a selfcentralizing normal q-subgroup; then G has only one q-block and for any p-block of G of positive defect we get a nontrivial inclusion. (Thus we get examples of nontrivial inclusions of a p-block in a q-block in semidirect products of an elementary abelian q-group of order q^d and a non-trivial p-subgroup of $\mathrm{GL}(d, q)$.)

We are dealing here with spin analogues of results in [10]. In that paper all nontrivial block inclusions for symmetric groups were classified; here we treat a similar problem for spin blocks of the covering groups $\tilde{S}_n$ of S_n. This translates to a question on block inclusions of bar partitions, where the blocks are determined by their bar cores.
As in [10], the primeness of p and q is not essential for our arguments. Thus on a more combinatorial level we classify for arbitrary different odd numbers $s, t \in \mathbb{N}$, $s > 1, t \nmid s$ the $\bar{s}$-blocks of bar partitions which are contained in a $\bar{t}$-block of bar partitions. We show that a nontrivial inclusion can happen only if the $\bar{s}$-block has bar weight 1 and its core has special properties. This leads us to the study of $\bar{t}$-good $\bar{s}$-cores and of $(\bar{s}, \bar{t})$-cores. (When p and q are odd primes, the $(\bar{p}, \bar{q})$-cores are exactly the labels of the spin characters which are simultaneously of defect 0 for p and q; see Remark 3.3 below.) We show for instance that $(\bar{s}, \bar{t})$-cores are parameterized by paths in a certain rectangular diagram of integers called the Yin-Yang diagram; in particular, their number is finite.
For any given different odd primes p and q it follows that whereas the number of nontrivial *spin block inclusions* $B_p \subseteq B_q$ is infinite, the number of *spin block equalities* $B_p = B_q$ is finite and in fact equal to $\binom{p_1+q_1}{p_1}$, with $p_1 = (p-1)/2, \quad q_1 = (q-1)/2$.

There are some significant differences between the results of this paper for bar partitions and those of [10] for partitions. For instance, in contrast to [10] there seems to be no simple generating function for the number of "good" partitions. On the other hand, an apparently difficult open problem about the containment of arbitrary (s, t)-cores in a maximal one [10, 4.11] is settled below for bar cores.

The paper is organized as follows: In section 2 we first give a brief exposition of the necessary facts from the theory of spin characters of $\tilde{S}_n$ and their labels, leading to the definition of $\bar{t}$-good $\bar{s}$-cores. They are the possible cores of the $\bar{s}$-blocks which may be contained non-trivially in a $\bar{t}$-block. The main result Theorem 2.3 describes exactly when this occurs. In Theorem 2.6 we list all possible non-trivial block inclusions in $\tilde{S}_n$. As a corollary, the Navarro-Willems question about block equality is answered positively for spin blocks and thus it is true for all blocks in $\tilde{S}_n$. Block equality for spin blocks can only happen for blocks which are simultaneously of defect 0 for two different odd numbers. This leads to the study of $(\bar{s}, \bar{t})$-cores in the final section. The main results are the enumeration of these partitions (Theorem 3.2) and the fact that there is a maximal such partition containing all the others (Theorem 3.6).

2 Block inclusions

For background on the theory of spin characters of $\tilde{S}_n$ we refer the reader to [4].
The associate classes of spin characters of a double covering group $\tilde{S}_n$ of S_n are labelled canonically by the partitions λ of n into distinct parts, i.e., $\lambda = (a_1, a_2, \cdots, a_m)$, $a_1 > a_2 > \cdots > a_m$ positive integers with $|\lambda| = a_1 + \ldots + a_m = n$; we call such

partitions *bar partitions*. Since we may identify a bar partition with the set of its parts, we adopt some set theoretic notions for it. Thus for example $a \in \lambda$ signifies that a is a part of the bar partition λ. In this case $\lambda \setminus \{a\}$ is then the partition obtained by deleting the part a from λ. For convenience, we will denote the partition obtained by reordering the parts of a composition α by α^o.
For spin characters there is an analogue to the so-called Nakayama conjecture which describes the distribution of the irreducible characters of S_n into p-blocks in terms of the p-cores of the partitions labelling them: The partition labels of characters in the same p-block should have the same p-core. This analogue to the Nakayama Conjecture, referred to as the Morris Conjecture, asserts that for an odd prime p, the $\bar{p}$-cores determine the p-blocks of spin characters of $\tilde{S}_n$; this was proved by Humphreys [5] and Cabanes [3].

A detailed description of the $\bar{p}$-combinatorics may be found in [9, Section 4]. A bar partition λ has a $\bar{p}$-core $\lambda_{(\bar{p})}$, which is obtained from λ by removing as many p-bars from it as possible; the number of removed p-bars is the $\bar{p}$-weight $w_{\bar{p}}(\lambda)$.

A tool for bar partitions corresponding to the well-known p-abacus is the $\bar{p}$-abacus. The $\bar{p}$-abacus has p runners, running from north to south. The i'th runner contains positions numbered $i, p+i, 2p+i, \ldots$ in increasing order from north to south. For $1 \leq i \leq \frac{p-1}{2}$ the runners numbered i and $p-i$ are called *conjugate*, and a pair of conjugate runners is simply referred to as a *runner pair*. The *abacus configuration* of a bar partition is obtained by placing the parts of the partition as beads on the $\bar{p}$-abacus. A bar partition is then a $\bar{p}$-core if (in the abacus configuration)
(i) the 0'th runner is empty,
(ii) at most one runner in a runner pair is non-empty and
(iii) a non-empty runner contains only beads in the top positions.

For $p = 2$, the block distribution of spin characters of $\tilde{S}_n$ was determined in [2]. Note that in contrast to the case of odd p, the 2-blocks are "mixed", i.e., they contain both ordinary and spin characters. In particular, no 2-block of $\tilde{S}_n$ is of defect 0. The combinatorics in this case may be viewed as a $\bar{4}$-combinatorics. Indeed, we have a $\bar{4}$-abacus with one runner for all even parts (the 0'th runner), on which we can slide by steps of 2, and two conjugate runners for the residues 1 and 3 modulo 4. A bar partition is then a $\bar{4}$-core exactly if the 0'th runner is empty (i.e., there are no even parts), at most one of the two conjugate runners is non-empty, and a non-empty runner has only beads at the top. We will call the corresponding combinatorial block of bar partitions of n a $\bar{4}$-block, and we will denote the $\bar{4}$-core of a bar partition λ by $\lambda_{\bar{4}}$.

