Block inclusions and cores of partitions

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Abstract

Necessary and sufficient conditions are given for an s-block of integer partitions to be contained in a t-block. The generating function for such partitions is found analytically, and also bijectively, using the notion of an (s,t)-abacus. The largest partition which is both an s-core and a t-core is explicitly given.

1 Introduction

The starting point for our investigation is the following question. Let n be a positive integer. Let s, t be *different* positive integers $\leq n$. Let B_s be an s-block and B_t a t-block of partitions of n. Thus B_s is the set of all partitions of n having a fixed given s-core (and similarly for B_t). Is it possible that $B_s \subseteq B_t$? A stronger question is: When is $B_s = B_t$?

The answer to the former question led us to study a class of s-core partitions called (s,t)-good partitions. It turns out that when s and t are relatively prime, there is a unique *minimal* (s,t)-good partition and this partition is also (t,s)-good. Thus it is an (s,t)-core, i.e. a partition which is simultaneously s-core and t-core.

Anderson [1] showed that the number of (s,t)-cores is finite and in fact equal to $\binom{s+t}{t}/(s+t)$. We show that the unique minimal (s,t)-good partition contains exactly $\frac{(s^2-1)(t^2-1)}{24}$ nodes and that it is also the unique *maximal* (s,t)-core. Thus any (s,t)-core has at most $\frac{(s^2-1)(t^2-1)}{24}$ nodes.

The question about equality of blocks for different integers originates from a problem formulated for a general finite group G in a paper by Navarro and Willems [8]. We consider for a prime p a p-block B_p in G simply as a subset of the set $\mathrm{Irr}(G)$ of irreducible characters of G. It was conjectured that if for different primes p, q we have $B_p = B_q$ then $|B_p| = 1$. This means that both blocks have defect 0 for their respective primes. (See [3, Lemma IV.4.19]). Recently it was noticed by C. Bessenrodt that the extension group $6.A_7$ of the alternating

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group A_7 provides a counterexample to the conjecture for non-principal blocks ($p, q = 5, 7$). However in the case of symmetric groups the conjecture is true. This is a consequence of a stronger statement.

In the case of a block inclusion $B_p \subseteq B_q$ in a finite group G we call the inclusion *trivial* if $|B_p| = 1$, i.e. if the smaller block has defect 0. This case is not particularly interesting and of course happens frequently: Whenever an irreducible character χ of G has p-defect 0, i.e. $p \nmid \frac{|G|}{\chi(1)}$, then for any other prime divisor q of $|G|$ there is a q-block B_q such that $\{\chi\} = B_p \subseteq B_q$. As noticed by Navarro there are also simple examples of non-trivial block inclusions in solvable groups, e.g. if the group has only one q-block.

By the so-called Nakayama conjecture ([5, Theorem 6.2.21]), which states that two irreducible characters of S_n are in the same p-block if and only if the partitions labelling them have the same p-core, there is then a natural correspondence between p-blocks of irreducible characters in S_n and p-blocks of partitions of n: The partitions are simply the natural labels of the irreducible characters in a block.

As shown in [7, Theorem 5.13] there is an analogue of the Nakayama conjecture for generalized blocks in S_n. This then gives a background for the more general questions about blocks of partitions formulated above.

In the case of blocks of partitions it seems reasonable to extend the concept of trivial inclusions as follows: If $t \mid s$ then two partitions which have the same s-core also have the same t-core. This is because an s-hook may be decomposed into $\frac{s}{t}$ t-hooks. Thus for any s-block there is always a t-block containing it. A *non-trivial block inclusion* $B_s \subseteq B_t$ is therefore defined as an inclusion where $t \nmid s$ and $|B_s| > 1$.

We define the *weight* $w = w(B_s)$ of an s-block of n by $w(B_s) = \frac{n-|\kappa|}{s}$, where κ is the common s-core of the partitions in B_s. It turns out that in S_n non-trivial block inclusions can only occur when B_s has weight 1 (Theorem 2.5). In particular, the Navarro-Willems conjecture is therefore valid in the symmetric groups, as is easily seen (see Corollary 2.8.)

The paper is organized as follows: In section 2 we classify the non-trivial block inclusions (and also all block equalities) and this leads to the definition of (s, t)-good partitions. In section 3 the generating function for the number of (s, t)-good partitions is computed showing that there is a unique minimal (s, t)-good partition $\kappa_{s,t}$. This partition is also (t, s)-good. In section 4 it is shown that $\kappa_{s,t}$ is also the maximal (s, t)-core. In the final section 5 we introduce an (s, t)-abacus which is used to establish a natural bijection between the sets of all (s, t)-good partitions and all s-cores. This provides also a bijective proof of the generating function identity of section 3.

2 Non-trivial block inclusions

In the following s, t are *different* positive integers.

Generally, a *β-set* is a finite subset X of $\mathbb{N}_0 = \{0, 1, 2, \cdots, \}$. Let X be a

β-set. An element $a \in X$ is called *s-maximal* if $a+s \notin X$. We call X *(s,t)-good* if all of its s-maximal elements have the same residue modulo t.

The *s-abacus* is defined as follows: It has s runners numbered $0, 1, \cdots, s-1$ running from north to south. On the i-th runner we place all non-negative integers of residue i modulo s in increasing order. A β-set may be represented by a *bead configuration* on the s-abacus by underlining on the abacus the numbers of the β-set. We refer to this also as the *s-abacus for X* and the underlined numbers are referred to as *beads*. For example, $X = \{1, 3, 4, 7, 8, 9\}$ is represented by the following bead configuration on the 3-abacus:

$$\begin{array}{ccc}
0 & \underline{1} & 2 \\
\underline{3} & \underline{4} & 5 \\
6 & \underline{7} & \underline{8} \\
\underline{9} & 10 & 11 \\
12 & 13 & 14 \\
\cdots\cdot & &
\end{array}$$

Let λ be a partition. Let $\beta(\lambda)$ be the β-set consisting of all first column hook lengths of λ. For $i \geq 0$ let $\beta(\lambda, i)$ be the set which is obtained from $\beta(\lambda)$ in the following way: It is the union of the set $\{0, 1, \cdots, i-1\}$ and the set obtained from $\beta(\lambda)$ by adding i to all its elements. (In the notation of [10] we have $\beta(\lambda, i) = \beta(\lambda)^{+i}$.) Note that $\beta(\lambda, 0) = \beta(\lambda)$. The sets $\beta(\lambda, i),\ i \geq 0$ are called the β-sets for λ. Trivially

Lemma 2.1 *Suppose that $X = \{c_1, c_2, \cdots, c_k\}$ is a β-set for the partition λ. Then λ contains*

$$|\lambda| = \sum_{i=1}^{k} c_i - \binom{k}{2}$$

nodes.

