COMPLEX GROUP ALGEBRAS OF THE DOUBLE COVERS OF THE SYMMETRIC AND ALTERNATING GROUPS

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Abstract. We prove that the double covers of the alternating and symmetric groups are determined by their complex group algebras. To be more precise, let $n \geq 5$ be an integer, G a finite group, and let $\hat{\mathsf{A}}_n$ and $\hat{\mathsf{S}}_n^{\pm}$ denote the double covers of A_n and S_n, respectively. We prove that $\mathbb{C}G \cong \mathbb{C}\hat{\mathsf{A}}_n$ if and only if $G \cong \hat{\mathsf{A}}_n$, and $\mathbb{C}G \cong \mathbb{C}\hat{\mathsf{S}}_n^{+} \cong \mathbb{C}\hat{\mathsf{S}}_n^{-}$ if and only if $G \cong \hat{\mathsf{S}}_n^{+}$ or $\hat{\mathsf{S}}_n^{-}$. This in particular completes the proof of a conjecture proposed by the second and fourth authors that every finite quasi-simple group is determined uniquely up to isomorphism by the structure of its complex group algebra. The known results on prime power degrees and relatively small degrees of irreducible (linear and projective) representations of the symmetric and alternating groups together with the classification of finite simple groups play an essential role in the proofs.

1. Introduction

The complex group algebra of a finite group G, denoted by $\mathbb{C}G$, is the set of formal sums $\{\sum_{g\in G} a_g g \mid a_g \in \mathbb{C}\}$ equipped with natural rules for addition, multiplication, and scalar multiplication. Wedderburn's theorem implies that $\mathbb{C}G$ is isomorphic to the direct sum of matrix algebras over $\mathbb{C}$ whose dimensions are exactly the degrees of irreducible complex representations of G. Therefore, the study of complex group algebras and the relation to their base groups is important in group representation theory.

In an attempt to understand the connection between the structure of a finite group and its complex group algebra, R. Brauer asked in his famous paper [3, Question 2]: *when do non-isomorphic groups have isomorphic complex group algebras*? Since this question might be too general to be solved completely, it is more feasible to study more explicit questions/problems whose solutions will provide a partial answer to Brauer's question. For instance, if two finite groups have isomorphic complex group algebras and one of them is solvable, is it true that the other is also solvable? Or, if two finite groups have isomorphic complex group algebras and one of them has a

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normal Sylow p-group, can we conclude the same with the other group? We refer the reader to Brauer's paper [3] or Section 9 of the survey paper [22] by G. Navarro for more discussions on complex group algebras.

A natural problem that arises from Brauer's question is the following: given a finite group G, determine all finite groups (up to isomorphism) having isomorphic complex group algebra with G. This problem is easy for abelian groups but difficult for solvable groups in general. If G is any finite abelian group of order n, then $\mathbb{C}G$ is isomorphic to a direct sum of n copies of $\mathbb{C}$ so that the complex group algebras of any two abelian groups having the same order are isomorphic. For solvable groups, the probability that two groups have isomorphic complex group algebra is often fairly 'high'. For instance, B. Huppert pointed out in [10] that among 2328 groups of order 2^7, there are only 30 different structures of complex group algebras. In contrast to solvable groups, simple groups or more generally quasi-simple groups seem to have a stronger connection with their complex group algebras. In [25], we have conjectured that every finite quasi-simple group is determined uniquely up to isomorphism by its complex group algebra and proved it for almost all quasi-simple groups except nontrivial perfect central covers of the alternating groups.

Let A_n denote the alternating group of degree n. (Throughout the paper we always assume that $n \geq 5$ unless otherwise stated.) The *Schur covers* (or *covering groups*) of the alternating groups as well as symmetric groups were first studied and classified by I. Schur [28] in connection with their projective representations. It is known that A_n has one isomorphism class of Schur covers, which is indeed the double cover except when n is 6 or 7. We are able to prove that every double cover of an alternating group of degree at least 5 is determined uniquely by its complex group algebra.

Theorem A. *Let $n \geq 5$. Let G be a finite group and $\hat{\mathsf{A}}_n$ the double cover of A_n. Then $G \cong \hat{\mathsf{A}}_n$ if and only if $\mathbb{C}G \cong \mathbb{C}\hat{\mathsf{A}}_n$.*

We note that besides the double covers, A_6 and A_7 also have triple and 6-fold covers. We prove a similar result for these (perfect central) covers and therefore complete the proof of the aforementioned conjecture.

Theorem B. *Let G be a finite group and H a quasi-simple group. Then $G \cong H$ if and only if $\mathbb{C}G \cong \mathbb{C}H$.*

Proof. This is a consequence of Theorem A, Theorem 6.2, and [25, Corollary 1.4]. □

The symmetric group S_n has two isomorphism classes of Schur's double covers, denoted by $\hat{\mathsf{S}}_n^-$ and $\hat{\mathsf{S}}_n^+$. (See [9] for the detailed presentation of $\hat{\mathsf{S}}_n^\pm$ in terms of generators and relations.) It turns out that the complex group algebras $\mathbb{C}\hat{\mathsf{S}}_n^+$ and $\mathbb{C}\hat{\mathsf{S}}_n^-$ are canonically isomorphic. Our next result solves the above problem for the double covers of the symmetric groups.

Theorem C. *Let $n \geq 5$. Let G be a finite group and $\hat{\mathsf{S}}_n^{\pm}$ the double covers of S_n. Then $\mathbb{C}G \cong \mathbb{C}\hat{\mathsf{S}}_n^{+}$ (or equivalently $\mathbb{C}G \cong \mathbb{C}\hat{\mathsf{S}}_n^{-}$) if and only if $G \cong \hat{\mathsf{S}}_n^{+}$ or $G \cong \hat{\mathsf{S}}_n^{-}$.*

This theorem might be compared to the main result of [30] where the fourth author studied the problem for the symmetric groups.

Let $\mathrm{Irr}(G)$ denote the set of all irreducible representations (or characters) of a group G over the complex field. As mentioned above, two finite groups have isomorphic complex group algebras if and only if they have the same set of degrees (counting multiplicities) of irreducible characters. Therefore, the proofs of our main results as expected depend heavily on the representation theories of the symmetric and alternating groups, their double covers, and quasi-simple groups in general. We particularly make use of the known results on relatively small degrees and prime power degrees of the irreducible characters of $\hat{\mathsf{A}}_n$ and $\hat{\mathsf{S}}_n^{\pm}$.

The remainder of the paper is organized as follows. In the next section, we give a brief overview of the representation theory of the symmetric and alternating groups as well as their double covers, and then collect some results on prime power character degrees of these groups. The results on minimal degrees are then presented in Section 3. In Section 4, we establish some useful lemmas that will be needed later in the proofs of the main results. The proof of Theorem A is carried out in Section 5 and exceptional covers of A_6 and A_7 are treated in Section 6.

The last four sections are devoted to the proof of Theorem C. Let G be a finite group such that $\mathbb{C}G \cong \mathbb{C}\hat{\mathsf{S}}_n^{\pm}$. We will show that $G' = G''$ and therefore there exists a normal subgroup M of G such that $M \subseteq G'$ and the chief factor G'/M is isomorphic to a direct product of k copies of a non-abelian simple group S. To prove that G is isomorphic to one of $\hat{\mathsf{S}}_n^{\pm}$, one of the key steps is to show that this chief factor is isomorphic to A_n. We will do this by using the classification of simple groups to eliminate almost all possibilities for k and S. As we will see, it turns out that the case of simple groups of Lie type in even characteristic is most difficult.

In Section 7, we prove a nonexistence result for particular character degrees of $\hat{\mathsf{S}}_n^{\pm}$ and apply it in Section 8 to show that S cannot be a simple group of Lie type in even characteristic. We then eliminate other possibilities of S in Section 9 and complete the proof of Theorem C in Section 10.

Notation. Since $\mathbb{C}\hat{\mathsf{S}}_n^{+} \cong \mathbb{C}\hat{\mathsf{S}}_n^{-}$, when working with character degrees of $\hat{\mathsf{S}}_n^{\pm}$, it suffices to consider just one of the two covers. For the sake of convenience, we will write $\hat{\mathsf{S}}_n$ to denote either one of the two double covers of S_n. If X and Y are two multisets, we write $X \subseteq Y$ and say that X is a sub-multiset of Y if the multiplicity of any element in X does not exceed that of the same element in Y. For a finite group G, the number of conjugacy classes of G is denoted by $k(G)$. We write $\mathrm{Irr}_{2'}(G)$ to mean the set of all irreducible characters of G of odd degree. The set and the multiset of character degrees of G are denoted respectively by $\mathrm{cd}(G)$ and $\mathrm{cd}^*(G)$. Finally, we denote by $d_i(G)$ the ith smallest nontrivial character degree of G. Other notation is standard or will be defined when needed.

2. Prime power degrees of $\hat{\mathsf{S}}_n$ and $\hat{\mathsf{A}}_n$

In this section, we collect some results on irreducible characters of prime power degree of $\hat{\mathsf{S}}_n$ and $\hat{\mathsf{A}}_n$. The irreducible characters of the double covers of the symmetric and alternating groups are divided into two kinds: faithful characters which are also known as spin characters, and unfaithful characters which can be viewed as ordinary characters of S_n or A_n.

2.1. **Characters of S_n and A_n.** To begin, let us recall some notation and terminologies of partitions and Young diagrams in connection with representation theory of the symmetric and alternating groups. A partition λ of n is a finite sequence of natural numbers $(\lambda_1, \lambda_2, ...)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\lambda_1 + \lambda_2 + \cdots = n$. If $\lambda_1 > \lambda_2 > \cdots$, we say that λ is a *bar partition* of n (also called a strict partition of n). The Young diagram associated to λ is an array of n nodes with λ_i nodes on the ith row. At each node (i, j), we define the *hook length* $h(i, j)$ to be the number of nodes to the right and below the node (i, j), including the node (i, j).

It is well known that the irreducible characters of S_n are in one-to-one correspondence with partitions of n. The degree of the character χ_λ corresponding to λ is given by the following *hook-length formula* of Frame, Robinson, and Thrall [6]:

$$f_\lambda := \chi_\lambda(1) = \frac{n!}{\prod_{i,j} h(i,j)} .$$

Two partitions of n whose Young diagrams transform into each other when reflected about the line $y = -x$, with the coordinates of the upper left node taken as $(0, 0)$, are called conjugate partitions. The partition conjugate to λ is denoted by $\overline{\lambda}$. If $\lambda = \overline{\lambda}$, we say that λ is self-conjugate. The irreducible characters of A_n can be obtained by restricting those of S_n to A_n. More explicitly, $\chi_\lambda \downarrow_{\mathsf{A}_n} = \chi_{\overline{\lambda}} \downarrow_{\mathsf{A}_n}$ is irreducible if λ is not self-conjugate. Otherwise, $\chi_\lambda \downarrow_{\mathsf{A}_n}$ splits into two different irreducible characters of the same degree. In short, the degrees of irreducible characters of A_n are labeled by partitions of n and are given by

$$\widetilde{f_\lambda} = \begin{cases} f_\lambda & \text{if } \lambda \neq \overline{\lambda}, \\ f_\lambda/2 & \text{if } \lambda = \overline{\lambda}. \end{cases}$$

Prime power degrees of irreducible representations of the symmetric and alternating groups are classified by A. Balog, C. Bessenrodt, J. B. Olsson, and K. Ono, see [1]. This result is critical in eliminating simple groups other than A_n involved in the structure of finite groups whose complex group algebras are isomorphic to $\mathbb{C}\hat{\mathsf{A}}_n$ or $\mathbb{C}\hat{\mathsf{S}}_n$.

Lemma 2.1 ([1], Theorem 2.4)**.** *Let $n \geq 5$. An irreducible character $\chi_\lambda \in \mathrm{Irr}(\mathsf{S}_n)$ corresponding to a partition λ of n has prime power degree $f_\lambda = p^r > 1$ if and only if one of the following occurs:*

(1) $n = p^r + 1$, $\lambda = (n-1, 1)$ *or* $(2, 1^{n-2})$, *and* $f_\lambda = n - 1$;
(2) $n = 5$, $\lambda = (2^2, 1)$ *or* $(3, 2)$, *and* $f_\lambda = 5$;
(3) $n = 6$, $\lambda = (4, 2)$ *or* $(2^2, 1^2)$, *and* $f_\lambda = 9$; $\lambda = (3^2)$ *or* (2^3), *and* $f_\lambda = 5$; $\lambda = (3, 2, 1)$ *and* $f_\lambda = 16$;
(4) $n = 8$, $\lambda = (5, 2, 1)$ *or* $(3, 2, 1^2)$, *and* $f_\lambda = 64$;
(5) $n = 9$, $\lambda = (7, 2)$ *or* $(2^2, 1^5)$, *and* $f_\lambda = 27$.

