COMBINATORIAL REMARKS ABOUT A "REMARKABLE IDENTITY"

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Abstract. In the June 2012 issue of this Magazine, Frumosu and Teodorescu–Frumosu proved that, for all integers $m \geq 2$,

$$\sum_{p=1}^{m} \left(\frac{(-1)^p}{p!} \sum_{k_1+\cdots+k_p=m} \frac{1}{k_1 \cdots k_p} \right) = 0$$

where the inner sum is taken over all Their proof is calculus–based, relying on power series manipulations. In this note, we provide a combinatorial proof of this identity (which they requested at the end of their article) by demonstrating that the identity of Frumosu and Teodorescu–Frumosu is closely related to Stirling number of the first kind, and we use the insights gained via this connection as well as knowledge from character theory to prove several other results of a similar type.

1. Introduction

In the June 2012 issue of this Magazine, Frumosu and Teodorescu–Frumosu [1] proved that, for all integers $m \geq 2$,

$$\sum_{p=1}^{m} \left(\frac{(-1)^p}{p!} \sum_{k_1+\cdots+k_p=m} \frac{1}{k_1 \cdots k_p} \right) = 0 \tag{1.1}$$

where the inner sum is taken over all p-term *ordered partitions* of m. Their proof is calculus–based, relying on power series manipulations. In this note, we provide a combinatorial proof of (a more general version of) this identity, which the authors requested at the end of their article. It is thus by specialization possible to state numerous other concrete results of a similar type. Related results may be proved using character theory of the symmetric groups. This is discussed in the final section.

2. First Combinatorial Approach - Stirling Numbers of the First Kind

The primary step in proving a generalization of (1.1) in a combinatorial way is to rewrite the inner sum so that the sum is taken over *partitions* rather than ordered partitions. We will utilize the *rising factorial* $x^{\overline{m}}$ which for $m \geq 1$ is defined by

$$x^{\overline{m}} := x(x+1)\cdots(x+m-1). \tag{2.1}$$

The quantity $x^{\overline{m}}$ is a polynomial for each $m \geq 1$, and so it can be written as a sum of ordinary powers:

$$x^{\overline{m}} = \sum_{p=1}^{m} s(m,p) x^p \tag{2.2}$$

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The coefficients $s(m,p)$ which appear in (2.2) are called the (unsigned) *Stirling numbers of the first kind*. These numbers $s(m,p)$ satisfy numerous properties [2, 4]. The key property which we need in this note is that $s(m,p)$ counts the number of permutations of the set $\{1,2,\cdots,m\}$ with exactly p cycles in their cycle decomposition.

We now show that

$$\frac{1}{p!}\sum_{k_1+\cdots+k_p=m}\frac{1}{k_1\cdots k_p}=\frac{1}{m!}s(m,p) \tag{2.3}$$

where as in (1.1) the sum is on p-term ordered partitions of m.

Indeed

$$\begin{aligned}
&\sum_{k_1+\cdots+k_p=m}\frac{1}{k_1\cdots k_p}\\
=&\sum_{\substack{k_1\ge k_2\ge\cdots\ge k_p\ge 1\\ k_1+\cdots+k_p=m}}\frac{1}{k_1\cdots k_p}\times(\text{number of ways to permute the parts})\\
=&\sum_{\substack{t_1,t_2,\ldots,t_m\ge 0\\ t_1\cdot 1+t_2\cdot 2+\cdots+t_m\cdot m=m\\ t_1+t_2+\cdots+t_m=p}}\frac{1}{1^{t_1}2^{t_2}\ldots m^{t_m}}\times\frac{p!}{t_1!t_2!\ldots t_m!}
\end{aligned}$$

where $t_i, 1\le i\le m$, is the number of occurrences of the part i in a given partition of m. By writing the partitions of m as $t_1\cdot 1+t_2\cdot 2+\cdots+t_m\cdot m$ we are able to get an explicit handle on those partitions of m which contain exactly p parts.