The following basic fact is needed ([5, Lemma 2.7.13], [10, Proposition (1.8)]):

Lemma 2.2 *Suppose that X is a β-set for the partition λ. Then λ contains a hook of length s if and only if there exists a $c \in X$ such that $c - s \geq 0$ and $c - s \notin X$. In this case $X \cup \{c-s\} \setminus \{c\}$ is the β-set for a partition obtained by removing an s-hook from λ.*

Let us write $X \sim_s X'$ for two β-sets if they have the same number of elements of any given residue modulo s. (Thus for example $\{1, 2, 4\} \sim_3 \{4, 5, 7\}$). The lemma above implies

Lemma 2.3 *Suppose that X and X' are β-sets for the partitions λ and λ'. If $|X| = |X'|$ then λ and λ' have the same s-core if and only if $X \sim_s X'$.*

This result is applied below in the following form:

Lemma 2.4 *Let Y be a β-set for the partition κ. Suppose that X is obtained from Y by a sequence of operations of the following form: Replace c in the set by a non-negative element $c+s$ (or $c-s$), which is not already in the set. If X is a β-set for the partition λ then this partition has the same s-core as κ.*

Theorem 2.5 *Assume $t \nmid s$. Let B_s be an s-block of n of weight $w \geq 1$ with core κ. If $w \geq 2$ then there exist two partitions in B_s with different t-cores. Thus B_s is not contained in any t-block B_t.*

If $w = 1$ then all partitions in B_s have the same t-core if and only if the β-set $\beta(\kappa, s)$ is (s,t)-good. This can only happen when s and t are relatively prime.

Remark 2.6 Before proving Theorem 2.5 we make a comment on the somewhat technical condition that the set $\beta(\kappa, s)$ is (s,t)-good. The addition of s to the β-set $\beta(\kappa)$ ensures that all runners on the s-abacus for $\beta(\kappa, s)$ are *non-empty* and that *0 is the unique bead on the 0-th runner*. Thus 0 is s-maximal in $\beta(\kappa, s)$. The condition that $\beta(\kappa, s)$ is (s,t)-good is then transformed into the following condition on the set of first column hook lengths for κ, $\beta(\kappa)$. Suppose that for $1 \leq i \leq s-1$, $\beta(\kappa)$ contains $c_i \geq 0$ integers which are congruent to i modulo s. Then $sc_i + i \equiv 0$ modulo t. This in particular forces $c_i \neq 0$ whenever $t \nmid i$. It also forces s and t to be relatively prime, e.g. since $sc_1 + 1 \equiv 0$ modulo t.

Proof of Theorem 2.5. We assume $w > 0$ and that $B_s \subseteq B_t$ for some t-block, i.e. that all partitions in B_s have the same t-core. Let $Y := \beta(\kappa, 2s)$, a β-set for κ. The set Y contains at least 2 elements of any given residue modulo s, since $\{0, 1, \cdots, 2s-1\} \subseteq Y$.

Since κ is an s-core there is by Lemma 2.2 for each i, $0 \leq i \leq s-1$ a unique element $m_i \in Y$ such that m_i is s-maximal in Y and $m_i \equiv_s i$. Note that also $m_i - s \in Y$. Consider the β-sets

$$Y_i = Y \cup \{m_i + ws\} \setminus \{m_i\}.$$

They are β-sets for partitions in B_s by Lemma 2.4.

We show that either all m_i's are congruent modulo t or $t \mid ws$.

Indeed, assume that $t \nmid ws$. Then $m_i \not\equiv_t m_i + ws$. Thus the number of elements in Y_i with the same residue as m_i is exactly one less than in Y. This is the unique residue where Y_i has fewer elements than Y. Since $Y_i \sim_t Y_j$ for all i, j, by Lemma 2.3 we must also have $m_i \equiv_t m_j$.

We should note that the case $m_i \equiv_t m_j$ for all i, j only occurs when $gcd(s,t) = 1$. Indeed if $u = gcd(s,t)$ and $m_0 \equiv_t m_1$ then $u \mid t \mid (m_1 - m_0)$. But since $m_i \equiv_s i$ we also have $m_1 - m_0 \equiv_s 1$ whence $u \mid s \mid (m_1 - m_0) - 1$ forcing $u = 1$.

If $w = 1$ the case $t \mid ws$ is not possible, since $t \nmid s$. Thus all m_i are congruent modulo t, so that $Y = \beta(\kappa, 2s)$ is (s,t)-good. The same is then true for $\beta(\kappa, s)$. As noticed above this can only happen when $gcd(s,t) = 1$. Conversely if $\beta(\kappa, s)$ and therefore Y is (s,t)-good, then $Y_i \sim_t Y_j$ for all i, j so that all partitions in B_s have the same t-core by Lemma 2.3.

Suppose that $w > 1$. Then also

$$Y_i' = Y \cup \{m_i + (w-1)s\} \setminus \{m_i - s\}$$

are β-sets for partitions in B_s and thus $Y_i \sim_t Y_j'$ for all i, j.

If $t \nmid ws$ then the number of elements in Y_i with the same residue as m_i is exactly one less than in Y and this then also has to be the case in Y_i'. Thus $m_i - s \equiv_t m_i$ contradicting $t \nmid s$.

If $t \mid ws$ then $Y_i \sim_t Y$. We then also have $Y_{i,j} \sim_t Y$, for all $i, j, i \neq j$, where

$$Y_{i,j} := Y \cup \{m_i + (w-1)s, m_j + s\} \setminus \{m_i, m_j\}.$$

This yields $m_i \equiv_t m_j + s$ for all $i \neq j$. These congruences are not compatible if $s > 2$, since $t \nmid s$. But for $s = 2$, m_0 and m_1 must have different parity. The equivalences $m_0 \equiv_t m_1 + s, m_1 \equiv_t m_0 + s$ force $t \mid 2s = 4$, i.e. $t = 4$. In that case m_0, m_1 cannot have different parity, a contradiction. $\diamond$

Corollary 2.7 *Suppose that B_s is an s-block and B_t is a t-block of n. If κ is the s-core of the partitions in B_s, then*

$$B_s \subseteq B_t \textit{ is nontrivial} \Leftrightarrow (s,t) = 1, w(B_s) = 1 \textit{ and } \beta(\kappa, s) \textit{ is } (s,t)-\textit{good}.$$

Corollary 2.8 *Suppose that B_s is an s-block and B_t is a t-block of n, where $s \neq t$. If $B_s = B_t$ then $|B_s| = 1$.*

Proof. After possibly interchanging s and t, assume that $t \nmid s$. Assume that $|B_s| \neq 1$. Apply Theorem 2.5 to the inclusion $B_s \subseteq B_t$. We get that B_s has weight 1 and thus $|B_s| = s$. Also in this case s and t are relatively prime. Thus $s \nmid t$ and the inclusion $B_t \subseteq B_s$ implies $|B_t| = t$. This is a contradiction to the assumption $B_s = B_t$. $\diamond$

The corollary shows that equality between an s-block and a t-block for $s \neq t$ happens exactly when the block contains a unique partition which is an *(s,t)-core*, i.e. simultaneously an s-core and a t-core.

We call a partition κ (s,t)-*good*, if κ is an s-core and the set $\beta(\kappa, s)$ is (s,t)-good. (See Remark 2.6 for details.)