Lemma 2.2 ([1], Theorem 5.1)**.** *Let $n \geq 5$. An irreducible character degree $\widetilde{f_\lambda}$ of A_n corresponding to a partition λ of n is a prime power $p^r > 1$ if and only if one of the following occurs:*

(1) $n = p^r + 1$, $\lambda = (n-1, 1)$ *or* $(2, 1^{n-2})$, *and* $\widetilde{f_\lambda} = n - 1$;
(2) $n = 5$, $\lambda = (2^2, 1)$ *or* $(3, 2)$, *and* $\widetilde{f_\lambda} = 5$; $\lambda = (3, 1^2)$ *and* $\widetilde{f_\lambda} = 3$;
(3) $n = 6$, $\lambda = (4, 2)$ *or* $(2^2, 1)$ *and* $\widetilde{f_\lambda} = 9$; $\lambda = (3^2)$ *or* (2^3) *and* $\widetilde{f_\lambda} = 5$; $\lambda = (3, 2, 1)$ *and* $\widetilde{f_\lambda} = 8$;
(4) $n = 8$, $\lambda = (5, 2, 1)$ *or* $(3, 2, 1^2)$, *and* $\widetilde{f_\lambda} = 64$;
(5) $n = 9$, $\lambda = (7, 2)$ *or* $(2^2, 1^5)$, *and* $\widetilde{f_\lambda} = 27$.

2.2. **Spin characters of S_n and A_n.** We now recall the spin representation theory of the symmetric and alternating groups, due to Schur [9, 28, 32]. For each bar partition $\mu = (\mu_1, \mu_2, ..., \mu_m)$ (i.e., $\mu_1 > \mu_2 > \cdots > \mu_m$) of n, there corresponds one or two irreducible characters (depending on whether $n - m$ is even or odd, respectively) of $\hat{\mathsf{S}}_n$ of degree

$$g_\mu = 2^{\lfloor (n-m)/2 \rfloor} \bar{g}_\mu$$

where $\bar{g}_\mu$ denotes the number of shifted standard tableaux of shape μ. This number can be computed by an analogue of the hook-length formula, the *bar formula* [9, Prop. 10.6]. The length $b(i,j)$ of the (i,j)-bar of μ is the length of the $(i, j+1)$-hook in the shift-symmetric diagram to μ (obtained by reflecting the shifted diagram to μ along the diagonal and pasting it onto μ; see [18, p.14] for details). Then

$$\bar{g}_\mu = \frac{n!}{\prod_{i,j} b(i,j)} \,.$$

The spin character degree may also be computed by the formula

$$g_\mu = 2^{\lfloor (n-m)/2 \rfloor} \frac{n!}{\mu_1! \mu_2! \cdots \mu_m!} \prod_{i<j} \frac{\mu_i - \mu_j}{\mu_i + \mu_j}.$$

Again, one can get faithful irreducible characters of $\hat{\mathsf{A}}_n$ by restricting those of $\hat{\mathsf{S}}_n^{\pm}$ to $\hat{\mathsf{A}}_n$ in the following way. If $n - m$ is odd, the restrictions of the two characters of $\hat{\mathsf{S}}_n^{\pm}$ labeled by μ to $\hat{\mathsf{A}}_n$ are the same and irreducible. Otherwise, the restriction of the one character labeled by μ splits into two irreducible characters of the same degree $g_\mu/2$. Let $\widetilde{g}_\mu$ be the degree of the irreducible spin character(s) of $\hat{\mathsf{A}}_n$ labeled by the bar partition μ; we then have

$$\widetilde{g}_\mu = \begin{cases} g_\mu & \text{if } n-m \text{ is odd,} \\ g_\mu/2 & \text{if } n-m \text{ is even.} \end{cases}$$

The classification of spin representations of prime power degree of the symmetric and alternating groups has been done by the first and third authors of this paper in [2].

Lemma 2.3 ([2], Theorem 4.2). *Let $n \geq 5$ and μ be a bar partition of n. The spin irreducible character degree g_μ of $\hat{\mathsf{S}}_n$ corresponding to μ is a prime power if and only if one of the following occurs:*

(1) $\mu = (n)$ *and* $g_\mu = 2^{\lfloor (n-1)/2 \rfloor}$;
(2) $n = 2^r + 2$ *for some* $r \in \mathbb{N}$, $\mu = (n-1, 1)$, *and* $g_\mu = 2^{2^{r-1}+r}$;
(3) $n = 5$, $\mu = (3, 2)$, *and* $g_\mu = 4$;
(4) $n = 6$, $\mu = (3, 2, 1)$, *and* $g_\mu = 4$;
(5) $n = 8$, $\mu = (5, 2, 1)$, *and* $g_\mu = 64$.

Lemma 2.4 ([2], Theorem 4.3). *Let $n \geq 5$ and μ be a bar partition of n. The spin irreducible character degree $\widetilde{g}_\mu$ of $\hat{\mathsf{A}}_n$ corresponding to μ is a prime power if and only if one of the following occurs:*

(1) $\mu = (n)$ *and* $\widetilde{g}_\mu = 2^{\lfloor (n-2)/2 \rfloor}$;
(2) $n = 2^r + 2$ *for some* $r \in \mathbb{N}$, $\mu = (n-1, 1)$, *and* $\widetilde{g}_\mu = 2^{2^{r-1}+r-1}$;
(3) $n = 5$, $\mu = (3, 2)$, *and* $\widetilde{g}_\mu = 4$;
(4) $n = 6$, $\mu = (3, 2, 1)$, *and* $\widetilde{g}_\mu = 4$;
(5) $n = 8$, $\mu = (5, 2, 1)$, *and* $\widetilde{g}_\mu = 64$.

3. Low degrees of $\hat{\mathsf{S}}_n$ and $\hat{\mathsf{A}}_n$

We present in this section some results on minimal degrees of both ordinary and spin characters of the symmetric and alternating groups. We start with ordinary characters.

Lemma 3.1 (Rasala [27]). *The following hold:*

(1) $d_1(\mathsf{S}_n) = n - 1$ *if* $n \geq 5$;
(2) $d_2(\mathsf{S}_n) = n(n-3)/2$ *if* $n \geq 9$;
(3) $d_3(\mathsf{S}_n) = (n-1)(n-2)/2$ *if* $n \geq 9$;
(4) $d_4(\mathsf{S}_n) = n(n-1)(n-5)/6$ *if* $n \geq 13$;
(5) $d_5(\mathsf{S}_n) = (n-1)(n-2)(n-3)/6$ *if* $n \geq 13$;
(6) $d_6(\mathsf{S}_n) = n(n-2)(n-4)/3$ *if* $n \geq 15$;

(7) $d_7(\mathsf{S}_n) = n(n-1)(n-2)(n-7)/24$ *if* $n \geq 15$.

Lemma 3.2 (Tong-Viet [30])**.** *If* $n \geq 15$ *then* $d_i(\mathsf{A}_n) = d_i(\mathsf{S}_n)$ *for* $1 \leq i \leq 4$ *and if* $n \geq 22$ *then* $d_i(\mathsf{A}_n) = d_i(\mathsf{S}_n)$ *for* $1 \leq i \leq 7$.

The minimal degrees of spin irreducible representations of $\hat{\mathsf{A}}_n$ and $\hat{\mathsf{S}}_n$ were obtained by A. Kleshchev and P. H. Tiep, see [13, 14]. These minimal degrees are indeed the degrees of the *basic spin* and *second basic spin* representations. Let $\mathfrak{d}_1(\hat{\mathsf{A}}_n)$ and $\mathfrak{d}_1(\hat{\mathsf{S}}_n)$ denote the smallest degrees of irreducible spin characters of $\hat{\mathsf{A}}_n$ and $\hat{\mathsf{S}}_n$, respectively.

Lemma 3.3 ([13], Theorem A)**.** *Let* $n \geq 8$. *The smallest degree of the irreducible spin characters of* $\hat{\mathsf{A}}_n$ *(*$\hat{\mathsf{S}}_n$*, resp.) is* $\mathfrak{d}_1(\hat{\mathsf{A}}_n) = 2^{\lfloor (n-2)/2 \rfloor}$ *(*$\mathfrak{d}_1(\hat{\mathsf{S}}_n) = 2^{\lfloor (n-1)/2 \rfloor}$*, resp.) and there is no degree between* $\mathfrak{d}_1(\hat{\mathsf{A}}_n)$ *(*$\mathfrak{d}_1(\hat{\mathsf{S}}_n)$*, resp.) and* $2\mathfrak{d}_1(\hat{\mathsf{A}}_n)$ *(*$2\mathfrak{d}_1(\hat{\mathsf{S}}_n)$*), resp.).*

Using the above results, we easily deduce the following.

Lemma 3.4. *The following hold:*

(1) *If* $n \geq 31$ *then* $d_i(\hat{\mathsf{S}}_n) = d_i(\mathsf{S}_n)$ *for* $1 \leq i \leq 7$.
(2) *If* $n \geq 34$ *then* $d_i(\hat{\mathsf{A}}_n) = d_i(\mathsf{A}_n)$ *for* $1 \leq i \leq 7$.

Proof. We observe that $\mathfrak{d}_1(\hat{\mathsf{S}}_n) = 2^{\lfloor (n-1)/2 \rfloor} > n(n-1)(n-2)(n-7)/24 = d_7(\mathsf{S}_n)$ if $n \geq 31$. Therefore part (1) follows by Lemmas 3.1 and 3.3. Similarly, we have $\mathfrak{d}_1(\hat{\mathsf{A}}_n) = 2^{\lfloor (n-2)/2 \rfloor} > n(n-1)(n-2)(n-7)/24 = d_7(\mathsf{A}_n)$ if $n \geq 34$ and thus part (2) follows. ☐

Lemma 3.5. *Let* G *be either* $\hat{\mathsf{A}}_n$ *or* $\hat{\mathsf{S}}_n$. *Then we have*

(1) *If* $n \geq 8$ *then* $d_1(G) = n-1$.
(2) *If* $n \geq 10$ *then* $d_2(G) = \min\{n(n-3)/2, \mathfrak{d}_1(G)\}$. *Furthermore, if* $n \geq 12$ *then* $d_2(G) > 2n$.
(3) *If* $n \geq 16$ *then* $d_3(G) = (n-1)(n-2)/2$ *and* $d_4(G) = \min\{n(n-1)(n-5)/6, \mathfrak{d}_1(G)\}$.
(4) *If* $n \geq 28$ *then* $d_4(G) = n(n-1)(n-5)/6$, $d_5(G) = (n-1)(n-2)(n-3)/6$, $d_6(G) = n(n-2)(n-4)/3$, *and* $d_7(G) = \min\{n(n-1)(n-2)(n-7)/24, \mathfrak{d}_1(G)\}$.

Proof. When $n \geq 8$, we see that $\mathfrak{d}_1(\hat{\mathsf{S}}_n) \geq \mathfrak{d}_1(\hat{\mathsf{A}}_n) = 2^{\lfloor (n-2)/2 \rfloor} > n-1$ and $d_1(\mathsf{S}_n) = d_1(\mathsf{A}_n) = n-1$, which imply that $d_1(\hat{\mathsf{A}}_n) = d_1(\hat{\mathsf{S}}_n) = n-1$ and part (1) follows. When $n \geq 10$, we observe that $d_2(\mathsf{A}_n) = d_2(\mathsf{S}_n) = n(n-3)/2$ and part (2) then follows from part (1).

Suppose that $n \geq 16$. It is easy to check that $2^{\lfloor (n-2)/2 \rfloor} > (n-1)(n-2)/3 = d_3(\mathsf{A}_n) = d_3(\mathsf{S}_n)$. It follows that $d_3(\hat{\mathsf{A}}_n) = d_3(\hat{\mathsf{S}}_n) = (n-1)(n-2)/2$ and $d_4(G) = \min\{n(n-1)(n-5)/6, \mathfrak{d}_1(G)\}$, as claimed.

Finally suppose that $n \geq 28$. We check that $2^{\lfloor (n-2)/2 \rfloor} > n(n-2)(n-4)/3 = d_6(\mathsf{A}_n) = d_6(\mathsf{S}_n)$ and part (4) then follows. ☐

4. Some useful lemmas

We begin with an easy observation.

