Symmetric groups, combinatorics and characters 2010
Second compulsory exercises

Solutions should be handed in at the latest on 17. March

Notation: In these exercises whenever λ, μ are partitions of n then $[\lambda](\mu)$ denotes the value of the irreducible character of S_n, labelled by λ on the conjugacy class, labelled by μ.

Exercise C2.1: Let h_{ij} denote the (i,j)'th hook length of the partition λ, when $(i,j) \in \mathcal{Y}(\lambda)$. Assume that $(2,2) \in \mathcal{Y}(\lambda)$. Show that $h_{11} + h_{22} = h_{12} + h_{21}$.

Exercise C2.2: When λ is a partition, we let f_λ denote the number of standard tableaux for λ. Let $n \in \mathbb{N}$, $0 \leq k \leq n-1$ and consider the partition $\rho_{n,k} = (n-k, 1^k)$ of n.

(1) Show that $f_{\rho_{n,k}}$ equals the binomial coefficient $\binom{n-1}{k}$. This should be done in two ways: One using the hook formula and one using the enumeration of standard tableaux for $\rho_{n,k}$.

(2) Explain a connection between the branching rule applied to $\lambda = \rho_{n,k}$ and the well known recursion formula for binomial coefficients, see e.g. [AE], page 66.

(3) Let now $\lambda = (n-1, 1)$. Let $\pi_t = (t, 1^{n-t})$ be a cycle of length t i S_n, where $2 \leq t \leq n$. Compute $[\lambda](\pi_t)$ using the Murnaghan-Nakayama formula. (Hint: Part (1) above may be relevant.)

Exercise C2.3: Assume that the partition $\lambda \vdash n$ has the conjugate partition λ^0. (See [AE] 3.2 or Chapter 1 in [LN].) For the hook lengths we obviously have

$$h_{ij}^{\lambda} = h_{ji}^{\lambda^0},$$

whenever $(i,j) \in \mathcal{Y}(\lambda)$. Let for $t \geq 1$ again π_t be the partition $\pi_t = (t, 1^{n-t})$.

(1) Show, using the Murnaghan-Nakayama formula, that

$$[\lambda](\pi_t) = (-1)^{t-1}[\lambda^0](\pi_t).$$

(Here it may be useful to consider the connection between the parities of the arm length and the leg length of a hook of length t in λ.)

(2) Show, using again the Murnaghan-Nakayama formula, that more generally for all $\mu \vdash n$ we have

$$[\lambda](\mu) = (-1)^{\mathrm{sgn}(\mu)}[\lambda^0](\mu).$$

Here $\mathrm{sgn}(\mu)$ denotes the sign of μ, i.e. the sign of the permutations with cycle type μ.

Exercise C2.4: It is shown in [LN] Chapter 3 (3.18) that the only 2-cores are the partitions $\kappa_k = (k, k-1, \cdots, 1)$ for $k \geq 0$.

(1) Assume that $e > 1$. Let κ be an e-core of n. Consider the set B of partitions of $n + 2e$ having κ as an e-core. Show that B contains exactly $2e + \binom{e}{2}$ partitions.

(2) Assume in addition that e is odd. Show that the partitions in B have at most 3 different 2-cores. (You may be inspired by (3.17) in [LN].)