Remark 2.9 *Whereas (s,t)-good partitions are not t-cores, it is true that all (s,t)-good partitions have the same t-core $\kappa_{s,t}$. Also, for any nontrivial inclusion $B_s \subseteq B_t$ the t-core of the partitions in B_t is always the same partition $\mu_{s,t}$. It is therefore independent of the particular (s,t)-good partition κ, which is the s-core of the partitions in B_s. These statements are proved in the final section of this paper.*

3 Generating functions

The goal of this section is to prove Theorem 3.1. Let

$$A_s(q) = \prod_{k=1}^{\infty} \frac{(1-q^{ks})^s}{1-q^k}$$

be the generating function for all partitions which are s-cores ([10], [4]).

Theorem 3.1 *Let s and t be positive integers which are relatively prime. The generating function for all partitions κ which are (s,t)-good is*

$$q^{(s^2-1)(t^2-1)/24} A_s(q^{t^2}).$$

Proof. Assume that κ is (s,t)-good. Let c_i, $0 \le i \le s-1$, be the number of elements of residue i modulo s in $\beta(\kappa)$, the first column hook lengths of κ. The number of nodes in κ can be found from the first column hook lengths. Since κ is an s-core, by Lemma 2.2 these hook lengths are

$$\bigcup_{i=1}^{s-1} \{i, i+s, \cdots, i+(c_i-1)s\},$$

so by Lemma 2.1 κ has

$$Q(c_1, c_2, \cdots, c_{s-1}) = \sum_{i=1}^{s-1} \left(s\binom{c_i}{2} + ic_i \right) - \binom{c_1 + \cdots + c_{s-1}}{2} \tag{1}$$

nodes. Thus we have

$$A_s(q) = \sum_{c_1, \cdots, c_{s-1} \ge 0} q^{Q(c_1, \cdots, c_{s-1})}. \tag{2}$$

By Remark 2.6 κ is (s,t)-good if, and only if,

$$sc_i + i \equiv 0 \mod t, \quad 1 \le i \le s-1. \tag{3}$$

Since s and t are relatively prime these equations always have unique solutions r_i, where $0 \le r_i \le t-1$, which we specify in Lemma 3.2.

Lemma 3.2 *The unique solution to* (3) *with $0 \le r_i \le t-1$ for $1 \le i \le s-1$ is given by*

$$r_i = \frac{1}{s}(t\sigma(i) - i) = \lfloor t\sigma(i)/s \rfloor,$$

where σ is the inverse of the permutation of $\{1, 2, \cdots, s-1\}$ induced by multiplication by t modulo s.

Proof. If $t = sT + m$, $0 < m \le s-1$, then m is relatively prime to s, and division by m modulo s does induce a permutation σ of $\{1, 2, \cdots, s-1\}$. For example, $\sigma(m) = 1$, and $r_m = (t-m)/s$ is a non-negative integer which is less than t.

First, since $sr_i + i = t\sigma(i)$ it is clear that (3) holds if $c_i = r_i$.

The second equality in Lemma 3.2 shows that r_i is a non-negative integer less than t. So it remains to prove this greatest integer form. Since

$$t\sigma(i) \equiv_s m\sigma(i) \equiv_s mi/m \equiv_s i,$$

$t\sigma(i) - i$ is divisible by s. Since $1 \le i < s$, the fractional part of $t\sigma(i)/s$ must be i/s. $\diamond$

For the proof of Theorem 3.1, it remains to prove the restricted generating function identity

$$\sum_{c_1, \cdots, c_{s-1} \ge 0,\ sc_i + i \equiv 0 \mod t} q^{Q(c_1, \cdots, c_{s-1})} = q^{(s^2-1)(t^2-1)/24} A_s(q^{t^2}).$$

Lemma 3.3 *If r_i is given by Lemma 3.2, then*

$$r_1 + r_2 + \cdots + r_{s-1} = (s-1)(t-1)/2.$$

Proof. This follows immediately from Lemma 3.2. $\diamond$

Lemma 3.4 *If r_i is given by Lemma 3.2, then*

$$Q(r_1, \cdots, r_{s-1}) = (s^2-1)(t^2-1)/24.$$

Proof. This follows from the explicit formula (1)

$$Q(r_1, \cdots, r_{s-1}) = \frac{s}{2} \sum_{i=1}^{s-1} r_i^2 - \frac{1}{2} \sum_{i,j=1}^{s-1} r_i r_j + \sum_{i=1}^{s-1} (-s/2 + i + 1/2) r_i.$$

When Lemma 3.2 is used in this equation, all sums are explicitly evaluable using

$$\sum_{i=1}^{s-1} i = \binom{s}{2}, \quad \sum_{i=1}^{s-1} i^2 = s(s-1)(2s-1)/6$$

except the sum

$$\frac{t}{s} \sum_{i=1}^{s-1} i\sigma(i),$$

which appears in the first and third terms, and cancels. $\diamond$

To complete the proof of Theorem 3.1, we need the following fact about Q.

Lemma 3.5 *If r_i is given by Lemma 3.2, then*

$$Q(r_1 + tn_1, \cdots, r_{s-1} + tn_{s-1}) = t^2 Q(n_{\sigma^{-1}(1)}, \cdots, n_{\sigma^{-1}(s-1)}) + Q(r_1, \cdots, r_{s-1}).$$

Proof. We have

$$Q(r_1+tn_1,\cdots,r_{s-1}+tn_{s-1}) = t^2Q(n_1,\cdots,n_{s-1})+Q(r_1,\cdots,r_{s-1})$$
$$+t\sum_{i=1}^{s-1}\Bigg(sr_i+(-s/2+i+1/2)(1-t)-(r_1+\cdots+r_{s-1})\Bigg)n_i.$$

Let $Q=Q'+L$, where

$$Q'(b_1,\cdots,b_{s-1}) = \frac{s-1}{2}\sum_{i=1}^{s-1}b_i^2-\sum_{1\le i<j\le s-1}b_ib_j$$
$$L(b_1,\cdots,b_{s-1}) = \sum_{i=1}^{s-1}(-s/2+i+1/2)b_i.$$

Since Q' is a symmetric function of $b_1,\cdots,b_{s-1}$, in order to prove Lemma 3.5, we need only consider the linear term and L, i.e.

$$\begin{aligned} tL(n_1,\cdots,n_{s-1})+&\sum_{i=1}^{s-1}\Bigg(sr_i+(-s/2+i+1/2)(1-t)-(r_1+\cdots+r_{s-1})\Bigg)n_i \\ &=tL(n_{\sigma^{-1}(1)},\cdots,n_{\sigma^{-1}(s-1)}). \end{aligned} \tag{4}$$

However (4) follows from Lemma 3.3 and Lemma 3.2. $\diamond$

Theorem 3.1 now follows from Lemma 3.5:

$$\begin{aligned} \sum_{n_1,\cdots,n_{s-1}\ge 0} q^{Q(r_1+tn_1,\cdots,r_{s-1}+tn_{s-1})} &= q^{Q(r_1,\cdots,r_{s-1})}\sum_{n_1,\cdots,n_{s-1}\ge 0} q^{t^2Q(n_{\sigma^{-1}(1)},\cdots,n_{\sigma^{-1}(s-1)})} \\ &=q^{Q(r_1,\cdots,r_{s-1})}\sum_{n_1,\cdots,n_{s-1}\ge 0} q^{t^2Q(n_1,\cdots,n_{s-1})} \\ &=q^{Q(r_1,\cdots,r_{s-1})}A_s(q^{t^2}). \end{aligned}$$

