

Algebra 3

Test exam

Exercise 1: Let G be a finite group of order $2 \cdot 5^2 \cdot 13 = 650$. Let $P \in \text{Syl}_5(G)$, $Q \in \text{Syl}_{13}(G)$.

(1) Use Sylow's theorems to determine the possibilities for the numbers $m_5(G)$ and $m_{13}(G)$ of 5- and 13-Sylow groups in G .

(2) Show that G has a normal subgroup H of index 2 with $m_5(G) = m_5(H)$ and $m_{13}(G) = m_{13}(H)$.

(Hint: Theorem 1.147 in the notes. Show that if H contains one p -Sylow group of G for some prime p , then it contains them all.)

(3) Show that $P \triangleleft G$ and $Q \triangleleft G$.

(4) Is G necessarily solvable?

Exercise 2: Let G be a finite p -group (p a prime). Suppose $N \triangleleft G$, $N \neq \{e\}$.

(1) Show that

$$N \cap Z(P) \neq \{e\}.$$

Hint: It is possible to modify the proof of Theorem 1.93 in the notes. Note that N is a union of conjugacy classes of G .

(2) Show that if $|N| = p$, then $N \subseteq Z(P)$.

(3) Suppose that $H \neq G$ is a subgroup. Show that $H \subset N_G(H)$, ie. $N_G(H)$ contains H properly.

Exercise 3: Let $L = \text{GF}(16)$ be the finite field with 16 elements.

(1) Determine the characteristic p of L .

Let $K = \text{GF}(p)$ and consider the polynomial $f(x) = x^3 + x + 1 \in K[x]$.

(2) Is $f(x)$ irreducible over K ?

(3) Is $f(x)$ separable over K ?

(4) Can L be obtained from K by adjoining a root of $f(x)$ over K ?

Exercise 4: Let M be the splitting field over \mathbb{Q} for $x^6 - 2$.

- (1) Find $[M : \mathbb{Q}]$.
- (2) Show that M/\mathbb{Q} is a normal extension.
- (3) Determine the Galois group G for M over \mathbb{Q} .
- (4) Decide if the G is abelian or not.
- (5) Decide if G is solvable or not.

Exercise 5: (7.4) ?? Let $f(x)$ be an irreducible polynomial in $\mathbb{Q}[x]$ of the form $f(x) = x^4 + ax^2 + b$, a and b rational numbers. Let M be the splitting field of $f(x)$ over \mathbb{Q} .

- (1) Prove that $[M : \mathbb{Q}]$ is 4 or 8. (Notice that $f(\alpha) = 0 \Rightarrow f(-\alpha) = 0$.)
- (2) Let M_1 , resp. M_2 be the splitting field of

$$f_1(x) = x^4 - 4x^2 + 2, \quad \text{resp.} \quad f_2(x) = x^4 - 10x^2 + 1.$$

- (3) Find $[M_1 : \mathbb{Q}]$ and $[M_2 : \mathbb{Q}]$.