

Algebra 3 Exercises, seventh week

10. October 2006

Each week there is a session with exercises tuesday afternoon starting at 1:30 PM (13:30). The following exercises are recommended for the session on 10. October.

Note: Some exercises are particularly recommended:
7.2, 7.3, 7.4, 7.9

Exercise 7.1: Discuss Example 3.50 and Exercise 3.51 in the notes. (Please note that there is a misprint in Exercise 3.51: in the third line from below b should be replaced by ib !)

Exercise 7.2: Let M be the splitting field over \mathbb{Q} for the polynomial $x^4 - 10x^2 + 20$. Show that M/\mathbb{Q} is a normal extension and determine the Galois group $\text{Gal}(M/\mathbb{Q})$.

Exercise 7.3: Let M_1 be the splitting field over \mathbb{Q} for $x^4 - 3$ and M_2 the splitting field over \mathbb{Q} for $x^4 + 3$.

- (1) Show that M_1 is a normal extension over \mathbb{Q} and determine the Galois group $\text{Gal}(M_1/\mathbb{Q})$. Find all subfields of M_1 whose dimension over \mathbb{Q} is 2.
- (2) Show that M_2 is a normal extension of \mathbb{Q} containing $\mathbb{Q}(i)$ and prove that $x^4 + 3$ is irreducible over $\mathbb{Q}(i)$.
- (3) Find the dimension $[M_2 : \mathbb{Q}]$ and the Galois group $\text{Gal}(M_2/\mathbb{Q})$.
- (4) Find all subfields of M_2 whose dimension over \mathbb{Q} is 2.
- (5) Find the dimension of $M_1 \cap M_2$ over \mathbb{Q} and the dimension of the compositum M_1M_2 over \mathbb{Q} .
- (6) Show that there exists a rational number a , such that $M_1M_2 = M_1(\sqrt{a})$ and use this to determine the Galois group $\text{Gal}(M_1M_2/\mathbb{Q})$.

Exercise 7.4: Let $f(x)$ be an irreducible polynomial in $\mathbb{Q}[x]$ of degree n . Assume that the number r of real roots of $f(x)$ satisfies $0 < r < n$. Let M be the splitting field for $f(x)$ over \mathbb{Q} (viewed as a subfield of the complex

number field \mathbb{C}). Show that $L = M \cap \mathbb{R}$ is a non-normal extension of \mathbb{Q} and that $[L : \mathbb{Q}] \geq n$.

(1) Show that $[M : \mathbb{Q}] \geq 2n$.

Now let $f(x)$ be $x^4 - 2x^3 - 2x + 1$.

(2) Show that $f(x) = x^4 - 2x^3 - 2x + 1$ is irreducible over \mathbb{Q} (consider $f(x+1)$).

(3) Show that $f(x)$ has exactly 2 real roots.

(4) Let M be the splitting field for $f(x)$ over \mathbb{Q} . Find $[M : \mathbb{Q}]$ and $\text{Gal}(M/\mathbb{Q})$. (Hint: Observe that a number α is a root of $f(x)$ if and only if $1/\alpha$ is a root of $f(x)$. Hence the roots of $f(x)$ have the form $\alpha, 1/\alpha, \beta, 1/\beta$ for suitable numbers α and β .)

Exercise 7.5: Let M/\mathbb{Q} be a normal extension whose Galois group is cyclic of order 4. Show that M cannot contain $i = \sqrt{-1}$. (Notice that a cyclic group of order 4 has exactly one element of order 2 and complex conjugation is an automorphism of order 2.)

Exercise 7.6: Let K/\mathbb{Q} be a finite extension.

(1) Show that K contains only finitely many roots of unity. (Hint: It can be shown that $\varphi(n) \geq \sqrt{n}/2$, where φ is Euler's function.)

(2) Suppose n is odd and that K contains a primitive n 'th root of unity. Show that K contains also a primitive $(2n)$ 'th root of unity.

Exercise 7.7: Let $F_n(x)$ be the n 'th cyclotomic polynomial. Show that if $n > 1$ is odd then $F_{2n}(x) = F_n(-x)$.

Exercise 7.8: For $n \in \mathbb{N}$ let M_n be the splitting field for the polynomial $x^n + 1$ over \mathbb{Q} . Determine the Galois group $\text{Gal}(M_n/\mathbb{Q})$.

Exercise 7.9: Are the Galois groups of the following extensions isomorphic?

(1) \mathbb{Q}_5/\mathbb{Q} and \mathbb{Q}_8/\mathbb{Q} .

(2) $\mathbb{Q}_{20}/\mathbb{Q}$ and $\mathbb{Q}_{16}/\mathbb{Q}$.

Exercise 7.10: Let M be the splitting field over \mathbb{Q} for the polynomial $x^4 + 5x^2 + 5$. Investigate whether $M = \mathbb{Q}_5$, the fifth cyclotomic field.

Exercise 7.11: Let ε_{17} be a primitive 17-th root of unity. Let

$$M = \mathbb{Q}(\varepsilon_{17} + \varepsilon_{17}^{-1}).$$

Show that M/\mathbb{Q} is normal and determine $\text{Gal}(M/\mathbb{Q})$.