ON THE OPERATOR SPACE UMD PROPERTY AND NON-COMMUTATIVE MARTINGALE INEQUALITIES

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Abstract

We prove that for $1 \leq p < q < \infty$ the analogue of the classical result $[BMO, L_p]_{\frac{p}{q}} = L_q$ holds in the setting of a finite von Neumann algebra $\mathcal{M}$, equipped with an increasing filtration $(\mathcal{M}_n)_{n\geq 1}$ of von Neumann subalgebras. We also obtain the corresponding results for the real method of interpolation. We discuss the appropriate operator space matrix norms and show that these interpolation results hold in the category of operator spaces. We apply further interpolation techniques to the study of the operator space UMD property, introduced by Pisier in the context of non-commutative vector-valued L_p-spaces, associated to a hyperfinite (and finite) von Neumann algebra. We discuss basic stability properties of UMD$_p$ operator spaces. It is unknown whether the property is independent of p in this setting. We show that for $1 < p, q < \infty$, the Schatten q-classes S_q are UMD$_p$. We provide further examples of UMD$_p$ (independent of p) operator spaces, including the non-commutative Lorentz spaces associated to a hyperfinite (and finite) von Neumann algebra.

To my parents,

Daniela and Nicolae

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Introduction

The interplay and deep connections between martingale theory and various fields of analysis are well-established (see, e.g., Burkholder [15]). Martingale inequalities and martingale transform techniques are powerful tools in shedding light on these connections.

Non-commutative conditional expectations and martingales arise in the setting of von Neumann algebras, which are the natural framework for non-commutative measure theory and integration. The study of these mathematical objects was and still is partly motivated by quantum mechanics, but non-commutative probability has also become an independent field of mathematical research. We refer to P.A. Meyer's exposition [59] on quantum probability, to the book of K.R. Parthasarathy [62] on quantum stochastic calculus, and to the book of Voiculescu, Dykema, Nica [84] concerning the rapidly developing area of free probability.

Pisier and Xu [69] proved the non-commutative analogues of the classical Burkholder-Gundy square function inequalities. Their work triggered a systematic investigation of non-commutative martingale inequalities, which had been previously considered only in special cases; see the results of Carlen and Krée [16] concerning fermionic versions of the square function inequalities. Since then remarkable progress has been made in establishing non-commutative analogues of other classical martingale inequalities (see Junge [45], and Junge and Xu [48]), in studying non-commutative martingale transforms and in finding the order of best constants in various martingale inequalities (see Randrianantoanina [74], and Junge and Xu [49]). Non-commutative martingale inequalities have connections to other areas of non-commutative probability; see, for example, the applications of the square function inequalities to the stochastic calculus with respect to free Brownian motion, developed by Biane and Speicher [5].

It should be pointed out that dealing with non-commutative martingales often requires an additional insight. For instance, stopping time arguments and maximal functions, which are often used in the classical proofs, appear unavailable in this setting. Indeed, note that the pointwise supremum of a sequence of bounded linear operators may not necessarelly represent a (possibly unbounded) linear operator at all. Therefore, functional-analytic methods, including interpolation, as well as concepts and appropriate constructions provided by operator space theory, become very important tools in this setting. An operator space is a norm closed linear space V of operators on some Hilbert space H, which is equipped with distinguished matrix norms obtained from $\mathcal{B}(H)$. An abstract characterization of operator spaces was given by Ruan [76]. The morphisms between operator spaces are completely bounded maps. References for the fundamentals of the subject are the books of Effros and Ruan [29], and Pisier [71].

More recently, Pisier [70] developed a theory of non-commutative vector-valued L_p-spaces $L_p(\mathcal{M};V)$, associated with a von Neumann algebra $\mathcal{M}$, equipped with a normal, faithful (abbreviated as n.f.) tracial state τ. Two restrictions are required for the theory to be satisfactory: $\mathcal{M}$ has to be a hyperfinite von Neumann algebra, and V an operator space (not just a Banach space). In Chapter 1, after reviewing the necessary background on operator spaces, we discuss the construction of the non-commutative vector-valued L_p-spaces and some of their stability properties (e.g. duality), and explain how non-commutative martingales naturally arise in this setting.