Lemma 4.1. *There always exists a prime number p with $n < p \leq d_2(G)$ for $G = \hat{\mathsf{A}}_n$ or $\hat{\mathsf{S}}_n$, provided that $n \geq 9$.*

Proof. It is routine to check the statement for $9 \leq n \leq 11$ by using [5]. So we can assume that $n \geq 12$. By Lemma 3.5, it suffices to prove that there exists a prime between n and $2n$. However, this is the well-known Bertrand-Chebyshev theorem [8]. □

The next three lemmas are critical in the proof of Theorem A.

Lemma 4.2. *Let S be a simple group of Lie type of rank l defined over a field of q elements with q even. Then*

$$|\mathrm{Irr}_{2'}(S)| \geq \begin{cases} q^l/(l+1, q-1) & \text{if } S \text{ is of type } A, \\ q^l/(l+1, q+1) & \text{if } S \text{ is of type } {}^2A, \\ q^l/3 & \text{if } S \text{ is of type } E_6, {}^2E_6, \\ q^l & \text{otherwise.} \end{cases}$$

Proof. Let S_{sc} be the finite Lie type group of simply-connected type corresponding to S. By [4, Corollary 3.6], S_{sc} has q^l semisimple conjugacy classes. For each semisimple class of S_{sc}, Lusztig's classification of complex characters of finite groups of Lie type says that there corresponds a semisimple character of the dual group, say S_{sc}^*, of S_{sc} of degree

$$|S_{sc}|_{2'}/|\mathbf{C}_{S_{sc}}(s)|_{2'} .$$

This means that S_{sc}^* of S_{sc} has at least q^l irreducible characters of odd degree.

If S is of type A, we have $S_{sc}^* = \mathrm{PGL}_{l+1}(q) = S.(l+1, q-1)$ and the lemma follows for linear groups. A similar argument works for unitary groups and E_6 as well as ${}^2E_6$. If S is not of these types, we will have $S = S_{sc} = S_{sc}^*$ and the lemma also follows. □

Lemma 4.3. *Let $n \geq 5$ and let $n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_t}$ be the binary expansion of n with $k_1 > k_2 > \cdots > k_t \geq 0$. Then*

$$|\mathrm{Irr}_{2'}(\hat{\mathsf{A}}_n)| \leq |\mathrm{Irr}_{2'}(\hat{\mathsf{S}}_n)| = 2^{k_1+k_2+\cdots+k_t} .$$

Proof. If $\mu = (\mu_1, \mu_2, ..., \mu_m)$ is a bar partition of n, the 2-part of the spin character degree g_μ of $\hat{\mathsf{S}}_n$ labeled by μ is at least

$$2^{\lfloor (n-m)/2 \rfloor},$$

which is at least 2 as $n \geq 5$ and $n - m \geq 3$. We note that if $n - m = 3$ then $\widetilde{g}_\mu = g_\mu$. Therefore, the 2-part of the degree $\widetilde{g}_\mu$ of $\hat{\mathsf{A}}_n$ is at least 2 as well. In particular, we see that every spin character degree of $\hat{\mathsf{A}}_n$ as well as $\hat{\mathsf{S}}_n$ is even. It follows that

$$|\mathrm{Irr}_{2'}(\hat{\mathsf{A}}_n)| = |\mathrm{Irr}_{2'}(\mathsf{A}_n)| \text{ and } |\mathrm{Irr}_{2'}(\hat{\mathsf{S}}_n)| = |\mathrm{Irr}_{2'}(\mathsf{S}_n)| .$$

As mentioned in [20], the number of odd degree irreducible characters of A_n does not exceed that of S_n. Now the lemma follows from the formula for the number of odd degree characters of S_n given in [17, Corollary 1.3]. □

Lemma 4.4. *Let $n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_t}$ be the binary expansion of n with $k_1 > k_2 > \cdots > k_t \geq 0$.*

(1) *If $k_1 + k_2 + \cdots + k_t \geq \sqrt{(n-3)/2}$ then $n < 2^{15}$;*
(2) *if $k_1 + k_2 + \cdots + k_t \geq \sqrt{n-3} - 3$ then $n < 2^{13}$;*
(3) *if $k_1 + k_2 + \cdots + k_t \geq (n-3)/18$ then $n < 2^{10}$;*
(4) *if $k_1 + k_2 + \cdots + k_t \geq (n-3)/30$ then $n < 2^{11}$.*

Proof. We only give here a proof for part (2). The other statements are proved similarly. As $n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_t}$, we get $k_1 = \lfloor \log_2 n \rfloor$ and hence

$$k_1 + k_2 + \cdots + k_t \geq \lfloor \log_2 n \rfloor (\lfloor \log_2 n \rfloor + 1)/2 .$$

However, it is easy to check that $\sqrt{(n-3)} - 3 > \lfloor \log_2 n \rfloor (\lfloor \log_2 n \rfloor + 1)/2$ if $n \geq 2^{14}$. For $2^{13} \leq n < 2^{14}$, the statement follows by direct computations. □

The following lemma is probably known but we include a short proof for the reader's convenience. It will be needed in the proof of Theorem C.

Lemma 4.5. *Let $k(\hat{\mathsf{A}}_n)$ and $k(\hat{\mathsf{S}}_n)$ denote the number of conjugacy classes of $\hat{\mathsf{A}}_n$ and $\hat{\mathsf{S}}_n$, respectively. Then*

$$k(\hat{\mathsf{S}}_n) < 2k(\hat{\mathsf{A}}_n) .$$

Proof. Let $\mathrm{Irr}_{\mathrm{faithful}}(G)$ and $\mathrm{Irr}_{\mathrm{unfaithful}}(G)$ denote the sets of faithful irreducible characters and unfaithful irreducible characters, respectively, of a group G. Let a and b be the numbers of conjugate partitions and non-conjugate partitions, respectively, of n. Also, let c and d be the numbers of bar partitions of n with $n - m$ even and odd, respectively. We have

$$|\mathrm{Irr}_{\mathrm{unfaithful}}(\hat{\mathsf{S}}_n)| = k(\mathsf{S}_n) = a + 2b \text{ and } |\mathrm{Irr}_{\mathrm{faithful}}(\hat{\mathsf{S}}_n)| = c + 2d$$

and

$$|\mathrm{Irr}_{\mathrm{unfaithful}}(\hat{\mathsf{A}}_n)| = k(\mathsf{A}_n) = 2a + b \text{ and } |\mathrm{Irr}_{\mathrm{faithful}}(\hat{\mathsf{A}}_n)| = 2c + d .$$

Therefore,

$$k(\hat{\mathsf{S}}_n) = a + 2b + c + 2d \text{ and } k(\hat{\mathsf{A}}_n) = 2a + b + 2c + d$$

and the lemma follows. □

5. Complex group algebras of $\hat{\mathsf{A}}_n$ - Theorem A

The aim of this section is to prove Theorem A. Let G be a finite group such that $\mathbb{C}G \cong \mathbb{C}\hat{\mathsf{A}}_n$. Then G has exactly one linear character, which is the trivial one, so that G is perfect. Let M be a maximal normal subgroup of G. We then have G/M is non-abelian simple and moreover

$$\mathrm{cd}^*(G/M) \subseteq \mathrm{cd}^*(G) = \mathrm{cd}(\hat{\mathsf{A}}_n).$$

To prove the theorem, it is clear that we first have to show $G/M \cong \mathsf{A}_n$. We will work towards this aim.

Proposition 5.1. *Let S be a non-abelian simple group such that $\mathrm{cd}^*(S) \subseteq \mathrm{cd}^*(\hat{\mathsf{A}}_n)$. Then S is isomorphic to A_n or to a simple group of Lie type in even characteristic.*

Proof. We will eliminate other possibilities of S by using the classification of finite simple groups. If $5 \le n \le 9$, then $\pi(S) \subseteq \pi(\hat{\mathsf{A}}_n) \subseteq \{2, 3, 5, 7\}$, hence by using [11, Theorem III] and [5], the result follows easily. From now on we assume that $n \ge 10$.

(i) Alternating groups: Suppose that $S = A_m$ with $7 \le m \ne n$. Since $\mathrm{cd}(A_m) \subseteq \mathrm{cd}(\hat{\mathsf{A}}_n)$, we get $d_1(\mathsf{A}_m) \ge d_1(\hat{\mathsf{A}}_n)$. As $d_1(\hat{\mathsf{A}}_n) = n - 1 \ge 9$ by Lemma 3.5, it follows that $d_1(\mathsf{A}_m) \ge 9$. Thus $m \ge 10$, and so $m - 1 = d_1(A_m) \ge d_1(\hat{\mathsf{A}}_n) = n - 1$. In particular, we have $m > n$ as $m \ne n$. We now deduce that $d_1(\mathsf{A}_m) > d_1(\hat{\mathsf{A}}_n)$ and hence $d_1(\mathsf{A}_m) \ge d_2(\hat{\mathsf{A}}_n)$.

By Lemma 4.1, there exists a prime p with $n < p \le d_2(\hat{\mathsf{A}}_n)$. As $d_2(\hat{\mathsf{A}}_n) \le d_1(\mathsf{A}_m) = m - 1$, we deduce that $p < m$. It follows that $p \mid |\mathsf{A}_m|$ and therefore A_m has an irreducible character of degree divisible by p by the Ito-Michler theorem (see [19, Remark 13.13]). However, this is a contradiction as $\mathrm{cd}(\mathsf{A}_m) \subseteq \mathrm{cd}(\hat{\mathsf{A}}_n)$ and $p \nmid |\hat{\mathsf{A}}_n|$.

(ii) Simple groups of Lie type in odd characteristic: Suppose that $S = G(p^k)$, a simple group of Lie type defined over a field of p^k elements with p odd. Since $|S|_p$ is the degree of the Steinberg character of S, we have $|S|_p \in \mathrm{cd}(\hat{\mathsf{A}}_n)$. As $|S|_p$ is an odd prime power, Lemma 2.4 implies that $|S|_p$ must be a degree of an unfaithful character of $\hat{\mathsf{A}}_n$. In other words, $|S|_p \in \mathrm{cd}(\mathsf{A}_n)$. Using Lemma 2.2, we deduce that $|S|_p = n-1$. Hence, $|S|_p = d_1(\hat{\mathsf{A}}_n)$ is the smallest nontrivial degree of $\hat{\mathsf{A}}_n$ by Lemma 3.5. However, by [31, Lemma 8] we have $d_1(S) < |S|_p = d_1(\hat{\mathsf{A}}_n)$, which is impossible as $\mathrm{cd}(S) \subseteq \mathrm{cd}(\hat{\mathsf{A}}_n)$.