$\diamond$

Corollary 3.6 *If s and t are relatively prime positive integers with $s\ge 5$, then an (s,t)-good partition of n exists exactly when $n=(s^2-1)(t^2-1)/24+mt^2$ for some non-negative integer m.*

Proof. This follows from the Granville-Ono theorem [4] on the existence of s-cores, $s\ge 5$. $\diamond$

4 The maximum (s,t)-core

Let $\kappa_{s,t}$ be the unique (s,t)-good partition of $(s^2-1)(t^2-1)/24$, whose c_i- values are given by r_i in Lemma 3.2. Any other (s,t)-good partition has more nodes than $\kappa_{s,t}$. In this section we prove that $\kappa_{s,t}$ also has a maximum property. It is both an s- and a t-core, and has more nodes than any other partition which is also an s- and t-core. We have called partitions which are both s- and t-cores, (s,t)-cores.

Theorem 4.1 *Let s and t be relatively prime positive integers. If λ is a partition of n which is both an s-core and a t-core, then $n \leq (s^2-1)(t^2-1)/24$. Moreover there is exactly one such partition for $n = (s^2-1)(t^2-1)/24$, which is $\kappa_{s,t}$.*

Let us note the following consequence of Theorem 4.1.

Corollary 4.2 *Let s and t be relatively prime positive integers. The minimal (s,t)-good partition $\kappa_{s,t}$ is also a t-core and $\kappa_{s,t} = \kappa_{t,s}$.*

Indeed, the first statement is immediate from Theorem 4.1. Clearly $\kappa_{s,t} = \kappa_{t,s}$ since they are both equal to the unique maximum (s,t)-core.

Proof of Theorem 4.1. Anderson [1] classified all partitions which are both s- and t-cores. We review the construction here.

We may assume that $t > s$. We construct a sequence of $s-1$ columns of integers, each column increasing from top to bottom, the columns aligned at the bottom elements. The integers in the columns are referred to as *elements* and the elements of a column all have the same residue modulo s. In the last, or $(s-1)^{st}$ column, place the integers $t-s, t-2s, \cdots, t-\lfloor t/s \rfloor s$, thus $\lfloor t/s \rfloor$ elements of residue t modulo s. In the $(s-k)^{th}$ column place

$$kt-s, kt-2s, \cdots, kt-\lfloor kt/s \rfloor s.$$

By Lemma 3.2, the $(s-k)^{th}$ column has $r_{\sigma^{-1}(k)}$ elements, each of residue $\sigma^{-1}(k)$ modulo s. An example with $s=5$, $t=7$ is given below.

$$\begin{array}{cccc} 3 & & & \\ 8 & 1 & & \\ 13 & 6 & & \\ 18 & 11 & 4 & \\ 23 & 16 & 9 & 2 \end{array}$$

Figure 1: β-set for $\kappa_{5,7}$

The β-set which is the union of these columns provides a partition λ which is both an s-core and a t-core which has $c_{\sigma^{-1}(k)} = r_{\sigma^{-1}(k)}$. We see that $\lambda = \kappa_{s,t}$ because $r_i = c_i$ for $1 \leq i \leq s-1$. In our example, $\kappa_{5,7} = (12,8,7,5,4,3,3,2,1,1,1,1)$.

Proposition 4.3 *$\kappa_{s,t}$ is an (s,t)-core.*

Any other partition which is an (s,t)-core has a β-set consisting of a subset of the β-set of $\kappa_{s,t}$ which is closed under moving up or to the right. We refer to the β-set as northeast justified. So if the largest element, $st-s-t$, which is at the bottom of the first column, is in the β-set, all elements of all columns must be in the β-set.

For example take $s=5$, $t=7$, and let the β-set be $\{1,3,4,6,8,11\}$, namely the elements above the underlined elements in the diagram below.

$$\begin{array}{cccc}
3 & & & \\
8 & 1 & & \\
\overline{13} & 6 & & \\
18 & 11 & 4 & \\
23 & \overline{16} & \overline{9} & \overline{2}
\end{array}$$

Figure 2: β-set for a (5,7)-core

We must show that any partition having such a β-set has fewer nodes than $\kappa_{s,t}$.

Suppose that the β-set for an (s,t)-core consists of the top $r_i - b_i$ elements of the column which contains the residue i modulo s. We must have $0 \le b_i \le r_i$, for $1 \le i \le s-1$. The bottom b_i elements of this column do not lie in the β-set. The lengths of these non-β-set columns must increase from right to left, because the β-set is northeast justified. In the example above, $b_2 = 1$, $b_4 = 1$, $b_1 = 1$, and $b_3 = 3$, $r_2 = 1$, $r_4 = 2$, $r_1 = 4$, and $r_3 = 5$, $\sigma^{-1}(i) \equiv 2i \mod 5$. So we have

$$b_{\sigma^{-1}(1)} \le b_{\sigma^{-1}(2)} \le \cdots \le b_{\sigma^{-1}(s-1)}, \quad 0 \le b_i \le r_i, \text{ for } 1 \le i \le s-1.$$

We must prove

$$Q(r_1 - b_{\sigma^{-1}(1)}, \cdots, r_{s-1} - b_{\sigma^{-1}(s-1)}) \le Q(r_1, \cdots, r_{s-1}), \tag{5}$$

for

$$0 \le b_{\sigma^{-1}(1)} \le b_{\sigma^{-1}(2)} \le \cdots \le b_{\sigma^{-1}(s-1)}, \quad b_i \le r_i,\ 1 \le i \le s-1. \tag{6}$$

We prove (5) under the inequalities (6) in 3 steps:

Step 1: Establish the $t = s+1$ case.
Step 2: Prove that the $t \equiv 1 \mod s$ case follows from the $t = s+1$ case.
Step 3: Prove the general case from the $t \equiv 1 \mod s$ case.

Steps 2 and 3 are technically less demanding than Step 1, so we do these steps first.

Proof of Step 2. Note that if $t = sT+1$, then $\sigma(i) = i$ and the values of r_i are given by $r_i = iT$, for $1 \le i \le s-1$. We must show that

$$Q(T - b_1, 2T - b_2, \cdots, (s-1)T - b_{s-1}) \le Q(T, 2T, \cdots, (s-1)T)$$

for the region $R(T)$

$$R(T) = \{(b_1, \cdots, b_{s-1} :\ 0 \le b_1 \le \cdots \le b_{s-1},\ b_i \le iT,\ 1 \le i \le s-1\}.$$

Lemma 4.4 *Assuming* Step 1, *L and Q' are both maximized in $R(T)$ uniquely at $(T, 2T, \cdots, (s-1)T)$.*

Proof. We see that

$$\begin{aligned} & L(T, \cdots, (s-1)T) - L(T-b_1, \cdots, (s-1)T - b_{s-1}) \\ &= \sum_{i=1}^{s-1} (-s/2 + i + 1/2) b_i \\ &= \sum_{i=1}^{\lfloor (s-2)/2 \rfloor} (s/2 - i - 1/2)(b_{s-2-i} - b_i) + (s/2 - 1/2) b_{s-1} \end{aligned}$$

is non-negative because $b_{s-2-i} \geq b_i$ for $1 \leq i \leq \lfloor (s-2)/2 \rfloor$ and $b_{s-1} \geq 0$. It is zero only when $b_{s-1} = 0$, thus all $b_i = 0$, so L has a unique maximum at $(T, \cdots, (s-1)T)$.

