Algebra 3 2010
Exercises in group theory

February 2010

Exercise 1*: Discuss the Exercises in the sections 1.1-1.3 in Chapter I of the notes.

Exercise 2: Show that an infinite group G has to contain a non-trivial subgroup, i.e. a subgroup $\neq G, \{e\}$.

Exercise 3: Suppose that $a^2b^2 = (ab)^2$ for all a, b in the group G. Show that G is abelian.

Exercise 4*: Show that

$$\langle (1,2,3,4), (1,2) \rangle = S_4.$$

Exercise 5: Let a, b and c be elements in a group G.

(1) Show that a and a^{-1} have the same order.

(2) Show that ab and ba have the same order.

(3) Show that abc and bca have the same order. (Try to generalize this to a general result.)

(4) Find three elements a, b, c in the symmetric group S_3, such that abc and bac have *different* orders.

Exercise 6*:
(1) Let a and b be commuting elements in G of finite orders m and n, respectively. Show that $(ab)^{mn} = e$.

(2) Suppose in addition that n and m are relatively prime. Prove that if a power a^i of a equals a power b^j of b, then $a^i = b^j = e$. Use this to prove that the order of ab is nm.

Exercise 7: Show that the elements of finite order in an *abelian* group G form a subgroup of G.

Exercise 8: Consider the following matrices as elements in the group $GL(2, \mathbb{R})$:

$$a = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Show that a has order 3 and that b has order 4. Show that the order of ab is infinite.

Exercise 9: Give examples of the following:

(1) An infinite group G with a subgroup $H \neq G$ and $|G : H|$ finite.

(2) An infinite group G with a subgroup $H \neq \{e\}$ and $|H|$ finite.

Exercise 10*: Suppose that the elements a, b in the group G satisfy the relation $aba^{-1} = b^2$, where $b \neq e$.

(1) Show that $a^5ba^{-5} = b^{32}$.

(2) Assume that $|a| = 5$. Compute $|b|$.

Exercise 11: Define an equivalence relation on the elements of the group G by

$$a \sim_c b \iff \langle a \rangle = \langle b \rangle.$$

Here $\langle a \rangle$ is the cyclic subgroup of G generated by a. Show that the $\sim_c$-equivalence class of an element a is always finite.

Exercise 12*: Let G be a finite group.

(1) Show that if $|G|$ is even then the number of elements of order 2 in G is *odd*. (Consider the mapping $x \mapsto x^{-1}$. Which elements are the fixed points of this map?)

(2) Show that the number of elements of order 3 in G is *even* (possibly 0).

(3) Show that the number of elements of order 4 in G is *even* (possibly 0).

(4) What can be said about the number of elements of order p in G, when p is a prime?

Exercise 13: Suppose that $G = \{a_1, a_2, ..., a_n\}$ is a finite *abelian* group of order n. Let $c = a_1 a_2 ... a_n$ be the product of all the elements in G. Show that $c^2 = 1$.

Exercise 14*: Consider the group $(\mathbb{Q}, +)$ and its subgroup $(\mathbb{Z}, +)$. Let p, q be two different prime numbers. Show that the cosets $\frac{1}{p} + \mathbb{Z}$ and $\frac{1}{q} + \mathbb{Z}$ in $\mathbb{Q}$ are different. Show that the index $|\mathbb{Q} : \mathbb{Z}|$ is infinite.

Exercise 15: Let S be a subgroup of the group G. Suppose that $a, b \in G$ satisfies $Sa = bS$. Thus the left coset of S containing a equals the right coset of S containing b. Show that $Sa = aS = bS = Sb$.

Exercise 16: It is wellknown that the subgroups of the group $(\mathbb{Z}, +)$ are exactly those on the form $(m\mathbb{Z}, +)$ for some $m \in \mathbb{Z},\ m \geq 0$. (Discuss this, if necessary.) Let $n_1, n_2 \in \mathbb{Z} \setminus \{0\}$ have greatest common divisor n. Show that the smallest subgroup of of $\mathbb{Z}$ which contains n_1 and n_2 is $n\mathbb{Z}$.