(iii) Sporadic simple groups and the Tits group: Using [7], we can assume that $n \ge 14$. To eliminate these cases, observe that $n \ge \max\{p(S), 14\}$, where $p(S)$ is the largest prime divisor of $|S|$, and that $d_i(S) \ge d_i(\hat{\mathsf{A}}_n)$ for all $i \ge 1$. With this lower bound on n, we find the lower bounds for $d_i(\hat{\mathsf{A}}_n)$ with $1 \le i \le 7$ using Lemmas 3.4 and 3.5. Choose $i \in \{2, 3, \cdots, 7\}$ such that $d_i(\hat{\mathsf{A}}_n) > d_j(S)$ for some $j \ge 1$ such that $|i - j|$ is minimal. If $j \ge i$, then we obtain a contradiction. If $j < i$, then $d_j(S) \in \{d_k(\hat{\mathsf{A}}_n)\}_{k=j}^{i-1}$. Solving these equations for n, we then obtain that either these equations have no solution or for each solution of n, we can find some $k \geq 1$ with $d_k(\hat{\mathsf{A}}_n) > d_k(S)$. As an example, assume that $S = O'N$. Then $n \geq 31$ since $p(S) = 31$. We have $d_7(\hat{\mathsf{A}}_n) = n(n-1)(n-2)(n-7)/24 \geq 26970$. As $d_4(S) = 26752 < d_7(\hat{\mathsf{A}}_n)$, it follows that $d_4(S) \in \{d_4(\hat{\mathsf{A}}_n), d_5(\hat{\mathsf{A}}_n), d_6(\hat{\mathsf{A}}_n)\}$. However, one can check that these equations have no integer solution. □

Proposition 5.2. *Let S be a non-abelian simple group such that $|S| \mid n!$ and $\mathrm{cd}^*(S) \subseteq \mathrm{cd}^*(\hat{\mathsf{A}}_n)$. Then $S \cong \mathsf{A}_n$.*

Proof. In light of Proposition 5.1 and its proof, it remains to assume that $n \geq 9$ and to prove that S cannot be a simple group of Lie type in even characteristic. Assume to the contrary that $S = G_l(2^k)$, a simple group of Lie type of rank l defined over a field of $q = 2^k$ elements. As above, we then have $|S|_2 \in \mathrm{cd}(\hat{\mathsf{A}}_n)$. By Lemmas 2.2 and 2.4, we have that $|S|_2 = n-1$, $|S|_2 = 2^{\lfloor (n-2)/2 \rfloor}$, or $|S|_2 = 2^{n/2+\log_2(n-2)-2}$ when $n = 2^r + 2$. Since the case $|S|_2 = n-1$ can be eliminated as in the proof of the previous proposition, we can assume further that

$$|S|_2 = 2^{\lfloor (n-2)/2 \rfloor} \text{ or } |S|_2 = 2^{n/2+\log_2(n-2)-2} \text{ when } n = 2^r + 2\,.$$

Recalling the hypothesis that $\mathrm{cd}^*(S) \subseteq \mathrm{cd}^*(\hat{\mathsf{A}}_n)$, we have

$$|\mathrm{Irr}_{2'}(S)| \leq |\mathrm{Irr}_{2'}(\hat{\mathsf{A}}_n)|. \tag{5.1}$$

(i) $S = B_l(2^k) \cong C_l(2^k)$, $D_l(2^k)$, or ${}^2D_l(2^k)$. Then $|S|_2 = 2^{kl^2}$ or $2^{kl(l-1)}$. In particular,

$$|S|_2 \leq 2^{kl^2}\,.$$

As $|S|_2 \geq 2^{\lfloor (n-2)/2 \rfloor}$, it follows that

$$kl^2 \geq \lfloor (n-2)/2 \rfloor \geq (n-3)/2\,.$$

Therefore

$$kl \geq \sqrt{(n-3)/2}\,.$$

Using Lemma 4.2, we then obtain

$$|\mathrm{Irr}_{2'}(S)| \geq q^l = 2^{kl} \geq 2^{\sqrt{(n-3)/2}}\,.$$

Now Lemma 4.3 and Equation (5.1) imply

$$k_1 + k_2 + \cdots + k_t \geq \sqrt{(n-3)/2},$$

where $n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_t}$ is the binary expansion of n. Invoking Lemma 4.4(1), we obtain that $n < 2^{15}$.

(ii) $S = A_l(2^k)$ or ${}^2A_l(2^k)$. Arguing as above, we have

$$k_1 + k_2 + \cdots + k_t \geq \sqrt{n-3} - 3,$$

which forces $n < 2^{13}$ by Lemma 4.4(2).

(iii) S is a simple group of exceptional Lie type. Using Lemma 4.4(3,4), we deduce that $n < 2^{11}$.

For each of the above cases, a computer program has checked that either $|S|$ does not divide $n!$ or S has an irreducible character degree not belonging to $\mathrm{cd}(\hat{\mathsf{A}}_n)$ for "small" n. This contradiction completes the proof. Let us describe one example where $S = \mathrm{Sp}_{2l}(2^k) \cong \Omega_{2l+1}(2^k)$. Then we have

$$kl^2 = \lfloor (n-2)/2 \rfloor \text{ or } kl^2 = n/2 + \log_2(n-2) - 2 \text{ when } n = 2^r + 2.$$

Moreover, the condition $|S| \mid n!$ is equivalent to

$$2^{kl^2} \prod_{i=1}^{l} (2^{2ki} - 1) \mid n!.$$

By computer computations, we can determine all triples (k, l, n) with $n < 2^{15}$ satisfying the above conditions. It turns out for each such triple, n is at most 170 and one of the three smallest character degrees of S is not a character degree of $\hat{\mathsf{A}}_n$. The low-degree characters of simple groups of Lie type can be found in [29, 15, 23]. □

We are now ready to prove the first main result.

***Proof of Theorem* A.** Recall the hypothesis that G is a finite group such that $\mathbb{C}G \cong \mathbb{C}\hat{\mathsf{A}}_n$. Therefore $\mathrm{cd}^*(G) = \mathrm{cd}^*(\hat{\mathsf{A}}_n)$. In particular, we have $|G| = |\hat{\mathsf{A}}_n|$ and $G = G'$ since $\hat{\mathsf{A}}_n$ has only one linear character. Let M be a maximal normal subgroup of G. Then G/M is a non-abelian simple group, say S. It follows that $\mathrm{cd}^*(S) = \mathrm{cd}^*(G/M) \subseteq \mathrm{cd}^*(G)$ and hence

$$\mathrm{cd}^*(S) \subseteq \mathrm{cd}^*(\hat{\mathsf{A}}_n)\,.$$

We also have

$$|S| \mid |G| = |\hat{\mathsf{A}}_n| = n!.$$

Applying Propositions 5.1 and 5.2, we deduce that $S \cong \mathsf{A}_n$.

We have shown that $G/M \cong \mathsf{A}_n$. Since $|G| = |\hat{\mathsf{A}}_n| = 2|\mathsf{A}_n|$, we obtain $|M| = 2$. In particular, M is central in G and therefore $M \subseteq \mathbf{Z}(G) \cap G'$. Thus $G \cong \hat{\mathsf{A}}_n$, as desired. □

6. Triple and 6-fold covers of A_6 and A_7

In this section, we aim to prove that every perfect central cover is uniquely determined up to isomorphism by the structure of its complex group algebra. To do that, we need the following result from [25, Lemma 2.5].

Lemma 6.1. *Let S be a nonabelian simple group different from an alternating group of degree greater than* 13. *Assume that S is different from* $\mathrm{PSL}_3(4)$ *and* $\mathrm{PSU}_4(3)$, *and let* $\mathrm{Mult}(S)$ *denote the Schur multiplier of S. Let G be a perfect group and*

$M \lhd G$ such that $G/M \cong S$, $|M| \leq |\mathrm{Mult}(S)|$, and $\mathrm{cd}(G) \subseteq \mathrm{cd}(\mathrm{Schur}(S))$. Then G is uniquely determined up to isomorphism by S and the order of G.

Now we prove the main result of this section.

Theorem 6.2. *Let G be a finite group and H a perfect central cover of A_6 or A_7. Then $G \cong H$ if and only if $\mathbb{C}G \cong \mathbb{C}H$.*

Proof. First, as $\mathsf{A}_6 \cong \mathrm{PSL}_2(9)$, every perfect central cover of A_6 can be viewed as a quasi-simple classical group, which is already studied in [24, Theorem 1.1]. So it remains to consider the perfect central covers of A_7. Let H be one of those and assume that G is a finite group such that $\mathbb{C}G \cong \mathbb{C}H$.

As before, we see that G is perfect and if M is a normal maximal subgroup of G, we have G/M is non-abelian simple and $\mathrm{cd}^*(G/M) \subseteq \mathrm{cd}^*(H)$. In particular,

$$\mathrm{cd}^*(G/M) \subseteq \mathrm{cd}^*(\mathrm{Schur}(\mathsf{A}_7)),$$

where $\mathrm{Schur}(\mathsf{A}_7)$ denotes the Schur cover (or the 6-fold cover) of A_7. It follows that $|G/M| \leq 6|\mathsf{A}_7| = 7,560$. Inspecting [5], we come up with

$$G/M \cong \mathrm{PSL}_2(q) \text{ with } 5 \leq q \leq 23, \text{ or } \mathrm{PSL}_3(3), \text{ or } \mathrm{PSU}_3(3), \text{ or } \mathsf{A}_7.$$

Since every possibility of G/M except A_7 does not satisfy the inclusion $\mathrm{cd}^*(G/M) \subseteq \mathrm{cd}^*(\mathrm{Schur}(\mathsf{A}_7))$, we deduce that $G/M \cong \mathsf{A}_7$.

On the other hand, as $\mathbb{C}G \cong \mathbb{C}H$, we have $|G| = |H|$. It follows that $|M| = |\mathbf{Z}(H)| \leq 6$. Using Lemma 6.1, we conclude that $G \cong H$. □

7. Excluding critical character degrees of $\hat{\mathsf{S}}_n$

In this section we prove a nonexistence result for special character degrees of $\hat{\mathsf{S}}_n$ which will be applied in the next section. Indeed, with the following proposition we prove a little more as only the case of even n will be needed (in fact, the proof shows that also versions with slightly modified 2-powers can be obtained).

Proposition 7.1. *Let $n \in \mathbb{N}$. If $2^{[\frac{n-2}{2}]}(n-1)$ is a degree of $\hat{\mathsf{S}}_n$, then $n \leq 8$ and the degree is an ordinary degree f_λ for $\lambda \in \{(2), (2,1), (4,2^2)\}$ (or their conjugates), or the spin degree g_μ for $\mu = (4,2)$.*

The strategy for the proof is inspired by the methods used in [1, 2] to classify the irreducible characters of prime power degrees. A main ingredient is a number theoretic result which is a variation of [1, Theorem 3.1].

First, we define $M(n)$ to be the set of pairs of finite sequences of integers $s_1 < s_2 < \cdots < s_r \leq n$, $t_1 < t_2 < \cdots < t_r \leq n$, with all numbers different from $n-1$, that satisfy

(i) $s_i < t_i$ for all i;
(ii) s_1 and t_1 are primes $> \frac{n}{2}$, different from $n-1$;

(iii) For $1 \leq i \leq r-1$, s_{i+1} and t_{i+1} contain prime factors exceeding $2n - s_i - t_i$ and not dividing $n-1$.

We then set $t(n) := \max\{t_r \mid ((s_i)_{i=1,\dots,r}, (t_i)_{i=1,\dots,r}) \in M(n)\}$, or $t(n) = 0$ when $M(n) = \emptyset$. Note that for all $n \geq 15$, there are at least two primes $p, q \neq n-1$ with $\frac{n}{2} < p < q \leq n$ (e.g., use [8]); hence for all $n \geq 15$ the set $M(n)$ is not empty.

Theorem 7.2. *Let $n \in \mathbb{N}$. Then $n - t(n) \leq 225$.*

For $15 \leq n \leq 10^9$, we have the tighter bounds

$$n - t(n) \begin{cases} = 7 & \text{for } n \in \{30, 54\} \\ = 5 & \text{for } n \in \{18, 24, 28, 52, 102, 128, 224\} \\ \leq 4 & \text{otherwise} \end{cases}$$

Proof. For $n > 3.9 \cdot 10^8$, the proof follows the lines of the arguments for [1, Theorem 3.1], noticing that in the construction given there the numbers in the sequences are below $n-1$ and that the chosen prime factors do not divide $n-1$; this then gives $n - t(n) \leq 225$.

A computer calculation (with Maple) shows that for all $n \leq 10^9$, we have the claimed bounds and values for $n - t(n)$. ☐

For a partition λ of n we denote by $l(\lambda)$ the length of λ, and we let $l_1(\lambda)$ be the multiplicity of 1 in λ. We put $h_i = h(i,1)$ for $1 \leq i \leq l(\lambda)$; these are the *first column hook lengths* of λ. We set $\mathrm{fch}(\lambda) = \{h_1, \dots, h_{l(\lambda)}\}$.

First we want to show that Proposition 7.1 holds for ordinary characters. Via computer calculations, the claim is easily checked up to $n = 44$, and in particular, we find the stated exceptions for $n < 9$. Thus we have to show that

$$(*) \qquad f_\lambda = 2^{[\frac{n-2}{2}]}(n-1)$$

cannot hold for $n \geq 9$; if necessary, we may even assume that $n > 44$.

To employ Theorem 7.2, we need some preparations that are similar to corresponding results in [1].