Since Q' is a homogeneous quadratic form, we have

$$Q'(Tb_1, \cdots, Tb_{s-1}) = T^2 Q'(b_1, \cdots, b_{s-1})$$

so maximizing Q' on $R(T) = TR(1)$ is equivalent to maximizing Q' on $R(1)$. By Step 1 this occurs uniquely at $(1, \cdots, s-1)$. $\diamond$

Proof of Step 3. For Step 3, suppose that $t = sT + m$, and r_i is given by Lemma 3.2. Again we want the maximum value to occur at $(r_1, \cdots, r_{s-1})$. We will use Lemma 3.5 to reduce the inequalities in (6) to the $\sigma =$ identity case and then apply Step 2.

By Lemma 3.5 we have

$$\begin{aligned} & Q(r_1, \cdots, r_{s-1}) - Q(r_1 - b_1, \cdots, r_{s-1} - b_{s-1}) \\ &= -t^2 Q(-b_{\sigma^{-1}(1)}/t, \cdots, -b_{\sigma^{-1}(s-1)}/t) \\ &= -t^2 Q(-d_1/t, \cdots, -d_{s-1}/t) \end{aligned}$$

where $d_k = b_{\sigma^{-1}(k)}$. The allowed values of the d_k are

$$\{(d_1, \cdots, d_{s-1}) : d_k \leq r_{\sigma^{-1}(k)},\ 1 \leq k \leq s-1,\ 0 \leq d_1 \leq d_2 \leq \cdots \leq d_{s-1}\}.$$

¿From Step 2 we know that the inequality holds for $m = 1$, i.e.

$$Q(-e_1, \cdots, -e_{s-1}) < 0$$

if $(e_1, \cdots, e_{s-1}) \neq (0, \cdots, 0)$ and

$$0 \leq e_1 \leq \cdots \leq e_{s-1}, \quad 0 \leq e_k \leq kT/(sT+1), \quad 1 \leq k \leq s-1.$$

If $T \to \infty$, since Q is independent of T, we see that $Q(-e_1, \cdots, -e_{s-1}) < 0$ if $(e_1, \cdots, e_{s-1}) \neq (0, \cdots, 0)$ and

$$0 \leq e_1 \leq \cdots \leq e_{s-1}, \quad 0 \leq e_k < k/s. \tag{7}$$

We verify that the allowed d_k/t satisfy (7). Since m and s are relatively prime, we have

$$\frac{1}{k}\lfloor\frac{mk}{s}\rfloor < \frac{m}{s}.$$

The inequality to verify for d_k/t when $t = sT + m$ is

$$\begin{aligned}\frac{d_k}{t} &\le \frac{r_{\sigma^{-1}(k)}}{t} = \frac{\lfloor tk/s\rfloor}{t} = \frac{\lfloor (sT+m)k/s\rfloor}{sT+m} = \frac{kT+\lfloor mk/s\rfloor}{sT+m}\\ &= \frac{k}{s}\left(\frac{T+\frac{1}{k}\lfloor\frac{mk}{s}\rfloor}{T+m/s}\right) < \frac{k}{s}.\end{aligned}$$

Thus $Q(-d_1/t, \cdots, -d_{s-1}/t) < 0$, for the allowed d_k which are not all 0.

This completes the proof of Step 3.

Proof of Step 1. Finally we come to Step 1. For $t = s+1$, both $\kappa_{s,s+1}$ and $\beta(\kappa_{s,s+1})$ can be given explicitly. We shall use these explicit representations to prove in Lemma 4.5 that the Ferrers diagram of any $(s, s+1)$-core must be contained in the Ferrers diagram of $\kappa_{s,s+1}$.

The $s-1$ columns of the diagram for $\beta(\kappa_{s,s+1})$ form a triangular shape. The $s-i$ elements in the i^{th} column are $s-i+as, 0 \le a \le (s-i)-1$. The parts of the partition $\kappa_{s,s+1}$ are triangular numbers

$$\kappa_{s,s+1} = \binom{s}{2}^1\binom{s-1}{2}^2\cdots\binom{3}{2}^{s-2}\binom{2}{2}^{s-1}. \tag{8}$$

For example if $s = 6$,

$$\kappa_{6,7} = (15, 10, 10, 6, 6, 6, 3, 3, 3, 3, 1, 1, 1, 1, 1). \tag{9}$$

The diagram for $\beta(\kappa_{6,7})$ is given below.

$$\begin{array}{lllll} 5 & & & & \\ 11 & 4 & & & \\ 17 & 10 & 3 & & \\ 23 & 16 & 9 & 2 & \\ 29 & 22 & 15 & 8 & 1 \end{array}$$

Figure 3: β-set for $\kappa_{6,7}$

A β-set $\beta(\mu)$ for an $(s, s+1)$-core μ consists of a subset of elements of $\kappa_{s,s+1}$, which are northeast justified. For example, the elements $\beta(\mu) = \{1, 2, 4, 5, 8, 11\}$ are northeast justified in $\kappa_{6,7}$, the corresponding partition is $\mu = (6, 4, 2, 2, 1, 1)$. There is one part of μ for each element of $\beta(\mu)$. The part of μ corresponding to the element $v \in \beta(\mu)$ is

$$v - |\{w \in \beta(\mu) : w < v\}|.$$

We say that an element $(a-1)s+j$ is on the a^{th} diagonal of $\kappa_{s,s+1}$. For example 16 is on the 3^{rd} diagonal of $\kappa_{6,7}$. The elements on the a^{th} diagonal consist of the $s-a$ consecutive integers,

$$(a-1)s+a, (a-1)s+a+1, \cdots, (a-1)s+s-1.$$

We shall prove a statement stronger than Theorem 4.1.

Proposition 4.5 *If μ is any $(s, s+1)$-core, then the Ferrers diagram of μ is contained in the Ferrers diagram of $\kappa_{s,s+1}$.*

The proof of Proposition 4.5 will proceed in the following way. The idea is to push the β-set $\beta(\mu)$ to the northwest, creating a canonical $(s, s+1)$-core, whose Ferrers diagram contains the Ferrers diagram of μ. Then we must check that the Ferrers diagram of $\kappa_{s,s+1}$ contains the Ferrers diagram of any of the canonical $(s, s+1)$-cores.

We use a simple sufficient condition on β-sets for Ferrers containment. If $0 \notin \beta(\mu) = S$, then increasing any particular element of S always produces a β-set S' for a partition μ' whose Ferrers diagram contains μ. In fact, by Lemma 2.2 μ' is obtained from the partition μ by adding a hook to it.