Exercise 17: Let H, K be finite subgroups of the group G. Let m be the least common multiple of $|H|$ and $|K|$. Show that $m \mid |HK|$.

Exercise 18: Suppose that $H \lhd G$ and that $|G : H| = p$, where p is a prime number. Let K be a subgroup of G. Show that either $K \subseteq H$ or $G = HK,\ \ |K : K \cap H| = p$.

Exercise 19: Find all subgroups of the dihedral group D_4 and draw a diagram illustrating their mutual positions. Find all normal subgroups of D_4.

Exercise 20*: Discuss the Exercises in the sections 1.5-1.8 in Chapter I of the notes.

Exercise 21*: Suppose that N_1 and N_2 are normal subgroups in the finite group G. Are the following claims *true or false*? (Please give either a proof or a counterexample.)

(i) If $N_1 \simeq N_2$, then $G/N_1 \simeq G/N_2$.

(ii) If $G/N_1 \simeq G/N_2$, then $N_1 \simeq N_2$.

Exercise 22*: Suppose that $S \lhd G$, that G is finite and that $\kappa : G \to \overline{G} = G/S$ is the canonical epimorphism. Suppose that $g \in G$ has an order, which is relatively prime to $|S|$. Show that $|g| = |\kappa(g)|$.

Exercise 23*: Discuss the Exercises in the sections 1.9-1.12 in Chapter I of the notes.

Exercise 24: Let G be a group. Show that the following subset T of $\mathrm{Aut}(G)$ is a *normal* subgroup in $\mathrm{Aut}(G)$

$$T = \{\alpha \in \mathrm{Aut}(G) |\ \alpha(U) = U \text{ for all subgroups } U \text{ in } G\}$$

Exercise 25: Show that for any (finite) group G we have:

$$|G| \leq 2 \Leftrightarrow \mathrm{Aut}(G) = \{1_G\}.$$

Exercise 26*: Show that the only finite group with 2 conjugacy classes is $\mathbb{Z}_2$.

Exercise 27: Let G be a group. Suppose that $|G : Z(G)| = n$ is finite. Show that any conjugacy class of G contains at most n elements.

Exercise 28: Show that the matrices

$$\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} |\ a, b, c \in \mathbb{Z}/3\mathbb{Z}\}$$

with usual matrix multiplication form a group of order 27, where every element $\neq e$ has order 3.

Use this to find two non-isomorphic finite groups for which for every t the number of elements of order t coincide for the two groups.

Exercise 29: Compute the center of the group of order 27 from the previous exercise.

Exercise 30*: Let G be a finite p-group (p a prime). Suppose $N \lhd G$, $N \neq \{e\}$.

(1) Show that

$$N \cap Z(P) \neq \{e\}.$$

Hint: It is possible to modify the proof of Theorem 1.111 in the notes. Note that N is a union of conjugacy classes of G.

Exercise 31*: Let G be a finite p-group (p a prime) and N a subgroup of order p. Show that $N \lhd G$ if and only if $N \subseteq Z(P)$.

Exercise 32: Let G be a p-group. Suppose that $H \neq G$ is a subgroup. Show that $H \subset N_G(H)$, ie. $N_G(H)$ contains H properly.

Exercise 33: Let G_4 be the abelian group consisting of all infinite sequences $(a_1, a_2, \cdots)$ where each a_i is some element in a cyclic group of order 4.

(1) Define an isomorphism $\phi : G_4 \to G_4 \times G_4$.

(2) Investigate whether the following subsets of G_4 are subgroups:

$$M_1 = \{(a_1, a_2, \cdots) \in G_4 \mid \text{There exists } k \in \mathbb{N} \text{ such that } a_i = e \text{ for all } i \geq k.\}$$

$$M_2 = \{(a_1, a_2, \cdots) \in G_4 \mid \text{Only finitely many } a_i's \text{ equal } e\} \cup \{(e, e, \cdots)\}.$$

Exercise 34: Let G_4 be the same group as in the previous exercise. Consider the (outer) direct product:

$$H_4 = C \times G_4$$

where C is a cyclic group of order 2. Show that G_4 is isomorphic to a proper subgroup of H_4 and that H_4 is isomorphic to a proper subgroup of G_4. Show that G_4 and H_4 are *not* isomorphic.