Proposition 7.3. *If q is a prime with $n - l_1(\lambda) \leq q \leq n$ and $q \nmid f_\lambda$, then*

$$q,\ 2q,\ \dots,\ \left[\frac{n}{q}\right] q \in \mathrm{fch}(\lambda)\,.$$

Proof. Put $w = \left[\frac{n}{q}\right]$, $n = wq + r$, $0 \leq r < q$. By assumption, we have $(w-1)q \leq (w-1)q + r = n - q \leq l_1 := l_1(\lambda)$. Thus $q, 2q, \dots, (w-1)q \in \mathrm{fch}(\lambda)$. If $wq \leq l_1$, then we are done. Assume that $l_1 < wq$. At most w hooks in λ are of lengths divisible by q (see e.g. [26, Prop.(3.6)]). If there are only the above $w-1$ hooks in the first column of length divisible by q, then $q | f_\lambda$ since $\prod_{i=1}^{w}(iq) \mid n!$, a contradiction. Let $h(i,j)$ be the additional hook length divisible by q. Since $\lambda \neq (1^n)$, $l_1 \leq h_2$. If $h_2 > l_1$, then $h(i,j) + h(2,1) > q + l_1 \geq n$. By [1, Cor. 2.8] we get $j = 1$. If $h_2 = l_1$, then

$\lambda = (n - l_1, 1^{l_1})$ and since $l_1 < wq$ there has to be a hook of length divisible by q in the first row. Since $n - l_1 \le q$ we must have $h_1 = wq$. □

In analogy to [1, Cor. 2.10], we deduce

Corollary 7.4. *Let $1 \le i < j \le l(\lambda)$. If $h \le n$ has a prime divisor q satisfying $2n - h_i - h_j < q$ and $q \nmid f_\lambda$, then $h \in \mathrm{fch}(\lambda)$.*

We now combine these results with Theorem 7.2, similarly as in [1]; as stated earlier we may assume that $n \ge 15$. Assuming $(*)$ for λ, the hook formula implies that there have to be hooks of length p and q in λ. As argued in [1], we then have $p, q \in \mathrm{fch}(\lambda)$ or $p, q \in \mathrm{fch}(\overline{\lambda})$; w.l.o.g., we may assume $p, q \in \mathrm{fch}(\lambda)$. Then the assumption $(*)$ forces *any* prime between $\frac{n}{2}$ and n, except $n-1$ if this is prime, to be in $\mathrm{fch}(\lambda)$. This gives an indication towards the connection with the sequences belonging to the pairs in $M(n)$.

Indeed, we have the following proposition which is proved similarly as the corresponding result in [1].

Proposition 7.5. *Let $n \ge 15$. Let $((s_i)_{i=1..r}, (t_i)_{i=1..r}) \in M(n)$. Let λ be a partition of n such that $(*)$ holds. Then $\{s_1, ..., s_r, t_1, ..., t_r\} \subset \mathrm{fch}(\lambda)$ or $\{s_1, ..., s_r, t_1, ..., t_r\} \subset \mathrm{fch}(\overline{\lambda})$.*

In particular, $n - h_1 \le 225$, and we have tighter bounds for $n - h_1$ when $n \le 10^9$ as given in Theorem 7.2.

Now we can embark on the **first part of the proof of Proposition 7.1**, showing the nonexistence of ordinary irreducible characters of the critical degree for $n \ge 9$. As remarked before, we may assume $n \ge 44$.

Set $m = [\frac{n-2}{2}]$, and assume that the partition λ of n satisfies

$$(*) \qquad f_\lambda = 2^m (n-1)\,.$$

Let $c = n - h_1$. By [1, Prop. 4.1] we have the following bound for the 2-part of the degree:

$$(f_\lambda)_2 \le n^2 \cdot ((2c+2)!)_2\,.$$

By Proposition 7.5, we have $c \le 225$, and hence $((2c+2)!)_2 \le (452!)_2 = 2^{448}$. Thus

$$2^m \le 2^{448} n^2 \le 2^{448} (2m+3)^2\,.$$

A short computation gives $m \le 467$, and hence $n \le 937$. By Proposition 7.5, $c \le 5$, unless $n = 54$, where we only get $c \le 7$. But for $n = 54$ we can argue as follows. As λ satisfies $(*)$, w.l.o.g. $43, 47 \in \mathrm{fch}(\lambda)$. Then $l_1(\lambda) \ge 35$ (by [1, Prop. 2.6]), and hence $17, 34 \in \mathrm{fch}(\lambda)$; since $54 > 3 \cdot 17$, by the hook formula there has to be one more hook of length divisible by 17 in λ. As $c \le 7$, this is in the first row or column; if it is not in the first column, we get a contradiction considering this hook and the one of length 43. Thus $51 \in \mathrm{fch}(\lambda)$, and hence $c \le 3$.

Hence for all $n \leq 937$ we have $((2c+2)!)_2 \leq (12!)_2 = 2^{10}$, and

$$2^m \leq 2^{10} n^2 \leq 2^{10}(2m+3)^2 \,.$$

This implies $m \leq 20$, and hence $n \leq 43$, where the assertion was checked directly. □

Next we deal with the spin characters of $\hat{\mathsf{S}}_n$. Recall that for a bar partition λ of n, $\bar{g}_\lambda$ is the number of shifted standard tableaux of shape λ and the spin character degree to λ is $g_\lambda = 2^{[\frac{n-l(\lambda)}{2}]}\bar{g}_\lambda$. Hence the condition on the spin degree translates into the condition (†) on $\bar{g}_\lambda$ given below.

Proposition 7.6. *Let λ be a bar partition of n. Then*

$$(\dagger) \qquad \bar{g}_\lambda = \begin{cases} 2^{\left\lceil \frac{l(\lambda)-2}{2} \right\rceil}(n-1) & \text{for } n \text{ even} \\ 2^{\left\lceil \frac{l(\lambda)-3}{2} \right\rceil}(n-1) & \text{for } n \text{ odd} \end{cases}$$

only for the following exceptions for $n \leq 6$: $\lambda = (2)$, $\lambda = (4,2)$.

We note that for $n \leq 34$ the assertion is easily checked by computer calculation (using John Stembridge's Maple package QF), so we may assume that $n > 34$ when needed.

We put $b_i = b(1,i)$ for the *first row bar lengths* of λ, and $\mathrm{frb}(\lambda)$ for the set of first row bar lengths of λ (see [26] for details on the combinatorics of bars).

In analogy to the case of ordinary characters where we have modified the results in [1], we adapt the results in [2] for the case under consideration now. Similarly as Proposition 7.3 before, we have a version of [2, Prop. 2.5] where instead of the prime power condition for $\bar{g}_\lambda$ the condition $q \nmid \bar{g}_\lambda$ is assumed for the prime q under consideration. For the corresponding variant of [2, Lemma 2.6] that says that any prime q with $\frac{n}{2} < q \leq n$ and $q \nmid \bar{g}_\lambda$ is a first row bar length of λ, we need two primes $p_1, p_2 \neq n-1$ with $p_1 + p_2 - n > \frac{n}{2}$. For $n \geq 33$, $n \neq 42$, we always find two primes $p_1, p_2 \neq n-1$ such that $\frac{3}{4}n < p_1 < p_2 \leq n$. But for $n = 42$, the primes $p_1 = 31$ and $p_2 = 37$ are big enough to have $p_1 + p_2 - n > \frac{n}{2}$.

The largest bar length of λ is $b_1 = b(1,1) = \lambda_1 + \lambda_2$. As before, the preparatory results just described together with our arithmetical Theorem 7.2 show that $n - b_1$ is small for a bar partition λ satisfying (†). More precisely, we obtain:

Proposition 7.7. *Let $n \geq 15$. Let $((s_1, \ldots, s_r), (t_1, \ldots, t_r)) \in M(n)$. Assume that λ is a bar partition of n that satisfies (†). Then $s_1, \ldots, s_r, t_1, \ldots, t_r \in \mathrm{frb}(\lambda)$.*

In particular, if λ satisfies (†), then $n - b_1 \leq 225$, and we have tighter bounds for $n - b_1$ when $n \leq 10^9$ as given in Theorem 7.2.

Now we can get into the **second part of the proof of Proposition 7.1**, showing the nonexistence of spin irreducible characters of the critical degree for $n \geq 7$. We have already seen that it suffices to prove Proposition 7.6, and that we may assume $n \geq 15$. Set

$$r = \begin{cases} \lceil \frac{l(\lambda)-2}{2} \rceil & \text{for } n \text{ even} \\ \lceil \frac{l(\lambda)-3}{2} \rceil & \text{for } n \text{ odd} \end{cases}$$

and assume that λ is a bar partition of n that satisfies (†).

Let $c = n - b_1$. As seen above, we have $c \leq 225$, and hence $l(\lambda) \leq 23$. Thus $r \leq 11$ in any case, and hence $\bar{g}_\lambda \leq 2^{11}(n-1)$.

Now, by [2, Prop. 2.2] we know that $\bar{g}_\lambda \geq \frac{1}{2}(n-1)(n-4)$ unless we have one of the following situations: $\lambda = (n)$, $\bar{g}_\lambda = 1$, or $\lambda = (n-1, 1)$, $\bar{g}_\lambda = n-2$. None of these exceptional cases is relevant here, and thus we obtain $n - 4 \leq 2^{12}$. But for $n \leq 4100$ we already know that $c \leq 7$. Then $l(\lambda) \leq 6$ and $r \leq 2$, and hence

$$\frac{1}{2}(n-1)(n-4) \leq \bar{g}_\lambda \leq 4(n-1)\,.$$

But then $n - 4 \leq 8$, a contradiction. Thus we have now completed the proof of Proposition 7.1. □

8. Eliminating simple groups of Lie type in even characteristic

Let G be a finite group whose complex group algebra is isomorphic to that of $\hat{\mathsf{S}}_n$. In order to show that G is isomorphic to one of the two double covers of S_n, one has to eliminate the involvement of all non-abelian simple groups other than A_n in the structure of G. The most difficult case turns out to be the simple groups of Lie type in even characteristic.

For the purpose of the next lemma, let $\mathcal{C}$ be the set consisting of the following simple groups:

$$\{{}^2F_4(2)', \mathrm{PSL}_4(2), \mathrm{PSL}_3(4), \mathrm{PSU}_4(2), \mathrm{PSU}_6(2), \\ \mathrm{P}\Omega_8^+(2), \mathrm{PSp}_6(2), {}^2B_2(8), G_2(4), {}^2E_6(2)\}.$$

Lemma 8.1. *If S is a simple group of Lie type in characteristic 2 such that $|S|_2 \geq 2^4$ and $S \notin \mathcal{C}$, then $|S|_2 < 2^{(e(S)-1)/2}$, where $e(S)$ is the smallest nontrivial degree of an irreducible projective representation of S.*

Proof. Assume that S is defined over a finite field of size $q = 2^f$. If $|S|_2 = q^{N(S)}$, then the inequality in the lemma is equivalent to

$$e(S) > 2N(S)f + 1. \tag{8.2}$$

The values of $e(S)$ are available in [29, Table II] for classical groups and in [15] for exceptional groups. The arguments for simple classical groups are quite similar. So let us consider the linear groups. Assume that $S = \mathrm{PSL}_m(q)$ with $m \geq 2$. We have $N(S) = m(m-1)/2$. First we assume that $m = 2$. As $|S|_2 = q \geq 2^4$, we deduce that $e(S) = q - 1$ and $f \geq 4$. In this case, we obtain $e(S) - 1 = 2^f - 2$ and $2N(S)f + 1 = 2f + 1$. As $f \geq 4$, we see that (8.2) holds. Next we assume that $m \geq 3$

and $S \neq \mathrm{PSL}_3(4), \mathrm{PSL}_4(2)$. As $|S|_2 \geq 2^4$, we deduce that $S \neq \mathrm{PSL}_3(2)$. We have $e(S) = (q^m - q)/(q-1)$. Now (8.2) is equivalent to

$$\frac{q^m - q}{q-1} > m(m-1)f + 1.$$

It is routine to check that this inequality holds for any $m \geq 3$ and $q \geq 2$.

The arguments for exceptional Lie type groups are also similar. For instance, if $S = {}^2B_2(2^{2m+1})$ with $m \geq 1$ then $|S|_2 = 2^{2(2m+1)}$ and $e(S) = 2^m(2^{2m+1} - 1)$. The inequality now can be easily checked. □

Proposition 8.2. *Let G be a finite group and S a simple group of Lie type in characteristic 2. Suppose that $M \subseteq G'$ is a normal subgroup of G such that $G'/M \cong S$ and $|G : G'| = 2$. Then $\mathrm{cd}^*(G) \neq \mathrm{cd}^*(\hat{\mathsf{S}}_n)$ for every integer $n \geq 10$.*

Proof. By way of contradiction, assume that $\mathrm{cd}^*(G) = \mathrm{cd}^*(\hat{\mathsf{S}}_n)$ for some $n \geq 10$. Let St_S be the Steinberg character of $G'/M \cong S$. As St_S extends to G/M and $|G/M : G'/M| = 2$, by Gallagher's Theorem (see [12, Corollary 6.17] for instance), G/M has two irreducible characters of degree $\mathrm{St}_S(1) = |S|_2$. As $n \geq 10$, Lemma 3.5(1) yields that $d_1(\hat{\mathsf{S}}_n) = n - 1$.