Let μ be an $(s, s+1)$-core with β-set $\beta(\mu)$. Let d_k be the number of elements of $\beta(\mu)$ on the k^{th} diagonal. Let $D(d_k)$ be the subset of $\beta(\kappa_{s,s+1})$ consisting of the largest d_k elements in the k^{th} diagonal of $\beta(\kappa_{s,s+1})$. Then

$$diag(\mu) = \bigcup_{k=1}^{s-1} D(d_k)$$

is the β-set for a partition whose Ferrers diagram contains μ. In fact it is not hard to see that $diag(\mu)$ is northeast justified, because $d_{k-1} \geq d_k + 1$ when $d_k > 0$, $k \geq 2$.

We show that by increasing elements of $diag(\mu)$ we can obtain a β-set for a canonical $(s, s+1)$-core, denoted by $\theta_{l,b}$. We let $\theta_{l,b}$ be the $(s, s+1)$-core such that $\beta(\theta_{l,b})$ consists of the top $l-1$ rows of $\beta(\kappa_{s,s+1})$ and the last b elements of the l^{th} row of $\beta(\kappa_{s,s+1})$. (We suppress the s-dependence of $\theta_{l,b}$ as s is fixed throughout this section.) The diagram below indicates by x's where $\beta(\theta_{4,2})$ is located in $\beta(\kappa_{s,s+1})$.

```
x
x  x
x  x  x
.  .  x  x
.  .  .  .  .
```

Figure 4: $\theta_{4,2}$

Lemma 4.6 *Let μ be an $(s, s+1)$-core with exactly $\binom{l}{2} + b$ parts, for some l, b, with $0 \leq b \leq l-1 \leq s-2$. Then the Ferrers diagram of μ is contained in the Ferrers diagram of the $(s, s+1)$-core partition $\theta_{l,b}$.*

Proof. Let μ be an $(s, s+1)$-core with d_k elements on the k^{th} diagonal. Note that $d_k = 0$ for $k > l-1$, since any element of $\beta(\mu)$ on the k^{th} diagonal produces at least $\binom{k+1}{2}$ elements in $\beta(\mu)$, and $\beta(\mu)$ has fewer than $\binom{l}{2}$ elements.

Let d'_k be the number of elements of $\beta(\theta_{l,b})$ on the k^{th} diagonal, so

$$d'_{l-k} = \begin{cases} k & \text{if } 1 \leq k \leq l-b-1, \\ k+1 & \text{if } l-b \leq k \leq l-1. \end{cases}$$

To show that $\beta(\theta_{l,b})$ can be obtained by increasing elements of $diag(\mu)$, we show that

$$d'_1 + d'_2 + \cdots + d'_p \leq d_1 + d_2 + \cdots + d_p, \quad 1 \leq p \leq l-1. \tag{10}$$

Suppose, by contradiction, that p is the smallest positive integer such that

$$d'_1 + d'_2 + \cdots + d'_{p+1} > d_1 + d_2 + \cdots + d_{p+1}. \tag{11}$$

Then we must have $d'_{p+1} > d_{p+1}$. Since $diag(\mu)$ is northeast justified, all elements of $diag(\mu)$ on diagonals $p+2, \cdots, l-1$ must be contained in $\beta(\theta_{l,b})$, so $d_k \leq d'_k$ for $k > p+1$. Using (11), we see that

$$\binom{l}{2} + b = d_1 + d_2 + \cdots + d_{l-1} < d'_1 + d'_2 + \cdots + d'_{l-1} = \binom{l}{2} + b,$$

which is a contradiction. $\diamond$

To complete the proof of Proposition 4.5, we show that the flattened partition $\theta_{l,b}$ is indeed inside $\kappa_{s,s+1}$. Let us take an example and compare $\theta_{5,0} = (14, 9, 9, 5, 5, 5, 2, 2, 2, 2)$ to $\kappa_{6,7}$. The elements of $\beta(\theta_{5,0})$ are given below.

5			
11	4		
17	10	3	
23	16	9	2

Figure 5: $\beta(\theta_{5,0})$ for $s = 6$

We see that the consecutive differences of parts for $\theta_{5,0}$ and $\kappa_{6,7}$ (see (9)) are the same. The reason for this behaviour is twofold: the i^{th} largest diagonal has length i in each case and the largest elements in consecutive diagonals differ by s. This shows that the Ferrers diagram of $\theta_{l,b}$ is contained in the Ferrers diagram of $\kappa_{s,s+1}$ and

$$\theta_{l,0} = \left(\binom{s}{2} - \binom{s-l+1}{2}\right)^1 \cdots \left(\binom{s-l+2}{2} - \binom{s-l+1}{2}\right)^{l-1}.$$

If $b > 0$, we take as an example $\theta_{5,2} = (12, 7, 7, 3, 3, 3, 3, 1, 1, 1, 1, 1)$ inside $\kappa_{6,7}$.

5				
11	4			
17	10	3		
23	16	9	2	
			8	1

Figure 6: $\beta(\theta_{5,2})$ for $s = 6$

Because we have two more elements, the largest part, 14, in $\theta_{5,0}$, has decreased by 2 to 12 in $\theta_{5,2}$. As before, the consecutive differences are the same until we come to the diagonals with an extra element, which will also add one to the multiplicities. The differences progressively decrease by one as the diagonals move northeast. In the example, the differences for $\theta_{5,2}$ are $(5, 4, 2)$ while those for $\theta_{5,0}$ are $(5, 4, 3)$. This becomes

$$\theta_{l,b} = p_1^1 p_2^2 \cdots p_{l-b-1}^{l-b-1} p_{l-b}^{l-b+1} \cdots p_{l-1}^{l}$$

where

$$p_i = \begin{cases} \binom{s-i+1}{2} - \binom{s-l+1}{2} - b, & 1 \leq i \leq l-b-1, \\ \binom{s-i+1}{2} - \binom{s-l+1}{2} + (i-l), & l-b \leq i \leq l-1. \end{cases} \tag{12}$$

To test for Ferrers containment, we need only check the last "extra part" in each diagonal against the next corresponding part of $\kappa_{s,s+1}$. This is

$$p_i \leq \binom{s-i}{2}, \quad l-b \leq i \leq l-1,$$

which holds because the difference in the two sides is $\binom{s-l}{2}$.

Thus after some routine calculations we have proven the following two results.

Lemma 4.7 *The Ferrers diagram of $\theta_{l,b}$ is contained in the Ferrers diagram of $\kappa_{s,s+1}$.*

Corollary 4.8 *Suppose μ is an $(s, s+1)$-core with exactly $\binom{l}{2} + b$ parts, where $0 \leq b \leq l-1 \leq s-2$. Then the Ferrers diagram of μ has at most*

$$\binom{l+2}{4} + (s-l)\binom{l+1}{3} - b\binom{l+1}{2} + s\binom{b+1}{2}$$

nodes, and must be contained in the Ferrers diagram of $\theta_{l,b}$ given by (12).

It is possible to use Corollary 4.8 to find the next largest $(s, s+1)$-core, which are $\theta_{s-1,s-2}$ and $\theta_{s-1,0}$. We do not give the details of the proof, which is a tedious checking of inequalities.

