The students in the course have to submit a written solution to the following exercises at the latest at 3PM on 27. October. The solutions may be submitted electronically (pdf-file) or given to one of the lecturers. The written solutions (in Danish or English) should be carefully written with references to quoted results from the notes and the exercises.

Exercise C2.1:
Let $P$ be a $p$-Sylow subgroup of $G$. Suppose that $M$ is a subgroup of $G$ containing $N_G(P)$. Show that $|G : M| \equiv 1 \pmod{p}$.

*Hint:* Note that $P \in Syl_p(M)$. Apply Sylow’s third theorem twice.

Exercise C2.2:

1. Let $G$ have order $2552 = 2^3 \cdot 11 \cdot 29$. Use Burnside’s normal complement theorem to show that for both $p \in \{11, 29\}$ we have either that $m_p(G) = 1$ or that $G$ has a normal $p$-complement.

2. Suppose in addition that $G$ has a normal $11$-complement $K$ and also a normal $29$-complement $L$. Let $P \in Syl_2(G)$. Show that $P$ is normal in $G$ and that $G/P$ is cyclic.

3. Let $Q \in Syl_{11}(G)$. Show that $Q$ is also normal in $G$. Let $R \in Syl_{29}(G)$. Show that $R$ is also normal in $G$.

*Hint:* Compute $m_{11}(L)$ and $m_{29}(K)$. Apply a Lemma on characteristic subgroups.

Exercise C2.3:

1. Prove that $\epsilon_8 = \frac{1+i}{\sqrt{2}}$ is a primitive 8-th root of unity and that $\mathbb{Q} (\epsilon_8)$ is the 8-th cyclotomic field $\mathbb{Q}_8$.

2. Let $L$ be the splitting field of $x^4 + 3$ over $\mathbb{Q}$. Prove that

$$L = \mathbb{Q} (\sqrt[4]{3} \cdot \epsilon_8, i)$$
(3) Let $M$ be the splitting field $x^4 - 12$ over $\mathbb{Q}$. Show that

$$M = \mathbb{Q}(\sqrt[4]{12}, i)$$

Determine $[M : \mathbb{Q}]$ and the Galois group $Gal(M/\mathbb{Q})$.

(4) Prove that $L = M$.

*Hint:* Use that $\sqrt[4]{12} = \sqrt[4]{3} \cdot \varepsilon_8 \cdot (1 - i)$.

(5) Let $N$ be the splitting field of $x^4 - 3$ over $\mathbb{Q}$. Prove that

$$N = \mathbb{Q}(\sqrt[4]{3}, i)$$

and determine the Galois group $Gal(N/\mathbb{Q})$.

(6) Prove that the compositum $MN = LN$ is the splitting field of the polynomial $(x^8 - 9) = (x^4 - 3)(x^4 + 3)$ and determine the Galois group $Gal(MN/\mathbb{Q})$.

*Hint:* Notice that $MN = N(\sqrt{2})$ and use Theorem 3.47.

**Exercise C2.4:**

Let $GF(3) = \mathbb{F}_3$ be the field with 3 elements.

(1) Show that $p(x) = x^3 - x + 1$ is an irreducible polynomial in $\mathbb{F}_3[x]$.

(2) Show that the field $\mathbb{F}_{27}$ with 27 elements is a splitting field of $p(x)$ over $\mathbb{F}_3$.

(3) Decide whether $p(x)$ divides the polynomial $x^{27} - x$ in $\mathbb{F}_3[x]$.

(4) Decide whether $p(x)$ divides the polynomial $x^{81} - x$ in $\mathbb{F}_3[x]$.

**Exercise C2.5:**

(1) Prove that $f(x) = x^5 - 25x + 5$ is irreducible in $\mathbb{Q}[x]$ and determine the number of real roots of $f(x)$.

(2) Let $M$ be the splitting field of $f(x)$ over $\mathbb{Q}$. Determine the Galois group $Gal(M/\mathbb{Q})$. Is $f(x)$ solvable by radicals?

(3) Does there exist an irreducible polynomial in $\mathbb{Q}[x]$ of degree 2 having a root in $M$?

(4) Does there exist an irreducible polynomial in $\mathbb{Q}[x]$ of degree 3 having a root in $M$?