Chapter 2 deals with the study of non-commutative BMO and its interpolation properties. In classical analysis and probability, the space BMO is a natural substitute for L_∞, thus a very useful tool for proving boundedness of martingale transforms and singular integral operators, such as the Hilbert transform. We will prove non-commutative analogues of the classical interpolation results (both for the complex and the real method) between BMO and L_p-spaces (respectively, Hardy spaces). Concerning the proofs, it is often highly non-trivial to transfer classical martingale inequalities to the non-commutative setting, for reasons already explained above. The proof of our main result in this chapter uses norm estimates for non-commutative analogues of the classical sharp functions introduced in [48], and their interpolation properties, as well as connections with the theory of Hilbert C^*-modules, discovered by Junge [45]. A non-commutative substitute for the maximal function, introduced in [45], is also used in the proof. In Section 2.3 we show that our results remain true for the real method of interpolation. The techniques used in the proof of the main result will enable us to define a natural operator space structure on the non-commutative

BMO space associated to a finite von Neumann algebra $\mathcal{M}$, equipped with an increasing filtration $(\mathcal{M}_n)_{n\geq 1}$ of von Neumann subalgebras. We refer to Section 2.3 for the discussion of the natural operator space structure of the non-commutative Hardy spaces, while the natural operator space structure of the non-commutative L_p-spaces was discussed in Section 1.4. We will prove that these interpolation results hold in the category of operator spaces.

Motivated by the classical notion of a UMD space (i.e., *unconditional for martingale differences*), which plays an important role in Banach space-valued martingale theory, Pisier [70] introduced the notion of operator space UMD_p. We discuss the definition and some basic stability properties of UMD_p operator spaces in Section 3.1. We show that, as in the classical setting, the UMD_p property is preserved under complex interpolation and by ultraproducts. If an operator space V is UMD_p, then its (standard) dual V^* is $\mathrm{UMD}_{p'}$, where p' is the conjugate exponent of p. Also, each matrix level $M_m(V)$ is UMD_p.

It is unknown whether the property is independent of p in this setting. By the Pisier-Xu non-commutative version of the Burkholder-Gundy square function inequalities, $L_p(\mathcal{N},\phi)$ is UMD_p, for every von Neumann algebra $(\mathcal{N},\phi)$, equipped with an n.f. tracial state ϕ. In Section 3.3 we show that for $1<q<\infty$, the Schatten q-classes S_q are UMD_p, independent of p. The proof relies on properties of the Haagerup tensor product and complex interpolation, and it was inspired by a question of Z.-J. Ruan whether the column Hilbert space C is UMD_p for some (all) $1<p<\infty$. As an application of ultraproduct results due to Junge [46], it follows that a large class of non-commutative L_q-spaces are UMD_p, independent of p. Namely, if $\mathcal{M}$ is a $QWEP$ von Neumann algebra (i.e., $\mathcal{M}$ is a quotient of a WEP von Neumann algebra, see Kirchberg [51]), equipped with an n.f. tracial state τ, then $L_q(\mathcal{M},\tau)$ is UMD_p, for $1<p,q<\infty$. Furthermore, we show that the class of operator spaces which are UMD_p, independent of p, contains the non-commutative Lorentz spaces associated to a hyperfinite (and finite) von Neumann algebra, as well as all finite dimensional operator spaces.

In Section A.1 of the Appendix we discuss some background results on ultraproduct theory for Banach spaces and, respectively, operator spaces. In Section A.2 we give a detailed proof to Kouba's interpolation result, which is an important tool in the proof of our main result from Chapter 3.

Chapter 1

Preliminaries

1.1 Non-commutative L_p-spaces and martingales

We use standard notation in operator algebras. We refer to Kadison and Ringrose [50] and Takesaki [81] for background on von Neumann algebras. Let us recall some basic definitions, and fix the notation that will be used throughout. Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space H, i.e., a unital $*$-subalgebra of $\mathcal{B}(H)$ (the algebra of bounded linear operators on H), which is closed under the σ-weak operator topology. Recall that $\mathcal{M}$ is called *semifinite* if there exists a *normal, semifinite, faithful* (abbreviated as *n.s.f.*) *trace*, i.e., a positive homogeneous, additive functional $\tau : \mathcal{M}_+ \to \mathbb{C}$, where

$$\mathcal{M}_+ \; = \; \{x^* x : x \in \mathcal{M}\}$$

is the positive cone of $\mathcal{M}$, satisfying the following properties:

$n)$ $\tau(\sup_i x_i) = \sup_i \tau(x_i)$, for all increasing nets (x_i) with supremum in $\mathcal{M}$;

$s)$ For all $x > 0$, there exists $0 < y < x$ such that $0 < \tau(y) < \infty$;

$f)$ If $x \in \mathcal{M}$ such that $\tau(x) = 0$, then $x = 0$;

$t)$ $\tau(xy) = \tau(yx)$, for all $x, y \in \mathcal{M}$.