Corollary 4.9 *There are exactly two $(s, s+1)$-cores which are partitions of $\binom{s+2}{4} - \binom{s}{2}$. Except for $\kappa_{s,s+1}$, every other $(s, s+1)$-core has fewer nodes.*

For general (s,t)-cores the appropriate number of nodes for Corollary 4.9 appears to be $(s^2-1)(t^2-1)/24-(s-1)(t-1)/2$.

Remark 4.10 *If λ and μ are (s,t)-cores, $\beta(\lambda) \subset \beta(\mu)$ does not imply $\lambda \subset \mu$. In this case one can have the size of μ less than the size of λ. An easy example is $\lambda = (3,2)$ $\mu = (2,1,1)$.*

Remark 4.11 *In view of Lemma 4.5 it seems reasonable to ask: If μ is any (s,t)-core, is the Ferrers diagram of μ contained in the Ferrers diagram of $\kappa_{s,t}$? This has been proven if $t = qs+r$ where $0 < r < s$ and $q \geq r-1$, by a group led by Anderson and Swisher* [2].

Remark 4.12 *Theorem 4.1 answers a question of Kane* [6, p. 10] *who found the size of $\kappa_{s,t}$. Puchta* [11] *stated that an (s,t)-core has at most s^2t^2 nodes. However, he applied a property of conjugates of t-cores to t-cores themselves, and his proof would appear to be incomplete.*

5 The (s,t)-abacus

Let again s,t be *relatively prime* positive integers.

We consider (s,t)-good partitions. Thus κ is (s,t)-good if it is an s-core and the set $\beta(\kappa,s)$ is (s,t)-good. This means that the block B_s of $|\kappa|+s$ with κ as core is contained in a t-block B_t.

We are going to define the (s,t)-abacus and are going to use it to establish a natural bijection between the set of all s-cores and the set of all (s,t)-good partitions. If λ is an s-core of n then the corresponding (s,t)-good partition has

$$\frac{(s^2-1)(t^2-1)}{24} + t^2 n$$

nodes. This gives then in particular a bijective proof of Theorem 3.1.

By Lemma 2.2 an s-core is uniquely determined by an $s-1$-tuple $(c_1,\cdots,c_{s-1})$ of non-negative integers, where c_i is the number of beads on the i-th runner of the s-abacus representing the first column hook lengths of the s-core.

Thus for example, if $s=5$ then the tuple (1,0,2,2) represents the following bead configuration, where again beads are shown as underlined numbers:

$$\begin{array}{ccccc} 0 & \underline{1} & 2 & \underline{3} & \underline{4} \\ 5 & 6 & 7 & \underline{8} & \underline{9} \\ 10 & 11 & 12 & 13 & 14 \\ \cdots & & & & \end{array}$$

Then the β-set $\{1,3,4,8,9\}$ of bead numbers represents the 5-core (5,5,2,2,1).

The s-core represented by $(c_1,\cdots,c_{s-1})$ contains a total of

$$\sum_{i=1}^{s-1}\left(s\binom{c_i}{2}+ic_i\right)-\binom{c_1+\cdots+c_{s-1}}{2}$$

nodes and is denoted by $\mathcal{C}(c_1, ..., c_{s-1})$.

To describe the s-cores which are (s,t)-good we need the following:

Let σ be the permutation of $\{1, \cdots, s-1\}$ described in Lemma 3.2. Thus σ is defined by the property that for $i = 1, \cdots, s-1$ we have $t\sigma(i) \equiv_s i$.

Put $r_i = \frac{t\sigma(i)-i}{s}$. Then $0 \le r_i \le t-1$ and the s-core $\kappa_{s,t} := \mathcal{C}(r_1, ..., r_{s-1})$ is (s,t)-good.

Let us consider the example $s = 5,\ t = 7$. We have $\sigma(1) = 3$, since $7 \cdot 3 \equiv_5 1$. Also $\sigma(2) = 1, \sigma(3) = 4, \sigma(4) = 2$. Moreover $r_1 = 4, r_2 = 1, r_3 = 5, r_4 = 2$. The bead configuration on the 5-abacus is

$$\begin{array}{ccccc}
0 & \underline{1} & \underline{2} & \underline{3} & \underline{4} \\
5 & \underline{6} & 7 & \underline{8} & \underline{9} \\
10 & \underline{11} & 12 & \underline{13} & 14 \\
15 & \underline{16} & 17 & \underline{18} & 19 \\
20 & 21 & 22 & \underline{23} & 24 \\
25 & 26 & 27 & 28 & 29 \\
.... & & & &
\end{array}$$

We arrange the first column hook lengths of $\kappa_{5,7}$ in the following diagram as was done in Figure 1 above:

$$\begin{array}{cccc}
3 & & & \\
8 & 1 & & \\
13 & 6 & & \\
18 & 11 & 4 & \\
23 & 16 & 9 & 2 \\
\hline
\end{array}$$

Generally we arrange the hook lengths of $\kappa_{s,t}$ in a similar way with $r_{\sigma^{-1}(s-1)}$ nodes in the first column, $r_{\sigma^{-1}(s-2)}$ nodes in the second column and so on. The nodes in the j-th column have residues $\sigma^{-1}(s-j)$ modulo s. Call this the *minimal* (s,t)*-diagram.* Put $r'_j = r_{\sigma^{-1}(s-j)}$.

We proceed to describe the (s,t)-abacus which is obtained from the usual s-abacus by changing the order of the runners using the permutation σ and adjusting the top of each runner. Let us specify here that for $1 \le j \le s-1$ runner j on the s-abacus is going to correspond to runner $\sigma(s-j)$ on the (s,t)-abacus. Going the other way runner j on the (s,t)-abacus corresponds to runner $\sigma^{-1}(s-j)$ on the s-abacus.

In detail the (s,t)-abacus is then defined as follows.

It has s runners, numbered $1, 2, ...s-1$ and 0, running from north to south. On the top of the first $s-1$ runners we place the minimal (s,t)-diagram as defined above. We extend these runners to the south by adding to the j-th runner in increasing order all remaining numbers which are congruent to the top r'_j numbers modulo s. The 0-th runner contains the numbers divisible by s in increasing order. This runner starts below and to the right of the minimal (s,t)-diagram.

As an example, here is part of the $(5,7)$-abacus.

Runner:	1	2	3	4	0
Row -5	3				
Row -4	8	1			
Row -3	13	6			
Row -2	18	11	4		
Row -1	23	16	9	2	
Row 0	28	21	14	7	0
Row 1	33	26	19	12	5
Row 2	38	31	24	17	10
Row 3	43	36	29	22	15
Row 4	48	41	34	27	20
Row 5	53	46	39	32	25
Row 6	58	51	44	37	30
Row 7	63	56	49	42	35
Row 8	68	61	54	47	40

The rows below the minimal (s,t)-diagram are numbered 0,1,2... as indicated in the example. Thus the i-th row contains a decreasing sequence of numbers which are congruent modulo t. The difference between neighbouring numbers is t and the rightmost number in the row (on the runner s) is $s \cdot i$. Clearly any non-negative integer is represented uniquely by a position on the (s,t)-abacus. If $k \geq 0$ the rows numbered $tk+i, i=0,...,t-1$ are said to be in the k-th *region* of the abacus. The rows of the minimal (s,t)-diagram are said to be in region -1 and are numbered upwards by -1,-2,... The (j,k)-*segment* is the positions on the j-th runner in the k-th region. In the above example the (3,0)-segment is the positions 14,19,24,29,34,39,44.