We claim that $|S|_2 > d_1(\hat{\mathsf{S}}_n) = n-1$. Suppose by contradiction that $|S|_2 = d_1(\hat{\mathsf{S}}_n)$. If $G/M \cong S \times C_2$, then $\mathrm{cd}(G/M) = \mathrm{cd}(S) \subseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$, which implies that $d_1(S) \geq d_1(\hat{\mathsf{S}}_n) = |S|_2$. However, this is impossible since S always has a nontrivial character degree smaller than $|S|_2$ (see [31, Lemma 8] for instance). Now assume that G/M is almost simple with socle S. If $S \not\cong \mathrm{PSL}_2(q)$ with $q \geq 4$, then $d_1(G/M) < |S|_2 = d_1(\hat{\mathsf{S}}_n)$ by [30, Lemma 2.4], which leads to a contradiction as before since $\mathrm{cd}(G/N) \subseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$. Therefore, assume that $S = \mathrm{PSL}_2(q)$ with $q = 2^f \geq 4$. Then $q = |S|_2 = n - 1 \geq 9$. If $q \equiv -1 \pmod 3$, then $d_1(G) = q - 1 < q = d_1(\hat{\mathsf{S}}_n)$ by [30, Lemma 2.5], which is impossible. Hence, $q \equiv 1 \pmod 3$ and $q+1 \in \mathrm{cd}(G/M) \subseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$. It follows that $q + 1 = (n-1) + 1 = n \in \mathrm{cd}(\hat{\mathsf{S}}_n)$. If $n \geq 12$, then $d_2(\hat{\mathsf{S}}_n) > 2n > n > d_1(\hat{\mathsf{S}}_n)$, hence n is not a degree of $\hat{\mathsf{S}}_n$. Thus $10 \leq n \leq 11$. However, we see that $n-1$ is not a power of 2 in both cases. The claim is proved.

Assume that $n = 2k+1 \geq 11$ is odd. By Lemmas 2.1 and 2.3, $|S|_2 = 2^k$ is the degree of the basic spin character of $\hat{\mathsf{S}}_n$. However, by [33, Table I] such a degree has multiplicity one, which contradicts the fact proved above that G has at least two irreducible characters of degree $|S|_2$.

Assume $n = 2k \geq 10$ is even. By Lemmas 2.1 and 2.3 and [33, Table I], $\hat{\mathsf{S}}_n$ always has the degree 2^{k-1} with multiplicity 2; and if $n = 2^r + 2$, then it has the degree $2^{k-1}(n-2) = 2^{2^{r-1}+r}$ with multiplicity 1. These are in fact the only nontrivial 2-power degrees of $\hat{\mathsf{S}}_n$. As in the previous case, by comparing the multiplicity, we see that $|S|_2 \neq 2^{2^{r-1}+r}$. Thus $|S|_2 = 2^{k-1}$ is the degree of the basic spin character of $\hat{\mathsf{S}}_n$ with multiplicity 2. Notice that $k \geq 5$ and hence $|S|_2 = 2^{k-1} \geq 2^4$.

Now let $\psi \in \mathrm{Irr}(G)$ with $\psi(1) = n-1$. As $|G : G'| = 2$ and $\psi(1)$ is odd, we deduce that $\phi = \psi_{G'} \in \mathrm{Irr}(G')$ and $\phi(1) = n-1$. Let $\theta \in \mathrm{Irr}(M)$ be an irreducible constituent of ϕ_M. Then $\phi_M = e(\theta_1 + \cdots + \theta_t)$, where $t = |G' : I_{G'}(\theta)|$, and each θ_i is conjugate to $\theta \in \mathrm{Irr}(M)$. If θ is not G'-invariant, then $\phi(1) = et\theta(1) = n-1 \geq \min(S)$, where $\min(S)$ is the smallest nontrivial index of a maximal subgroup of S. We see that $\min(S) > d_1(S) \geq e(S)$, where $e(S)$ is the minimal degree of a projective irreducible representation of S and so $n-1 \geq e(S)$. If θ is G'-invariant and $\phi_M = e\theta$ with $e > 1$, then e is the degree of a projective irreducible representation of S. It follows that $n-1 \geq e \geq e(S)$. In both cases, we always have

$$k-1 = \frac{n-2}{2} \geq \frac{e(S)-1}{2}\,.$$

Therefore,

$$|S|_2 = 2^{k-1} \geq 2^{(e(S)-1)/2}.$$

By Lemma 8.1, we deduce that $S \in \mathcal{C}$. Solving the equation $|S|_2 = 2^{(n-2)/2}$, we get the degree n. However, by using [5] and Lemma 3.5, we can check that $\mathrm{cd}(G/M) \nsubseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$ in any of these cases. For example, assume that $S \cong {}^2E_6(2)$. Then $|S|_2 = 2^{36} = 2^{(n-2)/2}$, so $n = 74$. By Lemma 3.5, we have $d_1(\hat{\mathsf{S}}_n) = n-1 = 73$ and $d_2(\hat{\mathsf{S}}_n) = n(n-3)/2 = 2627$. Using [5], we know that $\mathrm{cd}(G/M)$ contains a degree 1938. Clearly, $d_1(\hat{\mathsf{S}}_n) < 1938 < d_2(\hat{\mathsf{S}}_n)$, so $1938 \notin \mathrm{cd}(\hat{\mathsf{S}}_n)$, hence $\mathrm{cd}(G/M) \nsubseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$, a contradiction.

Finally we assume that $et = 1$. Then θ extends to $\phi \in \mathrm{Irr}(G')$ and to $\psi \in \mathrm{Irr}(G)$. Hence $\phi_M = \theta$ and so by Gallagher's Theorem, we have $\psi\tau \in \mathrm{Irr}(G)$ for every $\tau \in \mathrm{Irr}(G/M)$. In particular,

$$\psi(1)|S|_2 = 2^{k-1}(n-1) \in \mathrm{cd}(G) = \mathrm{cd}(\hat{\mathsf{S}}_n),$$

which is impossible by Proposition 7.1. □

9. Eliminating simple groups other than A_n

We continue to eliminate the involvement of simple groups other than A_n in the structure of G with $\mathbb{C}G \cong \mathbb{C}\hat{\mathsf{S}}_n$.

Proposition 9.1. *Let G be an almost simple group with non-abelian simple socle S. Suppose that $\mathrm{cd}^*(G) \subseteq \mathrm{cd}^*(\hat{\mathsf{S}}_n)$ for some $n \geq 10$. Then $S \cong \mathsf{A}_n$ or S is isomorphic to a simple group of Lie type in characteristic 2.*

Proof. We make use of the classification of finite simple groups.

(i) S is a sporadic simple group or the Tits group. Using [7], we can assume that $n \geq 19$. By Lemma 3.5(2), we have $d_2(\hat{\mathsf{S}}_n) = n(n-3)/2 \geq 152$. Since $d_2(G) \geq d_2(\hat{\mathsf{S}}_n) \geq 152$, using [5], we only need to consider the following simple groups:

$$J_3, Suz, McL, Ru, He, Co_1, Co_2, Co_3, Fi_{22}, O'N, HN, Ly, Th, Fi_{23}, J_4, Fi'_{24}, B, M.$$

To eliminate these cases, observe that $n \geq p(S)$, the largest prime divisor of $|S|$, and $d_i(G) \geq d_i(\hat{\mathsf{S}}_n)$ for all $i \geq 1$. Now with the lower bound $n \geq \max\{19, p(S)\}$, we can find the lower bounds for $d_i(\hat{\mathsf{S}}_n)$ with $1 \leq i \leq 7$ using Lemmas 3.4 and 3.5. Choose $i \in \{2, 3, \cdots, 7\}$ such that $d_i(\hat{\mathsf{S}}_n) > d_j(G)$ for some $j \geq 1$ such that $|i-j|$ is minimal. If $i \geq j$, then we obtain a contradiction. Otherwise, $d_j(G) \in \{d_k(\hat{\mathsf{S}}_n)\}_{k=j}^{i-1}$. Solving these equations for n, we then obtain that either these equations have no solution or for each solution of n, we can find some $k \geq 1$ with $d_k(\hat{\mathsf{S}}_n) > d_k(G)$.

For an example of such a demonstration, assume that $S = O'N$. In this case, we have $|\mathrm{Out}(S)| = 2$, so $G = S$ or $G = S.2$. Since $p(S) = 31$, we have $n \geq 31$. Assume first that $G = S = O'N$. Then $d_4(O'N) = 26752$ and since $n \geq 31$, by Lemma 3.4, $d_7(\hat{\mathsf{S}}_n) \geq 26970 > d_4(O'N)$. It follows that $d_4(O'N) \in \{d_i(\hat{\mathsf{S}}_n)\}_{i=4}^{6}$. However, we can check that these equations are impossible. Now assume $G = O'N.2$. Then $d_2(G) = 26752 < 26970 \leq d_7(\hat{\mathsf{S}}_n)$ so that $d_2(G) \in \{d_i(\hat{\mathsf{S}}_n)\}_{i=2}^{6}$. As above, these equations cannot hold for any $n \geq 31$. Thus $\mathrm{cd}(G) \nsubseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$.

For another example, let $S = M$. Since $|\mathrm{Out}(S)| = 1$, we have $G = S$ so that $p(S) = 71 \in \pi(\hat{\mathsf{S}}_n)$ and hence $n \geq 71$. As $d_1(M) = 196883 < 914480 \leq d_7(\hat{\mathsf{S}}_n)$, we deduce that $d_1(M) \in \{d_i(\hat{\mathsf{S}}_n) \mid i = 1, \cdots, 6\}$. Solving these equations, we obtain $n = 196884$. But then $d_2(\hat{\mathsf{S}}_n) > 21296,876 = d_2(M)$. Thus $\mathrm{cd}(M) \nsubseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$.

(ii) $S = \mathsf{A}_m$, with $m \geq 7$. Let $\lambda = (m-1, 1)$, a partition of m. Since $m \geq 7$, λ is not self-conjugate, hence the irreducible character χ_λ of S_m is still irreducible upon restriction to A_m. Note that $\mathrm{Aut}(\mathsf{A}_m) = \mathsf{S}_m$ as $m \geq 7$. Then $G \in \{\mathsf{A}_m, \mathsf{S}_m\}$ and G has an irreducible character of degree $m-1$. Since $\mathrm{cd}(G) \subseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$, we have $m - 1 \geq d_1(\hat{\mathsf{S}}_n) = n - 1$, so $m \geq n$. If $m = n$ then we are done. Hence, assume that $m > n \geq 10$. Then $d_1(G) = m - 1 > n - 1 = d_1(\hat{\mathsf{S}}_n)$. If $m \leq 17$ then $10 \leq n < m \leq 17$. These cases can be eliminated using [7]. Assume that $m \geq 18$. It follows that $17 \mid |G|$ and so $17 \mid |\hat{\mathsf{S}}_n|$, which implies that $n \geq 17$. Thus $17 \leq n < m$. It follows from Lemma 3.5(2) that $d_2(\hat{\mathsf{S}}_n) = n(n-3)/2$. Since $\mathrm{cd}(G) \subseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$ and $d_1(G) > d_1(\hat{\mathsf{S}}_n)$, we have $d_1(G) \geq d_2(\hat{\mathsf{S}}_n)$. Hence $m - 1 \geq n(n-3)/2$. Since $n \geq 17$, we have

$$m - 2n \geq n(n-3)/2 + 1 - 2n = n(n-7)/2 + 1 > 0,$$

so $m > 2n$. Therefore, $n < m/2 < m$. By Lemma 4.1, there exists a prime p such that $m/2 < p < m$. It follows that $p \mid |G|$ but $p \nmid |\hat{\mathsf{S}}_n|$ since $p > n$, which is a contradiction.