If $\tau(1) < \infty$, then the algebra $\mathcal{M}$ is called *finite*. In this case we will assume that τ is normalized, i.e., $\tau(1) = 1$, and τ will be called an n.f. tracial *state*.

Following [81], let $n_\tau = \{x \in \mathcal{M} : \tau(x^*x) < \infty\}$. Then n_τ is a two-sided ideal of $\mathcal{M}$. It follows that

$$m_\tau = \left\{ \sum_{i=1}^{k} x_i y_i : x_i, y_i \in n_\tau,\, k \in \mathbb{N} \right\}$$

is a two-sided ideal of $\mathcal{M}$, called the *definition ideal* of the trace τ. There is a unique linear extension of τ to m_τ preserving the tracial property, i.e., $\tau(xy) = \tau(yx)$ for all $x, y \in m_\tau$. For $1 \leq p < \infty$, define for all $x \in m_\tau$

$$\|x\|_p = \tau((x^*x)^{\frac{p}{2}}))^{\frac{1}{p}}.$$

Then $(m_\tau, \|\cdot\|_p)$ is a normed linear space, and $L_p(\mathcal{M}, \tau)$ denotes its Banach space completion. For $p = \infty$, let $L_\infty(\mathcal{M}, \tau) = \mathcal{M}$ (with the operator norm). The usual Hölder inequality holds, i.e., if $0 < r, p, q, \leq \infty$ are such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, then, for all $x \in L_p(\mathcal{M}, \tau), y \in L_q(\mathcal{M}, \tau)$ it follows that

$$xy \in L_r(\mathcal{M}, \tau) \quad \text{and} \quad \|xy\|_r \leq \|x\|_p \|y\|_q.$$

As a consequence, if $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then the pairing

$$\langle x, y \rangle \quad = \quad \tau(xy), \tag{1.1}$$

where $x \in L_p(\mathcal{M}, \tau)$ and $y \in L_{p'}(\mathcal{M}, \tau)$, defines a duality bracket between $L_p(\mathcal{M}, \tau)$ and $L_{p'}(\mathcal{M}, \tau)$. We have isometrically (see [61])

$$(L_p(\mathcal{M}, \tau))^* = L_{p\prime}(\mathcal{M}, \tau).$$

The non-commutative L_p-spaces can also be obtained by interpolation, as in the classical setting. In the sequel we will briefly recall some basic notions concerning the complex method of interpolation due to Calderon. Our main reference for interpolation theory is Bergh and Löfström [3]. A pair of Banach spaces (V_0, V_1) is called a *compatible couple* if they embed continuously in some topological vector space X. This allows us to consider the spaces $V_0 \cap V_1$ and $V_0 + V_1$ (by identifying them inside X). They are Banach spaces when equipped, respectively, with the following norms

$$\|x\|_{V_0 \cap V_1} \quad = \quad \max\{\|x\|_{V_0}, \|x\|_{V_1}\}, \tag{1.2}$$

$$\|x\|_{V_0+V_1} \quad = \quad \inf\{\|x_0\|_{V_0} + \|x_1\|_{V_1} : x = x_0 + x_1\}\,. \tag{1.3}$$

Following the notation from [3], let S denote the vertical strip $\{\, z \in \mathbb{C} : 0 \le \mathrm{Re} z \le 1\,\}$ and let S_0 denote the interior of S. Define $\mathcal{F}_{V_0+V_1}$ to be the family of all continuous and bounded functions $f : S \to V_0 + V_1$ satisfying the following properties

1. f is analytic in S_0 ,

2. $f(it) \in V_0$ and $f(1+it) \in V_1$ for all $t \in \mathbb{R}$,

3. $f(it) \to 0$ and $f(1+it) \to 0$ as $t \to \infty$.