Generally a β-set is then represented on the (s,t)-abacus by beads in the positions given by the β-set. Since the original runners of the s-abacus are preserved in the (s,t)-abacus the removal of an s-hook from the corresponding partition is registered on the (s,t)-abacus by moving a bead to a vacant position just above it on the same runner. The removal of a t-hook from the corresponding partition is registered on the (s,t)-abacus by moving a bead in a row to a vacant position just to the right of it, if the bead is on one of the runners 1,...,$s-1$. Beads on runner 0 and in row i have to be moved to a vacant position on the first runner in row $i-t$. Thus it is moved from a region (say region k) to the region numbered one below (region $k-1$).

Let $\beta(\kappa)$ be the set of first column hook lengths of the (s,t)-good partition κ. We represent $\beta(\kappa)$ by beads on the (s,t)-abacus. Since κ is an s-core there are no vacant positions above any bead on the runner containing it. Also runner 0 is empty, as 0 is not a hook length. The condition on the s-maximal elements in $\beta(\kappa)$ explained in Remark 2.6 then implies that the lowest bead on each runner has to be at a row, whose number is congruent -1 modulo t. Thus runner j has to contain at least r'_j beads for $1 \leq j \leq s-1$. We define then $a'_j(\kappa) \geq 0$ by the property that row $a'_j(\kappa)t-1$ contains the lowest bead on runner j. Thus the number of beads on that runner is $a'_j(\kappa)t + r'_j$. The (s,t)-good partition

κ is then uniquely determined by the numbers $a'_1(\kappa), ..., a'_{s-1}(\kappa)$, since these numbers determine $\beta(\kappa)$. Note that the beads fill up complete segments on the (s,t)-abacus, since the lowest beads on a runner is the lowest in the segment containing it. For technical reasons let us define

$$\mathbf{a}(\kappa) = (a_1(\kappa), ..., a_{s-1}(\kappa)),$$

where

$$a_j(\kappa) = a'_{s-j}(\kappa).$$

The t-quotient of a partition is read off its t-abacus by considering the bead positions on the runners. In the (s,t)-abacus the runners of the t-abacus are broken into pieces. They are just all the rows whose number have the same residue mod t.

Let us consider the t-quotient of the (s,t)-good partition κ. Then the beads on the rows whose numbers are congruent to i modulo t determine one of the t partitions in the t-quotient of κ. We have seen that beads fill up complete segments on the (s,t)-abacus. This means that the bead configurations on each runner of the t-abacus of κ represent the same partition, say $\rho(\kappa)$. We then get that

$$|\kappa| = t^2|\rho(\kappa)| + |\kappa'|,$$

where κ' is the t-core of κ. Moreover it is easily seen that

$$\rho(\kappa) = \mathcal{C}(\mathbf{a}(\kappa)).$$

In particular $\rho(\kappa)$ is an s-core. We reach a bead configuration for the t-core of κ by a sequence of $|\rho(\kappa)|$ operations, where each operation consists of moving all beads in a segment to the right to a neighbouring empty segment. Here we have to specify that the segment *to the right* of the segment numbered $(0,k)$ by definition is numbered $(1,k-1)$. After these operations we reach the bead configuration for the β-set of the form $\beta(\kappa_{s,t}, tl)$, where l is the number of parts in $\rho(\kappa)$. Thus $\kappa_{s,t}$ is the t-core of κ. We conclude that the maximal (s,t)-core $\kappa_{s,t}$ is the t-core of any (s,t)-good partition. Therefore the map $\kappa \mapsto \rho(\kappa)$ is a bijection between the set of all (s,t)-good partitions and the set of all s-cores.

We have shown:

Theorem 5.1 *There is a bijection ρ between the set of all (s,t)-good partitions and the set of all s-cores. For any (s,t)-good partition κ we have*

$$|\kappa| = t^2|\rho(\kappa)| + |\kappa_{s,t}| = \frac{(s^2-1)(t^2-1)}{24} + t^2|\rho(\kappa)|.$$

Moreover $\kappa_{s,t}$ is the t-core of κ and

$$(\rho(\kappa), \rho(\kappa), ..., \rho(\kappa))$$

is the t-quotient of κ. $\diamond$

Remark 5.2 *Suppose that λ^* denotes the conjugate (transpose) partition to the partition λ. In the notation of the above theorem we get easily that $\rho(\kappa)^* = \rho(\kappa^*)$.*

Let $\mu_{s,t}$ be the partition obtained by removing the maximal hook (of length $st - s - t$) from the maximal (s,t)-core $\kappa_{s,t}$. Clearly $\mu_{s,t}$ is again an (s,t)-core. It can be seen that the northeast justified subset of $\beta(\kappa_{s,t})$ which is a β-set for $\mu_{s,t}$ may be obtained as follows: After possibly conjugating the minimal (s,t)-diagram we may assume $s < t$. Then we have to remove the bottom two elements of the 1^{st} column and the bottom element of every other column of the diagram. Indeed you get a β-set X for $\mu_{s,t}$ from $\beta(\kappa_{s,t})$ by replacing $st - s - t$ by 0. Since $1, 2, \cdots, s-1 \in \beta(\kappa_{s,t})$, $s \notin \beta(\kappa_{s,t})$ we see that $X = \beta(\mu_{s,t}, s)$.

We finish with a somewhat surprising result, involving $\mu_{s,t}$.

Theorem 5.3 *Let κ be an (s,t)-good partition. Thus the s partitions obtained by adding an s-hook to κ all have the same t-core. This t-core equals $\mu_{s,t}$ and in particular is also an s-core.*

Proof. Denote the common t-core of the s partitions obtained by adding an s-hook to κ by μ. We show $\mu = \mu_{s,t}$.

Suppose that x is the maximal element in the β-set $\beta(\kappa)$ for κ. As we have seen the row containing x in the (s,t)-abacus has a number $\equiv_t -1$. Thus $x + s$ is in a row, whose number is divisible by t. Therefore $t \mid x + s$. The partition μ is the t-core of the partition with β-set $\beta(\kappa) \cup \{x + s\} \setminus \{x\}$.

We have argued above using simultaneous movements of segments that $\beta(\kappa_{s,t}, tl)$ is a β-set for the t-core of κ which is $\kappa_{s,t}$. A similar argument shows that the bead configuration on the t-abacus for a β-set of μ is obtained from that of $\beta(\kappa_{s,t}, tl)$ by moving the bead with the maximal number to the lowest empty position (which is tl) on runner 0. The result is the bead configuration of $\beta(\mu_{s,t}, tl + m)$, where $m = min(s,t)$. $\diamond$

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