From a combinatorial point of view, the notions of bar cores and bar blocks may in analogy to the case of odd primes be defined for any odd $s \in \mathbb{N}$, $s > 1$. The set of bar partitions of n having the same $\bar{s}$-core κ forms the $\bar{s}$-*block* to κ of ($\bar{s}$-)weight $\frac{1}{s}(n - |\kappa|)$. The number of bar partitions in a $\bar{s}$-block of weight w is denoted $\bar{k}(s, w)$. Below, we use the fact that $\bar{k}(s, 1) = \frac{s+1}{2}$ [9, Remark 4.8].

Remark 2.1 Let s, t be odd integers, $s, t > 1$.
(i) If $t \mid s$, then any two bar partitions with the same $\bar{s}$-core also have the same $\bar{t}$-core. Hence any $\bar{s}$-block is contained in a $\bar{t}$-block.
(ii) Any $\bar{s}$-block of weight 0 is contained in some $\bar{t}$-block.

Block inclusions of these two types are called *trivial*.

Let κ be an $\bar{s}$-core, $t \in \mathbb{N}$, $t \neq s$ (we emphasize that here we may also take $t = 4$). We call κ *$\bar{t}$-good*, if the following conditions are satisfied:
(i) If a part $a \in \kappa$ is s-maximal, i.e., $a + s \notin \kappa$, then $t \mid a$ or $t \mid a + s$.
(ii) If $i \in \{1, \ldots, s-1\}$ is such that $\{i, s-i\} \cap \kappa = \emptyset$, then $t \mid i$ or $t \mid s - i$.

Remark 2.2 We can have a $\bar{t}$-good $\bar{s}$-core κ only when $\gcd(s, t) = 1$: Note that if $\gcd(s, t) \neq 1$, then the runner pair to $1, s-1$ in the $\bar{s}$-abacus has to be empty for a $\bar{t}$-good $\bar{s}$-core κ by condition (i); but then condition (ii) forces $t \mid s - 1$, a contradiction.

Before we state the main result of this section we give an example to illustrate the non-trivial block inclusions appearing in Theorem 2.3:

Example. We take $s = 7, t = 3$; then $\kappa = (9, 2)$ is a $\bar{3}$-good $\bar{7}$-core. The $\bar{7}$-block B of weight 1 with this bar core consists of the bar partitions $(9, 7, 2)$, $(16, 2)$, $(9, 4, 3, 2)$, $(9, 6, 2, 1)$. They all have empty $\bar{3}$-core, and thus B is contained in a $\bar{3}$-block.

Theorem 2.3 *Let $s > 1$ be an odd integer. Let $t > 1$ be an odd integer with $t \nmid s$ or let $t = 4$. Let B be an $\bar{s}$-block, with $\bar{s}$-core κ, weight $w \geq 1$. If $w \geq 2$, then B is not contained in any $\bar{t}$-block.*
If $w = 1$, then B is contained in a $\bar{t}$-block if and only if κ is $\bar{t}$-good. This can only happen if $\gcd(s, t) = 1$.

Proof. First we discuss the case of blocks of weight 1. Let B be an $\bar{s}$-block with $\bar{s}$-core κ and weight $w = 1$. Certainly $\lambda = \kappa \cup \{s\} \in B$.
Assume first that B is contained in some $\bar{t}$-block, i.e., all bar partitions in B have the same $\bar{t}$-core. If $a \in \kappa$ is s-maximal, consider the bar partition $\mu = \kappa \cup \{a + s\} \setminus \{a\}$ in B. As $\mu_{(\bar{t})} = \lambda_{(\bar{t})}$, we have $(a, s)^o_{(\bar{t})} = (a + s)_{(\bar{t})}$. As $t \nmid s$, this can only happen if $a + s \equiv 0 \mod t$ or $a \equiv 0 \mod t$. Thus condition (i) of $\bar{t}$-good partitions is satisfied for κ. Now let $i \in \{1, \ldots, s-1\}$ be such that $\{i, s-i\} \cap \kappa = \emptyset$. Then $\mu = \kappa \cup \{i, s-i\} \in B$. Now $\mu_{(\bar{t})} = \lambda_{(\bar{t})}$ implies $(s)_{(\bar{t})} = (i, s-i)^o_{(\bar{t})}$, and hence we must have $t \mid i$ or $t \mid s - i$. Thus, κ is $\bar{t}$-good. By Remark 2.2 we have $(s, t) = 1$ in this case.
If B is an $\bar{s}$-block with $\bar{s}$-core κ and weight $w = 1$, then all bar partitions in B are of one of the forms just discussed, and the conditions of κ being $\bar{t}$-good guarantee that they all have the same $\bar{t}$-core as λ.

We now have to prove that an $\bar{s}$-block of weight $w \geq 2$ cannot be contained in any $\bar{t}$-block. Let κ be the $\bar{s}$-core of the $\bar{s}$-block B of weight $w \geq 2$; then $\lambda = \kappa \cup \{ws\} \in B$. We assume that all bar partitions in B have the same $\bar{t}$-core.
Let $a \in \kappa$ be s-maximal. Then we have the bar partition $\kappa \cup \{a + ws\} \setminus \{a\}$ as well as the bar partitions $\kappa \cup \{a + (w-l)s, ls\} \setminus \{a\}$, $1 \leq l \leq w-1$, in B. Since these have the same $\bar{t}$-core as λ, we deduce

$$(ws, a)^o_{(\bar{t})} = (a + ws)_{(\bar{t})} = (a + (w-l)s, ls)^o_{(\bar{t})} \ , \ 1 \leq l \leq w-1 \, .$$