From the above we see that

$$\frac{1}{p!}\sum_{k_1+\cdots+k_p=m}\frac{1}{k_1\cdots k_p}=\sum_{\substack{t_1,t_2,\ldots,t_m\ge 0\\ t_1\cdot 1+t_2\cdot 2+\cdots+t_m\cdot m=m\\ t_1+t_2+\cdots+t_m=p}}\frac{1}{1^{t_1}2^{t_2}\ldots m^{t_m}}\times\frac{1}{t_1!t_2!\ldots t_m!}$$

which is equivalent to

$$\frac{1}{p!}\sum_{k_1+\cdots+k_p=m}\frac{1}{k_1\cdots k_p}=\frac{1}{m!}\sum_{\substack{t_1,t_2,\ldots,t_m\ge 0\\ t_1\cdot 1+t_2\cdot 2+\cdots+t_m\cdot m=m\\ t_1+t_2+\cdots+t_m=p}}\frac{1}{1^{t_1}2^{t_2}\ldots m^{t_m}}\times\frac{m!}{t_1!t_2!\ldots t_m!}. \tag{2.4}$$

Now a summand in the sum on the right–hand side of (2.4) counts the number of permutations of the set $\{1,2,\cdots,m\}$ which, for each $1\le i\le m$, have exactly t_i cycles of length i in their unique cycle decomposition. This fact may be deduced directly using elementary counting methods. Therefore, the sum on the right–hand side of (2.4) equals $s(m,p)$, and this proves (2.3).

Clearly, (2.2) and (2.3) imply the following significant generalization of (1.1):

Theorem 2.1. *Let $m\ge 1$. We have the polynomial identity*

$$\sum_{p=1}^{m}\left(\frac{x^p}{p!}\sum_{k_1+\cdots+k_p=m}\frac{1}{k_1\cdots k_p}\right)=\frac{1}{m!}x^{\overline{m}}. \tag{2.5}$$

In particular Theorem 2.1 shows that the left–hand side of (1.1) equals $\frac{1}{m!}(-1)^{\overline{m}}$ which in turn equals 0, whenever $m \geq 2$, by (2.1). *Thus we have given a combinatorial proof of (1.1).*

Of course, (2.5) can be utilized to prove other combinatorial identities which are related to (1.1). For example, we see that

$$\sum_{p=1}^{m} \left(\frac{1}{p!} \sum_{k_1+\cdots+k_p=m} \frac{1}{k_1 \cdots k_p} \right) = 1 \tag{2.6}$$

for each $m \geq 1$ by substituting $x = 1$ into (2.5). Similarly, the substitution $x = 2$ in (2.5) yields

$$\sum_{p=1}^{m} \left(\frac{2^p}{p!} \sum_{k_1+\cdots+k_p=m} \frac{1}{k_1 \cdots k_p} \right) = m+1 \tag{2.7}$$

for $m \geq 1$. Lastly, for $m \geq 3$, (2.5) gives

$$\sum_{p=1}^{m} \left(\frac{(-2)^p}{p!} \sum_{k_1+\cdots+k_p=m} \frac{1}{k_1 \cdots k_p} \right) = 0 \tag{2.8}$$

via the substitution $x = -2$.

We return specifically to (2.6)–(2.8) below when we present our character–theoretic perspective.

3. Second Combinatorial Approach - Character Theory

In this section we refocus our attention to group theory, in particular to the characters of finite groups. (Character theory was invented by Frobenius as a help to study the structure of finite groups, but it has developed into an area of independent interest with applications outside of group theory. See [3] for a very thorough introduction.) In this context, we consider the permutations of Section 2 as elements of symmetric groups. We will see that from this point of view, the results in the previous section are all special cases of a more general formula involving characters. In addition, as often happens in abstraction, the more general formula quickly implies other results that aren't at all obvious from the results of Section 2.