(iii) S is a simple group of Lie type in odd characteristic. Suppose that $S = G(p^k)$, a simple group of Lie type defined over a field of p^k elements with p odd. Let St be the Steinberg character of S. Then, as St extends to G and $\mathrm{St}(1) = |S|_p$, we have $|S|_p \in \mathrm{cd}(\hat{\mathsf{S}}_n)$. Using Lemma 2.3, which says that all possible prime power degrees of spin characters of S_n are even, we deduce that $|S|_p \in \mathrm{cd}(\mathsf{S}_n)$. By Lemma 2.1, we then obtain that $|S|_p = n-1$ since $n \geq 10$. By Lemma 3.5, $n-1 = d_1(\hat{\mathsf{S}}_n)$ is the smallest nontrivial degree of $\hat{\mathsf{S}}_n$. Assume first that $S \neq \mathrm{PSL}_2(q)$. Then $d_1(G) < |S|_p = d_1(\hat{\mathsf{S}}_n)$ by [30, Lemma 2.4], which is a contradiction as $\mathrm{cd}(G) \subseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$. Now it remains to consider the case $S = \mathrm{PSL}_2(q)$. We have $q = n-1 \geq 9$. If G has a degree which is smaller than $|S|_p = q$, then we obtain a contradiction as before. So, by [30, Lemma 2.5], we have $p \neq 3$ and $q \equiv 1 \pmod 3$ or $p = 3$ and $q \equiv 1 \pmod 4$. In both cases G has an irreducible character of degree $q+1 = n = d_1(\hat{\mathsf{S}}_n)+1$. If $n \geq 12$, then $d_2(\hat{\mathsf{S}}_n) \geq 2n > n > d_1(\hat{\mathsf{S}}_n)$ by Lemma 3.5, so n is not a degree of $\hat{\mathsf{S}}_n$. Assume that $10 \leq n \leq 11$. Then $n = 10$ and $q = 9$. However, using [5], we can check that $\mathrm{cd}(G) \nsubseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$ for every almost simple group G with socle $\mathrm{PSL}_2(9) \cong \mathsf{A}_6$. □

Combining Propositions 9.1 and 8.2, we obtain the following results, which will be crucial in the proof of Theorem C.

Proposition 9.2. *Let G be a finite group and let $M \subseteq G'$ be a normal subgroup of G such that G/M is an almost simple group with socle $S \neq \mathsf{A}_n$, where $|G : G'| = 2$ and $G'/M \cong S$. Then $\mathrm{cd}^*(G) \neq \mathrm{cd}^*(\hat{\mathsf{S}}_n)$.*

Proof. If $n \geq 10$, then the result follows from Propositions 9.1 and 8.2. It remains to assume that $5 \leq n \leq 9$ and suppose by contradiction that $\mathrm{cd}^*(G) = \mathrm{cd}^*(\hat{\mathsf{S}}_n)$. Then $|G| = 2n!$ and so $|S| \mid 2n!$, hence $\pi(S) \subseteq \pi(\hat{\mathsf{S}}_n) \subseteq \{2,3,5,7\}$. By [11, Theorem III], S is one of the following simple groups:

(1) If $\pi(S) = \{2,3,5\}$, then $S \cong \mathsf{A}_5, \mathsf{A}_6$ or $\mathrm{PSp}_4(3)$.
(2) If $\pi(S) = \{2,3,7\}$, then $S \cong \mathrm{PSL}_2(7), \mathrm{PSL}_2(8)$ or $\mathrm{PSU}_3(3)$.
(3) If $\pi(S) = \{2,3,5,7\}$, then $S \cong \mathsf{A}_k$ with $7 \leq k \leq 10$, J_2, $\mathrm{PSL}_2(49), \mathrm{PSL}_3(4)$, $\mathrm{PSU}_3(5)$, $\mathrm{PSU}_4(3), \mathrm{PSp}_4(7), \mathrm{PSp}_6(2)$ or $\mathrm{P}\Omega_8^+(2)$.

Now it is routine to check that $\mathrm{cd}(G/M) \nsubseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$ unless $S \cong \mathsf{A}_n$, where G/M is almost simple with socle S. □

Proposition 9.3. *Let G be a finite group and let $M \subseteq G'$ be a normal subgroup of G. Suppose that $\mathrm{cd}^*(G) = \mathrm{cd}^*(\hat{\mathsf{S}}_n)$ and $G/M \cong G'/M \times C_2 \cong S^k \times C_2$ for some positive integer k and some non-abelian simple group S. Then $k = 1$ and $S \cong \mathsf{A}_n$.*

Proof. Since $\mathrm{cd}^*(S) \subseteq \mathrm{cd}^*(S^k)$, the hypothesis implies that

$$\mathrm{cd}^*(S) \subseteq \mathrm{cd}^*(\hat{\mathsf{S}}_n)\,.$$

Assume first that $5 \leq n \leq 9$. Since $|S^k| = |S|^k$ divides $|\hat{\mathsf{S}}_n| = 2n!$, we deduce that $\pi(S) \subseteq \pi(\hat{\mathsf{S}}_n)$ and in particular, $\pi(S) \subseteq \{2,3,5,7\}$. The possibilities for S are listed in the proof of Proposition 9.2 above. Observe that $|S|$ is always divisible by a prime r with $r \geq 5$. Hence, $r^k \mid |\hat{\mathsf{S}}_n|$, which implies that $k = 1$ as $|\hat{\mathsf{S}}_n|$ divides $2 \cdot 9!$. Now the fact that $S \cong \mathsf{A}_n$ follows easily.

¿From now on we can assume that $n \geq 10$. Using Proposition 9.1, we obtain $S = \mathsf{A}_n$ or S is a simple group of Lie type in even characteristic. It suffices to show that $k = 1$ and then the result follows from Proposition 8.2.

Assume that the latter case holds. Then S is a simple group of Lie type in characteristic 2. By Lemmas 2.1 and 2.3, $\hat{\mathsf{S}}_n$ has at most two distinct nontrivial 2-power degrees, which are $n-1$ and $2^{\lfloor (n-1)/2 \rfloor}$ or $2^{\lfloor (n-1)/2 \rfloor}$ and $2^{2^{r-1}+r}$ with $n = 2^r + 2$. By way of contradiction, assume that $k \geq 2$. If $k \geq 3$, then $G/M \cong S^k \times C_2$ has irreducible characters of degrees

$$|S|_2^k > |S|_2^{k-1} > |S|_2^{k-2} > 1.$$

Obviously, this is impossible as $\mathrm{cd}(G/M) \subseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$. Therefore, $k = 2$. In this case, G/M has character degree $|S|_2^2$ with multiplicity at least 2 and $|S|_2$ with multiplicity at least 4. It follows that either $2^{\lfloor (n-1)/2 \rfloor} = |S|_2^2$ and $n-1 = |S|_2$ or $2^{2^{r-1}+r} = |S|_2^2$ and $2^{\lfloor (n-1)/2 \rfloor} = |S|_2$. However, both cases are impossible by comparing the multiplicity.

It remains to eliminate the case $S \cong \mathsf{A}_n$ and $k \geq 2$. By comparing the orders, we see that

$$2(n!/2)^k |M| = 2n!\,.$$

After simplifying, we obtain

$$|M|(n!)^{k-1} = 2^k.$$

Since $n \geq 10$, we see that if $k \geq 2$ then the left side is divisible by 5 while the right side is not. We conclude that $k = 1$ and the proof is now complete. □

10. Completion of the proof of Theorem C

We need one more result before proving Theorem C.

Proposition 10.1. *Let G be a finite group and let S be a non-abelian simple group. Suppose that $|G : G'| = 2$ and $G' \cong S^2$ is the unique minimal normal subgroup of G. Then $\mathrm{cd}(G) \not\subseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$.*

Proof. By way of contradiction, assume that $\mathrm{cd}(G) \subseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$. Let $\alpha \in \mathrm{Irr}(S)$ with $\alpha(1) > 1$ and put $\theta = \alpha \times 1 \in \mathrm{Irr}(G')$. Observe that θ is not G-invariant so that $I_G(\theta) = G'$ hence $\theta^G \in \mathrm{Irr}(G)$ and so $\theta^G(1) = 2\alpha(1) \in \mathrm{cd}(\hat{\mathsf{S}}_n)$. On the other hand, if $\varphi = \alpha \times \alpha \in \mathrm{Irr}(G')$ then φ is G-invariant and since G/G' is cyclic, we deduce that φ extends to $\psi \in \mathrm{Irr}(G)$, so $\psi(1) = \alpha(1)^2 \in \mathrm{cd}(\hat{\mathsf{S}}_n)$. Thus, we conclude that

$$\text{if } a \in \mathrm{cd}(S)\backslash\{1\} \text{ then } 2a, a^2 \in \mathrm{cd}(\hat{\mathsf{S}}_n). \tag{10.3}$$

Let r be an odd prime divisor of $|S|$. The Ito-Michler theorem then implies that r divides some character degree, say a, of S. Since $a^2 \in \mathrm{cd}(\hat{\mathsf{S}}_n)$ by (10.3), we have $r^2 \mid 2n!$ and hence $n \geq 2r$ as $r > 2$. Thus, we have shown that

$$\text{if } r \in \pi(S) - \{2\}, \text{ then } r^2 \mid 2n! \text{ and } n \geq 2r. \tag{10.4}$$

Using the classification of finite simple groups, we consider the following cases.

(i) $S = \mathsf{A}_m$, with $m \geq 7$. As $7 \in \pi(S)$, it follows from (10.4) that $n \geq 14$. Since $m-1 \in \mathrm{cd}(S)$, both $2(m-1)$ and $(m-1)^2$ are in $\mathrm{cd}(\hat{\mathsf{S}}_n)$ by (10.3). As $m \geq 7$, we also have that $m(m-3)/2, (m-1)(m-2)/2 \in \mathrm{cd}(S)$ and so $m(m-3), (m-1)(m-2) \in \mathrm{cd}(\hat{\mathsf{S}}_n)$. We claim that $m < n$. Suppose by contradiction that $m \geq n$. As $n \geq 14$, by Lemma 4.1, there exists a prime r such that $n/2 < r \leq n$. Hence, the r-part of $2n!$ is just r. However, as $r \leq n \leq m$, r divides $|\mathsf{A}_m|$ and so $r^2 \mid 2n!$ by (10.4), a contradiction. Thus $m < n$ as wanted.

Since $m \geq 7$, we obtain that

$$1 < 2(m-1) < m(m-3) < (m-1)(m-2) < (m-1)^2. \tag{10.5}$$

By Lemma 3.5(1), we have $d_1(\hat{\mathsf{S}}_n) = n-1$, so $2(m-1) \geq n-1$ and thus $n \leq 2m-1$. As $n \geq 14$, we deduce that $m \geq 8$.

Assume first that $m \in \{8, 9, 10\}$. Then $(m-1)^2 \in \mathrm{cd}(\hat{\mathsf{S}}_n)$ is a prime power. As $3^3 \neq (m-1)^2 > d_1(\hat{\mathsf{S}}_n)$, Lemmas 2.1 and 2.3 yield that $(m-1)^2$ is a power of 2 and thus $m = 9$. Since $\{2^3, 3^3\} \subseteq \mathrm{cd}(\mathsf{A}_9)$, we have $\{2^4, 2^6, 3^6\} \subseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$ by (10.4) As $n \geq 14$, we have $2^{\lfloor (n-1)/2 \rfloor} > 2^4$ and $n(n-3)/2 > 2^4$, so $d_2(\hat{\mathsf{S}}_n) > 2^4$ by Lemma 3.5(2). This forces $2^4 = d_1(\hat{\mathsf{S}}_n) = n-1$ or equivalently $n = 17$. But then Lemmas 2.1 and 2.3 yield $3^6 = d_1(\hat{\mathsf{S}}_n)$, which is impossible.

Assume next that $m \geq 11$. Then $n \geq 22$ by (10.4). By Lemma 3.5(2), we have $d_2(\hat{\mathsf{S}}_n) > 2n > 2m$. In particular, $2(m-1) < d_2(\hat{\mathsf{S}}_n)$. By (10.3), we have $2(m-1) \in \mathrm{cd}(\hat{\mathsf{S}}_n)$, hence $2(m-1) = d_1(\hat{\mathsf{S}}_n) = n-1$, which implies that $n = 2m-1$. By Lemma 3.5(3), we have $d_3(\hat{\mathsf{S}}_n) = (n-1)(n-2)/2$ and thus by (10.5), we obtain that

$$(m-1)(m-2) \geq d_3(\hat{\mathsf{S}}_n) = (n-1)(n-2)/2 = (m-1)(2m-3).$$

After simplifying, we have $m-2 \geq 2m-3$ or equivalently $m \leq 1$, a contradiction.