We want to show: $(*)$ $a + ws \equiv 0 \mod t$.
First we consider the case of an odd t.
If $(*)$ does not hold, then the first equation above yields $a \equiv 0 \mod t$ or $ws \equiv 0 \mod t$ and the equation for $l = 1$ yields (using that $s \not\equiv 0 \mod t$)

$$a + (w-1)s \equiv 0 \mod t \, .$$

Now if $w = 2$, then we obtain $a \equiv 0 \equiv a + s \mod t$ (note that $2s \not\equiv 0 \mod t$), leading to the contradiction $s \equiv 0 \mod t$.
If $w \geq 3$, then we can also use the equation above for $l = 2$ to obtain $a + (w-2)s \equiv 0 \mod t$, leading again to the contradiction $s \equiv 0 \mod t$.
Now we turn to the case of $t = 4$. Again, we assume that $(*)$ does not hold.
If $w = 2$, then

$$(ws, a)^o_{(\bar{4})} = (a)_{(\bar{4})} = (a + ws)_{(\bar{4})} \ ,$$

and since $2s \equiv 2 \mod 4$, this can only hold for even a. Because of $(*)$, we must then have $a \equiv 0 \mod 4$. But now $l = 1$ also gives $\emptyset = (a)_{(\bar{4})} = (a + s, s)_{(\bar{4})}$, and this is a contradiction.
If $w \geq 3$, then we can also use the condition for $l = 2$ to obtain

$$(a + ws)_{(\bar{4})} = (a + (w-2)s, 2s)^o_{(\bar{4})} = (a + (w-2)s)_{(\bar{4})} \, .$$

Then $2s \equiv 2 \mod 4$ implies that $a+ws$ has to be even, and in fact $a+ws \equiv 2 \mod 4$ because of $(*)$. But now from $l = 1$ we obtain $\emptyset = (a+ws)_{(\bar{4})} = (a+(w-1)s, s)_{(\bar{4})} \neq \emptyset$, a contradiction. Hence we arrive in both case at the conclusion that all s-maximal $a \in \kappa$ satisfy $a + ws \equiv 0 \mod t$, and thus in particular, that they are all congruent mod t.

Now assume that κ has two s-maximal elements a, b (we want to show that this cannot occur). We then also have the bar partition $\kappa \cup \{a+s, b+s, (w-2)s\} \setminus \{a, b\}$ for $w \geq 3$ and $\kappa \cup \{a + s, b + s\} \setminus \{a, b\}$ for $w = 2$, respectively, in B, as well as the bar partitions $\kappa \cup \{a + (w-l)s, b + ls\} \setminus \{a, b\}$, for $1 \leq l \leq w-1$.
Since these have the same $\bar{t}$-core as λ, we deduce

$$(ws, a, b)^o_{(\bar{t})} = \begin{cases} (a + s, b + s, (w-2)s)^o_{(\bar{t})} & \text{for } w \geq 3 \\ (a + s, b + s)^o_{(\bar{t})} & \text{for } w = 2 \end{cases}$$

and

$$(ws,a,b)^o_{(\bar t)} = (a+(w-l)s, b+ls)^o_{(\bar t)}\ ,\ 1\le l\le w-1\ .$$

As $a+ws\equiv 0\equiv b+ws \mod t$, we also have $(ws,a,b)_{(\bar t)} = (a)_{(\bar t)}$. Now the first equation tells us that we cannot have $a\equiv 0 \mod t$, for $w\ge 2$, and indeed it gives a contradiction in the case $w=2$. For $w\ge 3$, it yields $a+s\equiv 0 \mod t$ when t is odd, and a odd, when $t=4$. Now, we can also use the equation above for $l=2$, which gives the desired contradiction in both cases.
Hence we know now that at most one conjugate pair of runners is non-empty.

Now assume that we have two empty conjugate runner pairs, i.e., there are $1\le i< j< s-j< s-i\le s-1$ such that $\kappa\cap\{i,s-i\}=\emptyset=\kappa\cap\{j,s-j\}$. We then find the following bar partitions in B:
$\kappa\cup\{i,ls-i,(w-l)s\}$, $1\le l\le w-1$, $\kappa\cup\{i+ls,(w-l)s-i\}$, $0\le l\le w-1$, and similarly for j instead of i, and furthermore we have in B the "mixed" bar partitions $\kappa\cup\{i,s-i,j,s-j\}$ for $w=2$ and $\kappa\cup\{i,s-i,j,s-j,(w-2)s\}$ for $w\ge 3$, respectively.
Again, we know that these bar partitions have the same $\bar t$-bar core as λ.
First assume that $ws\equiv 0 \mod t$ (then we must have $w\ge 3$); note that in the case $t=4$ the assumption w even implies $ws\equiv 0 \mod 4$ by using the bar partitions above.
When t is odd, for $l=1$, the first partition yields $i\equiv 0$ or $s-i\equiv 0 \mod t$ (but not both), and analogously for $j,s-j$. When $t=4$, the listed partitions also give $i\equiv 0$ or $s-i\equiv 0 \mod 4$, and analogously for $j,s-j$. But then the final partition (with both i,j appearing) immediately gives a contradiction in both cases.
Hence $ws\not\equiv 0 \mod t$ when t is odd, and when $t=4$, we even know that w is odd. But then, using the partitions of the second type at $l=0$ we obtain $(ws)_{(\bar t)} = (i,ws-i)_{(\bar t)}$, and thus we have $i\equiv 0$ or $i\equiv ws \mod t$ (and similarly for j). Using these partitions for $l=1$, we then obtain $i\equiv 0$ and $ws\equiv s$, or $i\equiv ws\equiv -s \mod t$, respectively. We have similar congruences for j, and we then deduce that we have either $ws\equiv s$ and $i\equiv 0\equiv j$, or $ws\equiv -s\equiv i\equiv j$. If $w=2$ or if $w\ge 3$ and $ws\equiv -s$, we easily arrive at a contradiction by using the mixed bar partitions, and in the case of $t=4$ we also use the partitions of the second type for $l=2$. If $w\ge 3$ and $ws\equiv s$ (and $i\equiv 0$), we use the first partition type at $l=2$ to reach a contradiction. Hence we cannot have two empty runner pairs. As we also have at most one non-empty runner pair, we can deduce now that $s=3$ or $s=5$.

Now assume that $a\in\kappa$ is s-maximal and that we have $1\le i< s-i\le s-1$ such that $\kappa\cap\{i,s-i\}=\emptyset$. Recall that we have already shown $a+ws\equiv 0 \mod t$. We consider the partition $\kappa\cup\{a+(w-1)s,s-i,i\}\setminus\{a\}\in B$. As it has the same $\bar t$-core as λ, we deduce $(ws,a)_{(\bar t)} = (a+(w-1)s,s-i,i)_{(\bar t)}$. When t is odd, we deduce immediately $i\equiv s$ or $i\equiv 0 \mod t$. But as $s\le 5$, we have $i\le 2$, and then the only possibility is $i=2$, $s=5$, $t=3$. But then the partition $\kappa\cup\{s,a+(w-2)s,i,s-i\}\setminus\{a\}\in B$ yields a contradiction. Hence we cannot have both a non-empty and an empty runner pair, and this forces $s=3$. When $t=4$, we argue similarly and we also use the partition

$\kappa \cup \{a+(w-2)s, s, s-i, i\} \setminus \{a\} \in B$ for $w \geq 3$, or $\kappa \cup \{s, s-i, i\} \in B$, for $w = 2$, to exclude the situation $s = 5$ and $i = 1$.