By the Phragmen-Lindelöf theorem, $\mathcal{F}_{V_0+V_1}$ is a Banach space under the norm

$$\|f\|_{\mathcal{F}_{V_0+V_1}} \quad = \quad \max\left\{\sup_{t\in\mathbb{R}} \|f(it)\|_{V_0},\ \sup_{t\in\mathbb{R}} \|f(1+it)\|_{V_1}\right\}. \tag{1.4}$$

For $0 \le \theta \le 1$, set $[V_0, V_1]_\theta = \{\, x \in V_0 + V_1 : x = f(\theta)\,,$ for some $f \in \mathcal{F}_{V_0+V_1}\,\}$. This is called the *complex interpolation space* (of exponent θ) between V_0 and V_1 , and it is a Banach space under the norm

$$\|x\|_\theta = \inf\{\|f\|_{\mathcal{F}_{V_0+V_1}} : x = f(\theta),\ f \in \mathcal{F}\}\,.$$

The complex method is an *exact* interpolation functor; i.e., if $T : V_0 + V_1 \to W_0 + W_1$ is a linear operator which is bounded both from V_0 to W_0 and from V_1 to W_1, then T is bounded from $[V_0, V_1]_\theta$ to $[W_0, W_1]_\theta$, and, moreover,

$$\|T : [V_0, V_1]_\theta \to [W_0, W_1]_\theta\| \quad \le \quad \|T : V_0 \to W_0\|^{1-\theta} \|T : V_1 \to W_1\|^{\theta}\,, \tag{1.5}$$

We now return to the construction of the non-commutative L_p-spaces by interpolation. Recall the following definition due to Nelson [61] (see also Terp [82]). For $\epsilon, \delta > 0$, let

$$N(\epsilon, \delta) = \{x \in \mathcal{M} : \text{for some projection } p \in \mathcal{M}\,,\ \|xp\| \le \epsilon \ \text{ and } \ \tau(1-p) \le \delta\,\}\,.$$

Consider on $\mathcal{M}$ the translation-invariant topology (called the *measure topology*) in which $N(\epsilon, \delta)$ is a fundamental system of neighborhoods of the origin. Let $\tilde{\mathcal{M}} = L_0(\mathcal{M})$ denote the completion of $\mathcal{M}$ with respect to this topology. Then $L_0(\mathcal{M})$ is a topological $*-$algebra, called the algebra of *τ-measurable operators* in $\mathcal{M}$. Moreover, for $1 \leq p \leq \infty$, $L_p(\mathcal{M}, \tau)$ embeds continuously into $L_0(\mathcal{M})$ (see [61]). Therefore $(\mathcal{M}, L_1(\mathcal{M}, \tau))$ is a compatible couple, and for all $1 \leq p < \infty$ we have isometrically

$$L_p(\mathcal{M}, \tau) = [\mathcal{M}, L_1(\mathcal{M}, \tau)]_{\frac{1}{p}} . \tag{1.6}$$

More generally, by the reiteration theorem for the complex method (Theorem 4.6.1 of [3]), for $1 \leq p_0, p_1, p \leq \infty$ and $0 < \theta < 1$ with $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, the following holds isometrically

$$L_p(\mathcal{M}, \tau) \quad = \quad [L_{p_0}(\mathcal{M}, \tau), L_{p_1}(\mathcal{M}, \tau)]_\theta . \tag{1.7}$$

For $1 \leq p < \infty$, let $L_p(\mathcal{M}, l_2^c) \subset L_p(\mathcal{M} \otimes \mathcal{B}(l_2))$ denote the space of column matrices with entries from $L_p(\mathcal{M})$. Similarly, denote by $L_p(\mathcal{M}, l_2^r)$ the space of row matrices with entries from $L_p(\mathcal{M})$. These spaces were introduced in [69], where they were denoted by $L_p(\mathcal{M}; l_2^c)$, respectively, $L_p(\mathcal{M}; l_2^r)$. As observed in [69], both $L_p(\mathcal{M}, l_2^c)$ and $L_p(\mathcal{M}, l_2^r)$ are 1-complemented subspaces of $L_p(\mathcal{M} \otimes \mathcal{B}(l_2))$. Hence, by (1.7) it follows that we have isometrically

$$L_p(\mathcal{M}, l_2^c) \quad = \quad [L_{p_0}(\mathcal{M}, l_2^c), L_{p_1}(\mathcal{M}, l_2^c)]_\theta , \tag{1.8}$$

$$L_p(\mathcal{M}, l_2^r) \quad = \quad [L_{p_0}(\mathcal{M}, l_2^r), L_{p_1}(\mathcal{M}, l_2^r)]_\theta . \tag{1.9}$$

