

Algebra 3

Compulsory exercises, first set Første sæt obligatoriske opgaver

10. February 2009

The students in the course have to submit a written solution to the following exercises in the mailbox of the instructor (Sune Nikolaj Precht Reeh) on tuesday 17. February at the latest at the start of the exercise classes. The written solutions (in Danish or English) should be carefully written with references to quoted results from the notes and the exercises.

Exercise C1.1: List three groups of order 42 which are not isomorphic. Your answer should include a proof for the fact that they are not isomorphic.

Hint: You may use the fact that two finite isomorphic groups must have the same number of elements of any given order. This follows for example from Exercise 1.58. Proof of this is not required.

Exercise C1.2: Let G be the subgroup of the symmetric group S_8 generated by the permutations

$$\alpha = (1, 2, 3, 4) \text{ and } \beta = (1, 5)(2, 6)(3, 7)(4, 8).$$

(1) Show that G contains the permutations

$$(1, 2, 3, 4)(5, 6, 7, 8) \text{ and } (1, 2, 3, 4)(5, 7)(6, 8).$$

(2) Show that one of the permutations from (1) commutes with α and β . Explain why this element is contained in the center $Z(G)$ of G .

Exercise C1.3: Let $\phi : G \rightarrow H$ be a homomorphism from the group G to the group H .

(1) Show ϕ maps a commutator in G onto a commutator in H .

(2) If $G^{(1)}$ and $H^{(1)}$ are the commutator subgroups of G and H respectively show that $\phi(G^{(1)}) \subseteq H^{(1)}$.

Hint: Use (1) and Remark 1.23 in the notes.

Suppose now in the rest of this exercise that ϕ is an *epimorphism*.

(3) Show that $\phi(G^{(1)}) = H^{(1)}$.

(4) Assume that G is a finite group. Show that $|H : H^{(1)}|$ divides $|G : G^{(1)}|$.

Hint: Let $\psi = \phi \circ \kappa$, where $\kappa : H \rightarrow H/H^{(1)}$ is the canonical epimorphism. Show that $G^{(1)} \subseteq \text{Ker}(\psi)$ and that ψ is an epimorphism. Then apply the Homomorphism theorem.