First we consider the case of an odd integer t. Let us assume that $\kappa \neq \emptyset$ and that $a \in \kappa$ is s-maximal. First, we also assume that $a - s \in \kappa$, i.e., the corresponding runner has at least two beads. The partition $\kappa \cup \{a+(w-1)s\} \setminus \{a-s\} \in B$ leads to the two possibilities $a \equiv 0 \equiv ws \mod t$ or $a \equiv s$ and $ws \equiv -s \mod t$. If $w \geq 3$, then the partition $\kappa \cup \{s, a+(w-2)s\} \setminus \{a-s\} \in B$ leads to a contradiction in both cases. Hence $w = 2$. But $2s \not\equiv 0 \mod t$, and $2s \equiv -s \mod t$ leads to $0 \equiv 3s = 9 \mod t$ and then $t = 9$; now, $a \equiv s = 3 \mod t = 9$ leads to $3 \mid a$ – a contradiction. Thus we can have only one bead on the runner, i.e., $\kappa = \{a\}$.
Now the partition $(a+(w-1)s, a, s-a)^o \in B$, and the $\bar{t}$-core of $\lambda = (ws, a)$ is empty, hence we have $a \equiv 0 \equiv ws$ or $a \equiv s \mod t$.
If $w \geq 3$, then also $\kappa \cup \{s, a+(w-2)s, s-a\} = (a+(w-2)s, s, a, s-a)^o \in B$, and we get a contradiction for both possibilities for a.
Hence $w = 2$. But as $ws = 2s \not\equiv 0 \mod t$, we then obtain $a \equiv s = 3 \mod t$ and $0 \equiv a + ws = a + 2s \equiv 9 \mod t$, and thus $t = 9$, but then $s = 3 \mid a$ gives a contradiction.

Hence the only possibility left is $\kappa = \emptyset$, $s = 3$. In this case $\lambda = (ws) = (3w) \in B$, and we also find the partition $(3(w-1), 2, 1) \in B$, for all $w \geq 2$. As these have the same $\bar{t}$-core, we cannot have $3w \equiv 0 \mod t$ but must have $3w \equiv 1$ or $2 \mod t$, and in the latter case we note that $w > 2$. If $3w \equiv 1 \mod t$, consider in B the partition $(3(w-2), 4, 2)^o$ for $w \geq 3$, or $(4, 2)$ for $w = 2$, respectively; as $t > 3$, this gives a contradiction. Hence $3w \equiv 2 \mod t$ and $w > 2$. In this case, the partition $(3(w-1), 3) \in B$ gives a contradiction.
This final contradiction proves the claim that no $\bar{s}$-block of weight $w \geq 2$ is contained in a $\bar{t}$-block when t is odd.

It remains to deal with the case $t = 4$. We keep in mind that we already know $s = 3$. If $w \geq 3$ is odd, we consider the partitions $\kappa \cup \{ws\}$ and $\kappa \cup \{(w-2)s, 2s\}$ in B, and we can deduce $(ws)_{(\bar{4})} = ((w-2)s, 2s)^o_{(\bar{4})} = ((w-2)s)_{(\bar{4})}$. But then we must have $ws \equiv (w-2)s \mod 4$, and hence $2s \equiv 0 \mod 4$, a contradiction. Thus w is even. If $w > 2$, we may also use the bar partition $\kappa \cup \{(w-1)s, s\}$ in B; comparing the $\bar{4}$-core with the one of $\kappa \cup \{ws\}$ implies $ws \equiv 0 \mod 4$ and hence $w \equiv 0 \mod 4$. If $w > 4$, consider $\kappa \cup \{(w-3)s, 2s, s\}$ in B. Comparing the $\bar{4}$-core again with that of $\kappa \cup \{ws\}$, we obtain $\emptyset = ((w-3)s, s)_{(\bar{4})}$, hence $(w-2)s \equiv 0 \mod 4$ and thus $2s \equiv 0 \mod 4$, a contradiction. Thus $w = 2$ or $w = 4$.
As in the case of odd t, we now assume first that $\kappa \neq \emptyset$. Again, we start with the situation where $a \in \kappa$ is s-maximal and where also $a - s \in \kappa$. Note that we have already shown $a + ws \equiv 0 \mod 4$, hence in particular a is even. If $w = 2$, we compare the $\bar{4}$-cores of $\kappa \cup \{2s\}$ and $\kappa \cup \{a+s\} \setminus \{a-s\}$. This yields $(a-s)_{(\bar{4})} = (a+s)_{(\bar{4})}$ and thus $-s \equiv s \mod 4$, a contradiction. If $w = 4$, we compare the $\bar{4}$-cores of $\kappa \cup \{4s\}$ and $\kappa \cup \{a+2s, s\} \setminus \{a-s\}$. This yields $(a-s)_{(\bar{4})} = (s)_{(\bar{4})}$ and thus $a \equiv 2s \mod 4$, a contradiction.

Next we consider the case $\kappa = \{a\}$; remember that a is even. For $w = 2$, we have $\emptyset = (2s, a)_{(\bar{4})} = (a+s, a, s-a)^o_{(\bar{4})} = (a+s, s-a)_{(\bar{4})}$, and thus $2s \equiv 0 \mod 4$, a contradiction. For $w = 4$, we have $\emptyset = (4s, a)_{(\bar{4})} = (a+2s, s, a, s-a)^o_{(\bar{4})} = (s, s-a)_{(\bar{4})}$, and thus $2s \equiv a \equiv 0 \mod 4$, again a contradiction.
Now assume $\kappa = \emptyset$. In this situation we use the fact that $s = 3$. Because of this, $(3w)$ and $(3(w-1), 2, 1)$ are in B, and then $\emptyset = (3w)_{(\bar{4})} = (3(w-1), 2, 1)_{(\bar{4})} = (3(w-1), 1)_{(\bar{4})}$ yields $w = 2$. In the remaining case where $w = 2$, we note that the bar partitions (6) and $(5, 1)$ in B have different $\bar{4}$-cores, thus reaching the final contradiction.
Thus we also have that no $\bar{s}$-block of weight $w \geq 2$ is contained in a $\bar{4}$-block. $\diamond$