For the remainder of this section we assume that $\mathcal{M}$ is finite. Let $(\mathcal{M}_n)_{n \geq 1}$ be an increasing sequence of von Neumann subalgebras of $\mathcal{M}$, whose union generates $\mathcal{M}$ in the w*-topology. Given $n \geq 1$, there is a unique normal conditional expectation $\mathcal{E}_n : \mathcal{M} \to \mathcal{M}_n$ such that $\tau \restriction \mathcal{M}_n \circ \mathcal{E}_n = \mathcal{E}_n$ (see Takesaki [81]). This extends to a norm 1 projection $\mathcal{E}_n : L_p(\mathcal{M}, \tau) \to L_p(\mathcal{M}_n, \tau_n)$, where $\tau_n = \tau \restriction \mathcal{M}_n$, satisfying the modular property

$$\mathcal{E}_n(axb) \quad = \quad a\mathcal{E}_n(x)b , \tag{1.10}$$

for all $a \in L_s(\mathcal{M}_n)$, $b \in L_r(\mathcal{M}_n)$ and $x \in L_p(\mathcal{M})$, where $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$.

A *non-commutative* $L_p(\mathcal{M})$*-martingale* relative to the filtration $(\mathcal{M}_n)_{n\geq 1}$ is a sequence $x = (x_n)_{n\geq 1}$ such that $x_n \in L_p(\mathcal{M}_n)$ and $\mathcal{E}_n(x_{n+1}) = x_n$, for all positive integers n . We say that x is a bounded $L_p(\mathcal{M})$-martingale if $\|x\|_p = \sup_n \|x_n\|_p < \infty$. The *difference sequence* of x is $dx = (dx_n)_{n\geq 1}$, where $dx_n = x_n - x_{n-1}$, with $x_0 = 0$. For $1 < p < \infty$, as a consequence of the uniform convexity of the space $L_p(\mathcal{M})$, we can and will identify the space of all bounded $L_p(\mathcal{M})$-martingales with $L_p(\mathcal{M})$ itself (see [69], Remark 1.3).

We now recall the norms in the non-commutative Hardy spaces of martingales, introduced in [69]. Let $1 \leq p < \infty$, and let $x = (x_n)_{n\geq 1}$ be an $L_p(\mathcal{M})$-martingale relative to $(\mathcal{M}_n)_{n\geq 1}$. Define

$$\begin{aligned}
\|x\|_{H_p(\mathcal{M})} &= \max\{\|(\sum_{n\geq 1} dx_n^* dx_n)^{1/2}\|_p \, , \, \|(\sum_{n\geq 1} dx_n dx_n^*)^{1/2}\|_p\} \qquad \text{if } 2 \leq p < \infty \, , \\
\|x\|_{H_p(\mathcal{M})} &= \inf\{\|(\sum_{n\geq 1} dy_n^* dy_n)^{1/2}\|_p + \|(\sum_{n\geq 1} dz_n dz_n^*)^{1/2}\|_p\} \qquad \text{if } 1 \leq p < 2 \, ,
\end{aligned}$$

where the infimum runs over all decompositions $x = y+z$, with y and z $L_p(\mathcal{M})$-martingales relative to $(\mathcal{M}_n)$. The Pisier-Xu non-commutative version [69] of the Burkholder-Gundy square function inequalities states that for $1 < p < \infty$, $L_p(\mathcal{M}) = H_p(\mathcal{M})$ with equivalent norms. More precisely, there exist $\alpha_p, \beta_p > 0$, depending only on p, such that for all finite $L_p(\mathcal{M})$-martingales x relative to the filtration $(\mathcal{M}_n)_{n\geq 1}$,

$$\alpha_p{}^{-1} \|x\|_{H_p(\mathcal{M})} \leq \quad \|x\|_p \quad \leq \beta_p \|x\|_{H_p(\mathcal{M})} \, . \tag{1.11}$$