(ii) S is a finite simple group of Lie type in characteristic p, with $S \neq {}^2F_4(2)'$. As $|S|$ is always divisible by an odd prime $r \geq 5$, we have $n \geq 2r \geq 10$ by (10.4). Let St denote the Steinberg character of S. We can check that $\mathrm{St}(1) = |S|_p \geq 4$. Since $\mathrm{St}(1) \in \mathrm{cd}(S)$, $2\mathrm{St}(1)$ and $\mathrm{St}(1)^2$ are degrees of $\hat{\mathsf{S}}_n$ by (10.3). As $2\mathrm{St}(1) < \mathrm{St}(1)^2$, we have $\mathrm{St}(1)^2 > d_1(\hat{\mathsf{S}}_n) = n-1$. Since $n \geq 10$, Lemmas 2.1 and 2.3 yield that $\mathrm{St}(1)^2$ is a 2-power. Hence, $2\mathrm{St}(1)$ is also a 2-power. By [31, Lemma 8], there exists a nontrivial degree x of S such that $1 < x < \mathrm{St}(1)$. It follows that $2x < 2\mathrm{St}(1)$ is also a degree of $\hat{\mathsf{S}}_n$. Therefore, $2\mathrm{St}(1) > d_1(\hat{\mathsf{S}}_n) = n-1$. Hence, $\hat{\mathsf{S}}_n$ has two distinct nontrivial 2-power degrees, none of which is $n-1$. It follows that $n = 2^r + 2 \geq 10$ and furthermore

$$2\mathrm{St}(1) = 2^{\lfloor (n-1)/2 \rfloor} \text{ and } \mathrm{St}(1)^2 = 2^{2^{r-1}+r}$$

by Lemma 2.3. Writing $\mathrm{St}(1) = 2^N$. Then $2^{r-1} + r = 2N$ and $2^{r-1} = N + 1$ since $\lfloor (n-1)/2 \rfloor = 2^{r-1}$. Solving these equations, we have $r = N - 1$ and $2^{r-1} = r + 2$. As $n \geq 10$, we deduce that $r \geq 3$. In this case, it is easy to check that the equation $2^{r-1} = r + 2$ has no integer solution.

(iii) S is a sporadic simple group or the Tits group. Recall that $p(S)$ is the largest prime divisor of $|S|$. By (10.4), we have $n \geq 2p(S)$. We consider the following cases:

Case $S \in \{M_{11}, M_{12}, J_1, M_{22}, M_{23}, HS, J_3, M_{24}, He,\ Ru,\ Co_3,\ Co_2,\ Co_1,\ {}^2F_4(2)'\}$. These groups can be eliminated as follows. Since $n \geq 2p(S) \geq 22$, we have $d_2(\hat{\mathsf{S}}_n) = n(n-3)/2 \geq p(S)(2p(S) - 3)$ by applying Lemma 3.5(3). For $i = 1, 2$, we have $2d_i(S) \in \mathrm{cd}(G) \subseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$ with $1 < 2d_1(S) < 2d_2(S)$. For each possibility of S, we can check using [5] that $p(S)(2p(S) - 3) > 2d_2(S)$, hence $d_2(\hat{\mathsf{S}}_n) > 2d_2(S) > 2d_1(S) > 1$, which is a contradiction.

Case $S = J_2$. As $p(S) = 7$, we have that $n \geq 14$. If $n \geq 15$, then $d_2(\hat{\mathsf{S}}_n) = n(n-3)/2 \geq p(S)(2p(S) - 3)$ by applying Lemma 3.5(2). Using the same argument as in the previous case, we obtain a contradiction. Hence $n = 14$. Then $d_2(\hat{\mathsf{S}}_n) = 64 > 2d_2(S) = 42 > 2d_1(S) = 28$, and we get a contradiction as above.

Case $S \in \{McL, Suz, Fi_{22}, HN, Ly, Th, J_4, B\}$. Since $n \geq 2p(S) \geq 22$, we have $d_2(\hat{\mathsf{S}}_n) = n(n-3)/2 \geq p(S)(2p(S) - 3)$. We have $d_2(\hat{\mathsf{S}}_n) \geq p(S)(2p(S) - 3) > 2d_1(S)$ and so $2d_1(S) = d_1(\hat{\mathsf{S}}_n) = n - 1$, hence $n = 2d_1(S) + 1$. But then $d_2(\hat{\mathsf{S}}_n) = n(n-3)/2 = (d_1(S) - 1)(2d_1(S) + 1) > 2d_2(S) > 2d_1(S) > 1$, which is impossible as $2d_i(S) \in \mathrm{cd}(G) \subseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$, where $i = 1, 2$.

Case $S = O'N$. Then $p(S) = 31$. We have $n \geq 2p(S) = 62$ and so by Lemma 3.4, $d_7(\hat{\mathsf{S}}_n) = n(n-1)(n-2)(n-7)/24 \geq 520025$. As $d_8(S) = 58311$, we have $2d_8(S) = 116622 \in \mathrm{cd}(\hat{\mathsf{S}}_n)$. Note that $2d_i(S) \in \mathrm{cd}(S_n)$ for $i = 1, 2, \cdots, 8$ and so $2d_i(S) \geq d_i(\hat{\mathsf{S}}_n)$ for all $1 \leq i \leq 8$. As $d_7(\hat{\mathsf{S}}_n) \geq 520025 > 116622 = 2d_8(S)$, we get a contradiction.

Case $S = Fi_{23}$. Then $p(S) = 23$ and so $n \geq 2p(S) = 46$. By Lemma 3.4, $d_4(\hat{\mathsf{S}}_n) = n(n-1)(n-5)/6 \geq 14145$. As $d_2(S) = 3588$, we obtain $2d_2(S) = 7176 \in \mathrm{cd}(\hat{\mathsf{S}}_n)$. Since $d_4(\hat{\mathsf{S}}_n) > 7176 > 2d_1(S)$, we must have that $7176 \in \{d_2(\hat{\mathsf{S}}_n), d_3(\hat{\mathsf{S}}_n)\}$. Using Lemma 3.4, we can check that these equations cannot happen.

Case $S = Fi_{24}'$. Then $p(S) = 29$ and $n \geq 2p(S) = 58$. By Lemma 3.4, $d_4(\hat{\mathsf{S}}_n) \geq 29203$. As $\{8671, 57477\} \subseteq \mathrm{cd}(S)$, we have $\{17342, 114954\} \subseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$. Since $d_4(\hat{\mathsf{S}}_n) > 17342$, we have $17342 \in \{d_1(\hat{\mathsf{S}}_n), d_2(\hat{\mathsf{S}}_n), d_3(\hat{\mathsf{S}}_n)\}$. It follows that $17342 \leq (n-1)(n-2)/2$ and hence $n \geq 188$. But then $d_2(\hat{\mathsf{S}}_n) \geq 17390 > 17342$. Thus $d_1(\hat{\mathsf{S}}_n) = n - 1 = 17342$ hence $n = 17343$ and so $d_2(\hat{\mathsf{S}}_n) \geq 150363810 > 114954$, a contradiction.

Case $S = M$. Then $p(S) = 71$ and $n \geq 2p(S) = 142$. By Lemma 3.4, $d_4(\hat{\mathsf{S}}_n) \geq 457169$. As $\{196883, 21296876\} \subseteq \mathrm{cd}(S)$, we obtain $\{393766, 42593752\} \subseteq \mathrm{cd}(\hat{\mathsf{S}}_n)$. Since $d_4(\hat{\mathsf{S}}_n) > 393766$, we have $393766 \in \{d_i(\hat{\mathsf{S}}_n)\}_{i=1}^{3}$. It follows that $393766 \leq$

$(n-1)(n-2)/2$ and hence $n \geq 889$. As $d_2(\hat{\mathsf{S}}_n) \geq 393827 > 393766$, we have $d_1(\hat{\mathsf{S}}_n) = n-1 = 393766$, hence $n = 393767$ and so $d_2(\hat{\mathsf{S}}_n) \geq 77525634494 > 42593752 > 393766$, a contradiction. The proof is complete. □

We are now ready to prove the main Theorem C.

Proof of Theorem C. By the hypothesis that $\mathbb{C}G \cong \mathbb{C}\hat{\mathsf{S}}_n$, we have $|G| = 2n!$ and as $\hat{\mathsf{S}}_n$ has two linear characters, we also have $|G : G'| = 2$.

First we claim that $G' = G''$. Assuming on the contrary that $G'' < G'$, then there is $K < G'$ normal in G and maximal such that G/K is solvable. Then $(G/K)'$ is the unique minimal normal subgroup of G/K and by [12, Lemma 12.3], all nonlinear irreducible characters of G/K have equal degree f and we are in one of the following two situations:

(i) G/K is an r-group for some prime r, $\mathbf{Z}(G/K)$ is cyclic, and $(G/K)/\mathbf{Z}(G/K)$ is elementary abelian of order f^2. Then we have
$$(G/K)' \leq \mathbf{Z}(G/K) < G/K\,.$$
Since $|G : G'| = 2$, we deduce that $(G/K)' = \mathbf{Z}(G/K)$. Now $(G/K)/\mathbf{Z}(G/K) \cong C_2$, which contradicts the fact that $(G/K)/\mathbf{Z}(G/K)$ is elementary abelian of order f^2.

(ii) G/K is a Frobenius group with an abelian Frobenius complement of order f and $(G/K)'$ is the Frobenius kernel and is an elementary abelian r-group. Then we have $f = |G/K : (G/K)'| = 2$ and so $2 \in \mathrm{cd}(G/K) \in \mathrm{cd}(\hat{\mathsf{S}}_n)$, which is a contradiction as $n \geq 5$. The claim is proved.

As G is quasi-perfect and G' is nontrivial, one can choose a normal subgroup M of G such that $M < G'$ and
$$G'/M \cong S^k,$$
where S is a non-abelian simple group and S^k is a chief factor of G. Let
$$C/M := \mathbf{C}_{G/M}(G'/M)\,.$$

(A) First we consider the case $C = M$. Then G'/M is the unique minimal normal subgroup of G/M. Therefore G/M permutes the direct factors of G'/M (which is isomorphic to S^k). It then follows that $k \leq 2$ as $|G : G'| = 2$. Invoking Proposition 10.1, we deduce that $k = 1$ and thus $G'/M \cong S$. Therefore, G'/M is the socle of G/M. As $\mathrm{cd}^*(G/M) \subseteq \mathrm{cd}^*(\hat{\mathsf{S}}_n)$, Proposition 9.2 then implies that $G'/M \cong \mathsf{A}_n$. Thus $G/M \cong \mathsf{S}_n$ and also $|M| = 2$. In particular, M is central in G and therefore $M \subseteq \mathbf{Z}(G) \cap G'$. We conclude that G is one of the two double covers of S_n, as desired.

(B) It remains to consider the case $C > M$. Since $C/M \lhd G/M$ and $\mathbf{Z}(G'/M) = 1$, it follows that G' is a proper subgroup of $G'C$. As $|G : G'| = 2$, we then deduce that $G = G'C$ and hence
$$G/M = G'/M \times C/M \text{ where } C/M \cong C_2\,.$$

Applying Proposition 9.3, we obtain that $k = 1$ and S is isomorphic to A_n. In other words, $G'/M \cong \mathsf{A}_n$. So $G/M \cong \mathsf{A}_n \times C_2$. Comparing the orders, we get $|M| = 2$ and so $M \subseteq \mathbf{Z}(G)$. As $M \leq G' = G''$, it follows that $M \leq \mathbf{Z}(G') \cap G''$, which in turn implies that G' is the double cover of A_n. We have proved that

$$G' \cong \hat{\mathsf{A}}_n. \tag{10.6}$$

Moreover, as $C/M \cong C_2$ and $|M| = 2$, we have

$$C \cong C_4 \text{ or } C_2 \times C_2. \tag{10.7}$$

Now we claim that G is an (internal) central product of G' and C with amalgamated central subgroup M. To see this, let $x, y \in G'$ and $c \in C$. Then the facts $C/M = \mathbf{C}_{G/M}(G'/M)$ and $M \leq \mathbf{Z}(G)$ imply

$$[x, y]^c = [x^c, y^c] = [xm_1, ym_2] = [x, y],$$

for some $m_1, m_2 \in M$. Therefore, C centralizes $G' = G''$ and the claim follows. This claim together with (10.6) and (10.7) yield

$$4k(\hat{\mathsf{A}}_n) = 4k(G') = k(G' \times C) \leq k(M)k(G) = 2k(G),$$

where the inequality comes from the well-known result that $k(X) \leq k(N)k(X/N)$ for N a normal subgroup of X (see [21] for instance). Since $\mathrm{cd}^*(G) = \mathrm{cd}^*(\hat{\mathsf{S}}_n)$, we have $k(G) = k(\hat{\mathsf{S}}_n)$. It follows that $2k(\hat{\mathsf{A}}_n) \leq k(\hat{\mathsf{S}}_n)$. This however contradicts Lemma 4.5 and the proof is now complete. □

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