Corollary 2.4 *Let s, t be odd integers, $s, t > 1$, $t \neq s$. If B is an $\bar{s}$-block and a $\bar{t}$-block of bar partitions, then $|B| = 1$.*

Proof. W.l.o.g. we may assume $t \nmid s$. Suppose that $|B| > 1$. Then Theorem 2.3 implies that the $\bar{s}$-block B is of $\bar{s}$-weight 1, and that $\gcd(s, t) = 1$. But then we can apply Theorem 2.3 again, now to the $\bar{t}$-block B, to deduce that also its $\bar{t}$-weight is 1. But then

$$\bar{k}(s, 1) = \frac{s+1}{2} = |B| = \bar{k}(t, 1) = \frac{t+1}{2}\,,$$

contradicting $s \neq t$. $\diamond$

Remark 2.5 Given the connection to spin blocks of the double covers of the symmetric groups this shows that the Navarro-Willems conjecture [8] holds for spin blocks of these groups (at odd primes). As noted before, the case of 2-blocks of $\tilde{S}_n$ was treated in [2]. All such blocks are "mixed", i.e., they contain both ordinary irreducible characters and spin characters, and thus there can be no equalities between 2-blocks and spin blocks for odd primes in $\tilde{S}_n$. As the conjecture also holds for ordinary blocks of these groups by [10], we conclude that the conjecture holds for the double covers of the symmetric groups. As we have seen above, it may happen that for odd p the spin characters in a $\bar{p}$-block of positive defect are all contained in the same 2-block of $\tilde{S}_n$, but only in the case where the small block is of weight 1 and the $\bar{p}$-core is $\bar{4}$-good. For instance, the $\bar{3}$-block of weight 1 to $\kappa = (4, 1)$ consists of the bar partitions $(7, 1)$ and $(4, 3, 1)$ which both have empty $\bar{4}$-core, i.e., the corresponding spin characters are both in the principal 2-block of $\tilde{S}_8$.

For the sake of completeness we list all possible non-trivial block inclusions in $\tilde{S}_n$. We call a p-block of $\tilde{S}_n$ *ordinary* if it does not contain a spin character. Otherwise is it called a *spin* block. Each block has a *core* as described above.

Theorem 2.6 *Let p, q be different primes, B_p a p-block and B_q a q-block of $\tilde{S}_n$, both of positive defect. Then $B_p \subset B_q$ if and only if one of the following occurs:*

- *p and q are both odd, B_p and B_q are both ordinary, B_p has defect 1 and its core is q-good. ([10])*
- *p and q are both odd, B_p and B_q are both spin, B_p has defect 1 and its core is $\bar{q}$-good.*
- *$q = 2$, B_p is ordinary, B_p has defect 1 and its core is 2-good.*
- *$q = 2$, B_p is spin, B_p has defect 1 and its core is $\bar{4}$-good.*

3 $(\bar{s}, \bar{t})$-cores and block equalities

In this section we classify the situations which occur in Corollary 2.4; in particular, this includes the classification of the (labels of) the spin characters of the double cover groups $\tilde{S}_n$ which are of defect 0 for two distinct odd primes.

Let $s, t \in \mathbb{N}$, $s, t > 1$, be *odd* and $\gcd(s, t) = 1$.
Throughout this section we also assume that $s < t$.

A bar partition which is simultaneously an $\bar{s}$-core and a $\bar{t}$-core is called an $(\bar{s}, \bar{t})$-core. If κ is such a partition and a is one of its parts, then a cannot be represented in the form $ks + lt$,where k, l are nonnegative integers. Indeed, in that case s or t would be a part of κ, but by the definition of bar cores this is not possible. Thus the parts of κ are contained in the set

$$X_{s,t} = \mathbb{N} \setminus \{ks + lt | k, l \geq 0\} .$$

It is known (see [1], [10]) that this is the finite set consisting of all *positive* numbers of the form $(st - s - t) - (ks + lt)$, $k, l \geq 0$. Thus there is only a finite number of possibilities for the parts of κ and therefore also only a finite number of $(\bar{s}, \bar{t})$-cores.

We note that $\frac{s+t}{2} = (st - s - t) - (\frac{t-3}{2}s + \frac{s-3}{2}t) \in X_{s,t}$. However, $\frac{s+t}{2}$ cannot be a part of an $(\bar{s}, \bar{t})$-core κ. Indeed, if this were the case, then $\frac{t-s}{2} = \frac{s+t}{2} - s$ would be a part of κ, since it is an $\bar{s}$-core. Adding the parts $\frac{t-s}{2}$ and $\frac{s+t}{2}$ we get t, contradicting that κ is an $\bar{t}$-core.

Arrange the elements of $X_{s,t}$ in the *(s,t)-diagram* (see e.g. [10], where this has also appeared). Start with the largest entry $st - s - t$ in the lower left hand corner and subtract multiples of s along the rows and multiples of t along the columns as long as possible.

The (s, t)-diagram incorporates parts of the runners of the usual $\bar{s}$-abacus (arranged horizontally and reordered) and of the usual $\bar{t}$-abacus (arranged vertically and reordered). Not all runners of the $\bar{s}$- and $\bar{t}$-abacus are represented. As usual, for $0 \leq i \leq s - 1$ we refer to the i-runner of the $\bar{s}$-abacus as the runner containing numbers $\equiv_s i$; similar notation is used for the $\bar{t}$-abacus. On the $\bar{s}$-abacus, only

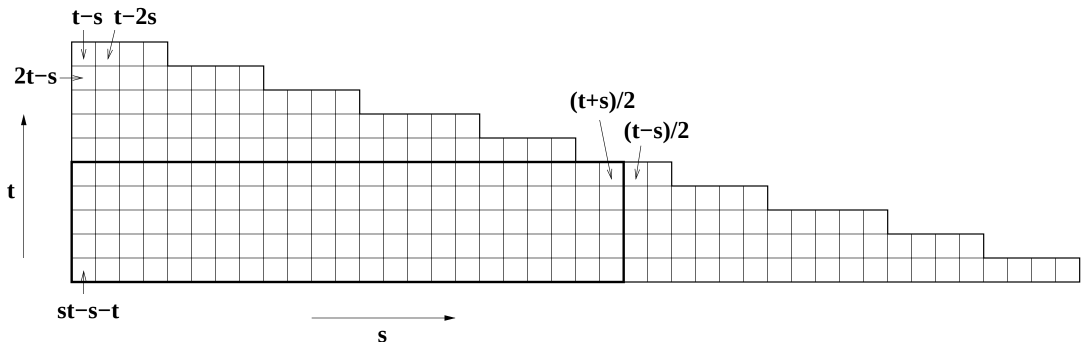

Figure 1: The (s,t)-diagram

the 0-runner is missing, but all the $\bar{t}$-runners numbered sk, $0 \leq k \leq \lfloor \frac{t}{s} \rfloor$ are not represented in the (s,t)-diagram.