If G is a (finite) group then a representation T of G is a map, associating to each group element $g \in G$ an invertible square matrix $T(g)$ with complex entries such that T "repects multiplication". This means that $T(g_1 g_2) = T(g_1)T(g_2)$ for all $g_1, g_2 \in G$, i.e. T is a homomorphism of groups. The character χ_T of T is defined by $\chi_T(g) = \mathrm{trace}(T(g))$, the sum of the diagonal entries in $T(g)$. A character of G is the character of some representation. One of the amazing facts about characters is that in a sense you may recover a representation from its character. You are not losing information by looking only at traces. The fact that similar matrices have the same trace implies that character values are constant on the conjugacy classes of the group. It is known that the sum and the product of two characters is again a character. A character is called *irreducible* if it cannot be written as a sum of two other characters. The simplest irreducible character of G is the *trivial character* 1_G, which maps all elements of G to 1. Any character χ may be decomposed into a sum of (not necessarily distinct) irreducible characters and it can be shown that this decomposition is unique. We denote by $a(\chi)$ the multiplicity (number of occurrences) of the trivial character 1_G in the decomposition of the character χ. It is well–known [3, Theorem 14.17] that

$$a(\chi) = \frac{1}{m!}\sum_{g\in G}\chi(g) \tag{3.1}$$

so $a(\chi)$ is really just the *average of all the values of the character* χ.

The set of all permutations of $\{1, 2, \cdots, m\}$ considered above forms a group, which is called the symmtric group S_m. The characters of S_m turn out to be non-zero integer-valued functions on the elements of S_m which take the same value on permutations having the same cycle decomposition. This is because these elements are in the same conjugacy class of S_m.

To involve the characters of S_m we start by replacing the term $(-1)^p$ in the left–hand side of (1.1) with a "weight function" $w(k_1, k_2, ..., k_p)$. That is, we define

$$\sigma(m, w) := \sum_{p=1}^{m}\left(\frac{1}{p!}\sum_{k_1+\cdots+k_p=m}\frac{w(k_1, ..., k_p)}{k_1\cdots k_p}\right)$$

for $m \geq 1$ and suitable choices of the function w. Choosing w to be identically 1 or to be $(-1)^{m-p}$, we see that $\sigma(m, w)$ equals 1 or 0, respectively, by (2.6) and (1.1). These are also examples of a special kind of weight function which we now consider.

Any character of S_m gives rise to a weight function. Thus, if χ is any character of S_m, then we may define a weight function w_χ as follows:

> $w_\chi(k_1, ..., k_p)$ is the value of χ on an element which is a product of disjoint cycles of lengths $k_1, ..., k_p$.

Apart from the the trivial character 1_{S_m} the simplest irreducible character is the sign character sgn_{S_m}. It maps an even permutation to 1 and and an odd permutation to -1. If a permutation in S_m is a product of p disjoint cycles then it is an even permutation exactly when $m - p$ is even, i.e. when $(-1)^{m-p} = 1$. Thus $w_{sgn_{S_m}}(k_1, ..., k_p) = (-1)^{m-p}$.

We have then

Theorem 3.1. *For any character χ of S_m,*

$$\sigma(m, w_\chi) = a(\chi).$$

Proof. By definition $w_\chi(k_1, ..., k_p)$ is independent of the ordering of $k_1, ..., k_p$. Therefore the calculation in the previous section shows that

$$\sigma(m, w_\chi) = \sum_{p=1}^{m}\left(\frac{1}{p!}\sum_{k_1+\cdots+k_p=m}\frac{w_\chi(k_1, \ldots, k_p)}{k_1\cdots k_p}\right) = \frac{1}{m!}\sum_{g\in S_m}\chi(g) = a(\chi). \quad \square$$

In view of the above remarks the left-hand sides of (2.6) and (1.1) equal $\sigma(m, w_{1_{S_m}})$ and $(-1)^m\sigma(m, w_{sgn_{S_m}})$, repectively. Thus (2.6) and (1.1) follow from Theorem 3.1. We also have that $a(sgn_{S_m}) = 0$. This is because 1_{S_m} obviously cannot occur in the decomposition of the irreducible character sgn_{S_m}. Thus we have gained the following additional insight: *The original identity* (1.1) *is equivalent to the well–known fact that there are equally many even and odd permutations of* $\{1, 2, \dots, m\}$.