(Thus there is no "Bernstein's equivalence theorem" for groups.)

Exercise 35: Describe all the 2-Sylow subgroups and all the 3-Sylow subgroups of D_6 and of $S_3 \times S_3$.

Exercise 36*: Show that $m_p(D_n) = 1$ for all $n \geq 2$ and all odd primes p.

Exercise 37*: Which of the following groups are isomorphic?
$\mathbb{Z}_{24}$, $\mathbb{Z}_4 \times \mathbb{Z}_6$, S_4, $A_4 \times \mathbb{Z}_2$, $\mathbb{Z}_8 \times \mathbb{Z}_3$, D_{12}, $D_6 \times \mathbb{Z}_2$.

Exercise 38*: Show that a group of order 45 is abelian. Determine the number of isomorphism classes of groups of order 45.

Exercise 39*: Show that if $|G| = 80 = 2^4 \cdot 5$ then either $m_5(G) = 1$ or $m_2(G) = 1$. Is G solvable? Give examples of three non-isomorphic non-abelian groups of order 80, each with a normal 5-Sylow subgroup.

Exercise 40: Show that a group of order 200 has a normal 5-Sylow subgroup and that it is solvable.

Exercise 41*: Let G have order $231 = 3 \cdot 7 \cdot 11$. Show that $m_7(G) = m_{11}(G) = 1$. Show that G has a cyclic subgroup of order 33. Show that if $P \in Syl_{11}(G)$, then $P \subseteq Z(G)$.

Exercise 42: Lad $n = pa$, where p is a prime and $1 \leq a < p$. Compute the order of a p-Sylow group of the symmetric group S_{pa}. Show that such a group is abelian.

Exercise 43: Consider the permutations

$$\alpha = (1,2,3) \quad \beta = (1,4,7)(2,5,8)(3,6,9) \in S_9.$$

Show that α and β generate a subgroup of order $81 = 3^4$ in S_9. Show that this is a non-abelian 3-Sylow subgroup in S_9.

Exercise 44: (*The Frattini argument*). Suppose that $H \lhd G$ and that $P \in \mathrm{Syl}_p(H)$. Show that

$$G = HN_G(P).$$

(This result will be presented by the instructor. It is sometimes very useful, for instance in the study of finite solvable groups.)

Exercise 45: Show that any group of order $132 = 2^2 \cdot 3 \cdot 11$ contains subgroups of order $33 = 3 \cdot 11$, $44 = 2^2 \cdot 11$ and $12 = 2^2 \cdot 3$. (This is an example of P. Hall's generalization of Sylow's theorems for finite *solvable* groups. Ask your instructor!)
Hint: The final Example in the notes discusses groups of order 132. The previous exercise may also be used to show the existence of a subgroup of order 12.

Exercise 46*: Discuss the Exercises in the sections 1.14-1.21 in the notes.

Exercise 47*: Let G be a subgroup of S_n. Suppose that we can find elements $\tau_2, \cdots, \tau_n \in G$, such that $\tau_i(1) = i$ for $2 \leq i \leq n$. Show that G is transitive.

Exercise 48*: Show, eg. using the previous exercise, that for $n \geq 3$ is A_n a transitive subgroup of S_n.

Exercise 49*: For $m \in \mathbb{N}$ let

$$s(m) = \min\{n \in \mathbb{N} \mid m \text{ divides } n!\}.$$

Thus for example if $m = 700 = 2^2 5^2 7$ then $s(m) = 10$. Show that a simple group of order m cannot have a proper subgroup of index $< s(m)$.

Hint: Use Theorem 1.169 in the notes.

Exercise 50*: Let p be the largest prime number dividing the order of the simple group G. Show, e.g. using the previous exercise, that G cannot have a proper subgroup of index $< p$.

Exercise 51*: Let G have order $2552 = 2^3 \cdot 11 \cdot 29$. Use Burnside's normal complement theorem to show for $p \in \{11, 29\}$ we have that either $m_p(G) = 1$ or G has a normal p-complement.