As an example, the diagram for $s = 7,\ t = 17$ is shown below. We have marked the rectangle with the lower left hand corner $st - s - t = 95$ and the upper right hand corner $\frac{s+t}{2} = 12$; none of these numbers can be parts of a $(\bar{7}, \bar{17})$-core, i.e., these cores may only contain parts which are among the non-bold numbers in the diagram. There are still further restrictions. For example, if a $(\bar{7}, \bar{17})$-core has a part equal to 4 (from the final row) then none of the numbers 3, 10, 13, 30 from the top rows can be a part. (For instance 4+30 is divisible by 17.)

10	3												
27	20	13	6										
44	37	30	23	16	9	2							
61	54	47	40	33	26	19	12	5					
78	71	64	57	50	43	36	29	22	15	8	1		
95	88	81	74	67	60	53	46	39	32	25	18	11	4

Figure 2: The (7,17)-diagram

Removing the numbers from the marked rectangular subdiagram of the (s,t)-diagram containing $u = \frac{s-1}{2}$ rows and $v = \frac{t-1}{2}$ columns leaves two smaller diagrams, which we refer to as the *Yin diagram* and the *Yang diagram*. The top Yin diagram has $st - s - t\frac{s+1}{2}$ as its largest element and the bottom Yang diagram has $st - t - s\frac{t+1}{2}$ as its largest element.

Remark 3.1 There is a simple connection between the numbers in the Yin diagram and the Yang diagram: remove all entries less than or equal to $\frac{t-s}{2}$ from the Yin

10	3					
27	20	13	6			
44	37	30	23	16	9	2

5					
22	15	8	1		
39	32	25	18	11	4

Figure 3: The (7,17) Yin and Yang diagrams

diagram and subtract $\frac{t-s}{2}$ from the remaining entries to get the Yang diagram. This is easily seen from the definition of the (s,t)-diagram. Indeed we just need to notice that the difference between the largest elements in the Yin and Yang diagram is $\frac{t-s}{2}$.

Let us discuss the lengths of the rows and columns in the diagrams. Define a permutation σ of $\{1, 2, ..., s-1\}$ by the property that for $i = 1, ..., s-1$

$$\sigma(i)t \equiv_s i.$$

Note that the definition of σ shows that for all i we have

$$\sigma(i) + \sigma(s-i) = s\ . \tag{1}$$

Put

$$c_i = \frac{\sigma(i)t - i}{s}\ . \tag{2}$$

The c_i's are numbers between 1 and $t-1$ satisfying

$$c_i + c_{s-i} = t - 1\ . \tag{3}$$

The (s,t)-diagram contains $s-1$ rows. For each i, $1 \leq i \leq s-1$, there is a row containing the numbers $i + sj$, $0 \leq j \leq c_i - 1$ (see [10, Lemma 3.2]). Therefore the diagram contains a total of $\frac{(s-1)(t-1)}{2}$ numbers. Half of these (namely the $\frac{s-1}{2} \cdot \frac{t-1}{2}$ numbers in the rectangle) are removed to get the Yin and Yang diagrams.
The first $\frac{s-1}{2}$ rows contain the numbers of the Yin diagram and the bottom $\frac{s-1}{2}$ rows contain the numbers of the Yang diagram.

Equation (3) shows that if the Yin diagram contains numbers of residue i modulo s then the Yang diagram contains numbers of residue $s-i$ modulo s, and these come in opposite order. Moreover the total number of numbers of residue $i, s-i$ in both diagrams is $\frac{t-1}{2}$.

Therefore the Yang diagram may be rotated 180 degrees and combined with the Yin diagram to give a new *rectangular* $\frac{s-1}{2} \times \frac{t-1}{2}$-diagram. We refer to this as the *Yin-Yang diagram.* Conjugate runners are joined at their lowest entries, as shown in this example:

10	3	4	11	18	25	32	39
27	20	13	6	1	8	15	22
44	37	30	23	16	9	2	5

Figure 4: The (7,17) Yin-Yang diagram

Theorem 3.2 *Let $s, t \in \mathbb{N}$, $s, t > 1$, $\gcd(s,t) = 1$. Set $u = \frac{s-1}{2}$, $v = \frac{t-1}{2}$. There are only finitely many $(\bar{s}, \bar{t})$-cores, and their number is*

$$\binom{u+v}{u}.$$

Proof. As we have seen, the Yin-Yang diagram contains all integers which may occur as parts in an $(\bar{s}, \bar{t})$-core. Thus an $(\bar{s}, \bar{t})$-core κ has parts which form a subdiagram $T(\kappa)$ of the Yin-Yang diagram. If $a \in T(\kappa)$, $a > s$, then $a - s \in T(\kappa)$, and if $a > t$, then $a - t \in T(\kappa)$.
We draw a line separating the Yin and the Yang part of the Yin-Yang diagram. If $a \in T(\kappa)$ then also all numbers in the area between a and the separating line are in $T(\kappa)$.

Consider one of the $\binom{u+v}{u}$ "diagonal" paths between the entries of the Yin-Yang diagram. It is composed of vertical (north-to-south) and horizontal (west-to-east) moves from the top left hand corner to the bottom right hand corner of the diagram.

10	3	4	11	18	25	32	39
27	20	13	6	1	8	15	22
44	37	30	23	16	9	2	5

Figure 5: A $(\bar{7}, \bar{17})$-core in the Yin-Yang diagram

Together with the separating line the path forms the border of two subdiagrams (possibly empty) of the Yin-Yang diagram. One subdiagram is contained in the Yin diagram and the other in the Yang diagram. The subdiagrams taken together can have a nonempty intersection only with one of a pair of conjugate runners (both with respect to the $\bar{s}$- and $\bar{t}$-abacus), and also they are closed under subtraction of multiples of s, t; therefore the entries of these subdiagrams form then the parts of an $(\bar{s}, \bar{t})$-core. In the example shown in Figure 5, this core partition κ is $(18, 11, 8, 4, 2, 1)$; here, $1, 4, 8, 11, 18$ are in the Yang part of the diagram $T(\kappa)$ and only 2 is in the Yin part of $T(\kappa)$.