This character–theoretic viewpoint also provides a new way to view (2.7). Namely, consider the action of S_m on the power set $\mathcal{P}_m$ of $\{1, 2, \dots, m\}$. The corresponding character χ_{pow} has the property that $\chi_{pow}(g) = 2^p$ where p is the number of cycles in g. Using [5, Example 7.18.8], we see that $a(\chi_{pow}) = m+1$. Also $a(sgn_{S_m}\chi_{pow}) = 0$. (Note that $sgn_{S_m}\chi_{pow}$ is a product of two characters and thus also a character.) Now (2.7) and (2.8) follow from Theorem 3.1.

We now discuss one final example of a "relative" of the original identity (1.1) which does not follow from Theorem 2.1. Consider the weight function w_1 defined by

$$w_1(k_1, \dots, k_p) := |\{i \mid k_i = 1\}|.$$

Then $w_1 = w_{\chi_{nat}}$ where χ_{nat} is the character of S_m acting naturally on the set $\{1, 2, \dots, m\}$. (Thus $\chi_{nat}(g)$ equals the number of fixed points of g.) As noted in [3, Corollary 29.10], χ_{nat} is a sum of 1_{S_m} and another irreducible character. Thus $a(\chi_{nat}) = 1$ and $a(sgn_{S_m}\chi_{nat}) = 0$. Therefore, we have the following "relatives" of (2.6) and (1.1) respectively:

Theorem 3.2. *For all* $m \geq 2$,

$$\sum_{p=1}^{m} \left(\frac{1}{p!} \sum_{k_1+\cdots+k_p=m} \frac{|\{i \mid k_i = 1\}|}{k_1 \cdots k_p} \right) = 1,$$

and for all $m \geq 3$,

$$\sum_{p=1}^{m} \left(\frac{(-1)^p}{p!} \sum_{k_1+\cdots+k_p=m} \frac{|\{i \mid k_i = 1\}|}{k_1 \cdots k_p} \right) = 0.$$

We close by highlighting that the first equation in Theorem 3.2 is equivalent to the following combinatorial statement:

The total number of fixed points in all permutations of $\{1, 2, \dots, m\}$ *equals* $m!$.

This is because the left-hand side of the equation is equal to $a(\chi_{nat})$ and $\chi_{nat}(g)$ counts the fixed points of g. We see also by (3.1) that in average the permutations of $\{1, 2, \dots, m\}$ have one fixed point.

There is a simple direct proof of the combinatorial statement: List the $m!$ permutations in an $(m! \times m)$-matrix where the i^{th} row contains the i^{th} permutation in some arbitrary ordering of the permutations. For example, the corresponding matrix for the case $m = 3$ can be written as follows:

$$\begin{array}{ccc}
\mathbf{1} & \mathbf{2} & \mathbf{3} \\
\mathbf{1} & 3 & 2 \\
2 & 1 & \mathbf{3} \\
2 & 3 & 1 \\
3 & 1 & 2 \\
3 & \mathbf{2} & 1
\end{array}$$

The fixed points corresponds to the occurrencies of an integer $\mathbf{j}$ in the j^{th} column of this matrix. Clearly each column contains each of the integers $1, 2, \dots, m$ with the same multiplicity of $(m-1)!$.

In particular the j^{th} column contains j with this multiplicity. Thus there is a total of $m\cdot(m-1)! = m!$ fixed points in all the permutations of $\{1, 2, \ldots, m\}$.

References

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