On the other hand, if κ is an $(\bar{s},\bar{t})$-core, then consider the intersection of $T(\kappa)$ with the i-th row in the Yin-Yang diagram. Suppose that the rightmost element r_i is in the j_i'th column. If r_i, r_{i+1} are both in the Yin part or in the Yang part then clearly $j_i \leq j_{i+1}$. If r_i is in Yin and r_{i+1} in Yang then again $j_i \leq j_{i+1}$ since the Yang part is to the right of the Yin part of row $i+1$. If r_i is in Yang and r_{i+1} in Yin and $j_i > j_{i+1}$ then the element in position $(i+1, j_i)$ would be in $T(\kappa)$ forcing κ to have a mixed s- or t-bar. This is not possible. Thus the positions of the rightmost elements form an increasing sequence and therefore describe a diagonal path. $\diamond$

Remark 3.3 The above theorem implies that the number of associate classes of spin characters of $\tilde{S}_n$ which are simultaneously of defect 0 for two odd primes p and q is

$$\binom{\frac{p-1}{2}+\frac{q-1}{2}}{\frac{p-1}{2}}.$$

The problem of enumerating the number of selfassociate and non-selfassociate characters with this property is still open.

Recall that an $\bar{s}$-core κ is called $\bar{t}$-good if
(i) If a part $a \in \kappa$ is s-maximal, then $t \mid a$ or $t \mid a+s$.
(ii) If $i \in \{1, \ldots, s-1\}$ is such that $\{i, s-i\} \cap \kappa = \emptyset$, then $t \mid i$ or $t \mid s-i$.
The $(\bar{s},\bar{t})$-core, whose parts are the elements in the Yin diagram (resp. Yang diagram) will be referred to as the *Yin partition* (resp. *Yang partition*).
These partitions play a special rôle:

Proposition 3.4 *The Yin partition is the minimal $\bar{t}$-good $\bar{s}$-core (i.e., any other such partition is a partition of a larger integer). Also the Yang partition is the minimal $\bar{s}$-good $\bar{t}$-core.*

Proof. The s-maximal elements of the Yin partition are the elements of the first column of the Yin diagram. Adding s to any of these elements gives a number divisible by t, so (i) is satisfied. For any pair of conjugate runners in the $\bar{s}$-abacus, the Yin diagram contains one of the runners, and thus (ii) is satisfied. Hence the Yin partition is a $\bar{t}$-good $\bar{s}$-core. As $s < t$, a minimal $\bar{t}$-good $\bar{s}$-core has to have at least one of any two conjugate runners non-empty. Such a nonempty runner (say the i-th) contains an s-maximal element which then has to satisfy (i). As we have seen the minimal possibility for the number of beads is then c_i (as defined by equation (2)). Since the Yin diagram contains the shorter of any two conjugate runners, the Yin partition is minimal among all $\bar{t}$-good $\bar{s}$-cores.

The proof in the case of the Yang partition is similar, but slightly more complicated. The t-maximal elements of the Yang partition are the elements of the last row of the Yang diagram; adding t to any of them gives a number divisible by s, so (i) is satisfied. Let $1 \leq i \leq t-1$. If $s \mid i$ then the i-th $\bar{t}$-runner in the (s,t)-diagram is

empty. If $s \mid t-i$, say $i = t - sk$, then the i-th $\bar{t}$-runner is not in the Yang part but in the Yin part of the (s,t)-diagram (see Figure 1).
For all other i the Yang diagram contains exactly one of the $\bar{t}$-runners i or $t-i$ (namely, the shorter one in the (s,t)-diagram). Thus (ii) is fulfilled. The argument for minimality is essentially the same as above, as the $\bar{t}$-runners which are allowed to be empty are not represented in the Yang diagram. $\diamond$

Remark 3.5 We proceed to give a description of *all* $\bar{t}$-good $\bar{s}$-cores. By condition (ii), such a partition has to have at least one of any two conjugate $\bar{s}$-runners non-empty. The other conjugate runner has to be empty, since we have an $\bar{s}$-core. The minimal number of beads on $\bar{s}$-runner j is c_j (defined as above by equation (2)), if this runner is non-empty. Since a non-empty runner contains an s-maximal element, the other possibilities for the non-zero lengths of the runner are limited by condition (i) to be $d_j = c_j + k$,where k is congruent to 0 or 1 modulo t. Thus any $\bar{t}$-good $\bar{s}$-core is uniquely described by an $(s-1)$-tuple $(d_1, d_2, .., d_{s-1})$ of nonnegative integers satisfying

$$d_i d_{s-i} = 0,\ d_i + d_{s-i} > 0$$

If $d_i \neq 0,$ then there exists $m \geq 0$ such that $d_i = c_i + mt$ or $d_i = c_i + mt + 1$.

It is possible to label any $\bar{t}$-good $\bar{s}$-partition uniquely by an $(s-1)$-tuple of integers $(z_1, ..., z_{s-1})$.
For $1 \leq i \leq u = (s-1)/2$ we put

$$\begin{array}{rclrcll} d_i & = & c_i + z_i t, & d_{s-i} & = & 0 & \text{if } \ z_i \geq 0, \\ d_{s-i} & = & c_{s-i} - z_i t - t, & d_i & = & 0 & \text{if } \ z_i < 0. \end{array}$$

For $u+1 \leq i \leq s-1$ we put

$$\begin{array}{rclrcll} d_i & = & c_i + z_i t + 1, & d_{s-i} & = & 0 & \text{if } \ z_i \geq 0, \\ d_{s-i} & = & c_{s-1} - z_i t - t + 1, & d_i & = & 0 & \text{if } \ z_i < 0. \end{array}$$

Although it is possible in principle to compute $|\kappa|$ for a partition κ corresponding to $(z_1, ..., z_{s-1})$ in terms of these integers, there seems to be no nice generating function like the one for t-good s-cores in [10].

We may also describe *all* $\bar{s}$-good $\bar{t}$-cores. Here the runners starting with numbers of the form $t - sk$, sk with $1 \leq k \leq \lfloor \frac{t}{s} \rfloor$, create a special problem, since they are allowed to be empty by condition (ii). It is however in analogy with the previous case possible to label any $\bar{s}$-good $\bar{t}$-partition uniquely by a $(t-1)$-tuple of integers $(v_1, ..., v_{t-1})$. We omit the details.

Next we show that among all $(\bar{s}, \bar{t})$-cores, the Yin partition is maximal with respect to containment of the corresponding Young diagrams.

Theorem 3.6 *Any $(\bar{s}, \bar{t})$-core is contained in the Yin partition.*

Proof. As before we set $u = \frac{s-1}{2}$, $v = \frac{t-1}{2}$. Suppose that the i-th row in the Yin-Yang diagram contains α_i numbers in the Yin part and β_i numbers in the Yang part, $1 \le i \le u$. Then we have

$$\alpha_i + \beta_i = v,\ 1 \le i \le u,$$

$$\alpha_1 \le \alpha_2 \le \cdots \le \alpha_u, \quad \beta_u \le \beta_{u-1} \le \cdots \le \beta_1.$$

Moreover, as observed in Remark 3.1, adding $\frac{t-s}{2}$ to any of the β_i Yang numbers from the i-th row gives a number among the α_{u+1-i} Yin numbers in row $u+1-i$. In particular,

$$\beta_i \le \alpha_{u+1-i},\ 1 \le i \le u.$$

We refer now to the proof of Theorem 3.2. Let κ be an $(\bar{s}, \bar{t})$-core. The subdiagram $T(\kappa)$ of the Yin-Yang diagram is divided into a Yin part $T(\kappa)_1$ and a Yang part $T(\kappa)_2$. Suppose $T(\kappa)_1$ contains segments of length $\gamma_1, \gamma_2, ..., \gamma_k$ in the rows numbered $a_1, a_2, ..., a_k$ with $1 \le a_1 < a_2 < ... < a_k \le u$ and that $T(\kappa)_2$ contains segments of length $\delta_1, \delta_2, ..., \delta_l$ in the rows numbered $b_1, ..., b_l$ with $1 \le b_1 < b_2 < ... < b_l \le u$. Since the row numbers are all different we have in particular for all relevant i, j that $a_i \ne b_j$. Note that $\gamma_j \le \alpha_{a_j}$ and $\delta_j \le \beta_{b_j} \le \alpha_{u+1-b_j}$.

Our strategy is to replace Yang numbers in $T(\kappa)_2$ with larger Yin numbers which are not in $T(\kappa)_1$. This will produce a new partition containing κ, which is contained in the Yin partition, and this proves the assertion.

The δ_j numbers from $T(\kappa)_2$ are smaller than the δ_j largest numbers among the β_{b_j} Yang numbers in row b_j which in turn are smaller than the δ_j largest numbers among the α_{u+1-b_j} Yin numbers in row $u+1-b_j$.

Replace *if possible* the δ_j numbers from row b_j in $T(\kappa)_2$ with the δ_j largest numbers among the α_{u+1-b_j} Yin numbers in row $u+1-b_j$ for all $j = 1, ..., l$.

The replacement procedure may fail for some δ_j if some nodes in the Yin part of row $u+1-b_j$ are in $T(\kappa)_1$. This problem occurs if $u+1-b_j = a_i$ for some i and in addition $\gamma_i + \delta_j > \alpha_{a_i}$.

The path describing κ ends in row b_j at the position $\alpha_{b_j} + \delta_j$. Therefore, if $b_j < a_i$, we must have $\alpha_{a_i} - \gamma_i \ge \alpha_{b_j} + \delta_j$, which is not possible in the critical situation above. If $a_i < b_j$, then $\alpha_{b_j} > \alpha_{a_i}$ and the largest Yin numbers in row b_j are larger than the largest Yin numbers in row $a_i = u+1-b_j$. As $b_j = u+1-a_i$, the Yin numbers in row b_j have not been used in the procedure before; hence in the critical case we may replace the δ_j numbers from row b_j in $T(\kappa)_2$ by the δ_j largest Yin numbers in the same row b_j. $\diamond$

Proposition 3.7 *Set $u = \frac{s-1}{2}$, $v = \frac{t-1}{2}$. The sum of the numbers in the Yin-Yang diagram is*

$$\frac{1}{6} uv(ut + vs - 2)\ .$$

Proof. The sum of the numbers in the (s,t)-diagram is $\frac{(s^2-1)(t^2-1)}{24} + \binom{2uv}{2}$, by [10, Lemma 3.4]. We subtract the sum of the numbers in the rectangular part with corners $st - s - t$ and $\frac{s+t}{2} = u + v + 1$, and size $u \times v$. This sum is

$$uv(u+v+1) + u\sum_{i=0}^{v-1} is + v\sum_{j=0}^{u-1} jt = uv(u+v+1) + us\binom{v}{2} + vt\binom{u}{2}.$$

The proof is finished by a routine calculation. $\diamond$

Acknowledgement. The authors gratefully acknowledge the support by The Danish National Research Council.

References

[1] J. Anderson, Partitions which are simultaneously t_1- and t_2-core. *Discrete Math.* **248** (2002), 237–243.

[2] C. Bessenrodt, J. B. Olsson, The 2-blocks of the covering groups of the symmetric groups. Adv. Math. **129** (1997), 261–300

[3] M. Cabanes, Local structure of the p-blocks of $\tilde{S}_n$, *Math. Z.* **198** (1988), 519–543

[4] P. Hoffman, J. F. Humphreys, *Projective representations of the symmetric groups.* Oxford, 1992

[5] J. F. Humphreys, Blocks of projective representations of the symmetric groups, *J. London Math. Soc. (2)* **33** (1986), 441–452

[6] G. James, A. Kerber, *The representation theory of the symmetric group*, Encyclopedia of Mathematics and its Applications, 16, Addison-Wesley 1981

[7] A. O. Morris, The spin representation of the symmetric group, *Canad. J. Math.* **17** (1965), 543–549

[8] G. Navarro, W. Willems, When is a p-block a q-block? Proc. Amer. Math. Soc. 125 (1997) 1589-1591

[9] J. B. Olsson, *Combinatorics and representations of finite groups*, Vorlesungen aus dem FB Mathematik der Univ. Essen, Heft 20, 1993

[10] J. B. Olsson, D. Stanton, Block inclusions and cores of partitions, preprint 2005