1.2 Operator spaces

We refer to Effros and Ruan [29] and Pisier [71] for details on operator spaces and completely bounded maps. We shall briefly recall some definitions. A (*concrete*) *operator space* on a Hilbert space H is a norm closed linear subspace E of $\mathcal{B}(H)$. For all positive integers m, the natural inclusion

$$M_m(E) \subseteq M_m(\mathcal{B}(H)) = \mathcal{B}(H^m)$$

induces a norm $\|\cdot\|_m$ on $M_m(E)$. Ruan [76] gave an abstract characterization of operator spaces in terms of their matrix norms. Namely, an (*abstract*) *operator space* is a vector space E equipped with matrix norms $\|\cdot\|_m$ on $M_m(E)$, for each positive integer m, satisfying the following axioms

$$\begin{aligned} \|x \oplus y\|_{m+n} &= \max\{\|x\|_m, \|y\|_n\}, \\ \|\alpha x \beta\|_m &\leq \|\alpha\|\|x\|_m\|\beta\|, \end{aligned} \tag{1.12}$$

for all $x \in M_m(E)$, $y \in M_n(E)$ and $\alpha, \beta \in M_m(\mathbb{C})$. The morphisms in the category of operator spaces are *completely bounded maps*. Given a linear map between two (abstract) operator spaces $\phi : E_0 \to E_1$, define $\phi_m : M_m(E_0) \to M_m(E_1)$ by

$$\phi_m([v_{ij}]) = [\phi(v_{ij})].$$

for all $[v_{ij}]_{i,j=1}^m \in M_m(E_0)$. Let $\|\phi\|_{cb} = \sup\{\|\phi_m\| : m \in \mathbb{N}\}$. The map ϕ is called *completely bounded* (respectively, *completely contractive*) if $\|\phi\|_{cb} < \infty$ (respectively, $\|\phi\|_{cb} \leq 1$). The map ϕ is said to be a *complete isometry* if all ϕ_m are isometries, and a *complete isomorphism* if it is a linear isomorphism with $\|\phi\|_{cb}, \|\phi^{-1}\|_{cb} < \infty$. Finally, ϕ is called a *complete quotient map* if each ϕ_m is a quotient map.

Due to the following representation theorem, we shall not distinguish between *concrete* and *abstract* operator spaces.

Theorem 1.2.1 (Ruan [76]) *If E is an abstract operator space, then there is a Hilbert space H, a concrete operator space $F \subseteq \mathcal{B}(H)$, and a complete isometry ϕ from E onto F. If E is separable as a normed space, then we can let $H = l_2$.*

The space of all completely bounded maps from E_0 to E_1 is denoted by $\mathcal{CB}(E_0, E_1)$. It turns out that $\mathcal{CB}(E_0, E_1)$ is an operator space, with matrix norms defined for all positive integers m by

$$M_m(\mathcal{CB}(E_0, E_1)) = \mathcal{CB}(E_0, M_m(E_1)). \tag{1.13}$$

If E is an operator space, then the dual space $E^* = \mathcal{B}(E, \mathbb{C}) = \mathcal{CB}(E, \mathbb{C})$ endowed with matrix norms given by

$$M_m(E^*) \quad = \quad \mathcal{CB}(E, M_m(\mathbb{C})) \tag{1.14}$$

is again an operator space, called the *standard operator space dual* of E. This was independently discovered by Effros-Ruan [26] and Blecher-Paulsen [6]. Furthermore, if F is a closed subspace of E, then both F and E/F are operator spaces; F is equipped with the induced operator space structure from E, while on E/F the matrix norms are defined for all poositive integers m by

$$M_m(E/F) \quad = \quad M_m(E)/M_m(F)\,.$$

Let (E_0, E_1) be a compatible couple of operator spaces. Following Xu [89], we equip the space $E_0 + E_1$ with the Banach space norm

$$\|x\|_{E_0+E_1} \quad = \quad \inf\{\max\{\|x_0\|_{E_0}\,, \|x_1\|_{E_1}\} : x = x_0 + x_1, x_0 \in E_0, x_1 \in E_1\}\,. \tag{1.15}$$

Let us also recall the spaces

$$\begin{aligned} E_0 \oplus_p E_1 \quad &= \quad \{x = (x_0, x_1) \in E_0 \oplus E_1 : \|x\|_p = (\|x\|_{E_0}^p + \|x_1\|_{E_1}^p)^{1/p}\}\,, \quad 1 \le p < \infty\,; \\ E_0 \oplus_\infty E_1 \quad &= \quad \{x = (x_0, x_1) \in E_0 \oplus E_1 : \|x\|_\infty = \max\{\|x\|_{E_0}, \|x_1\|_{E_1}\}\}\,. \end{aligned}$$

As observed in [68], $E_0 \oplus_\infty E_1$ is an operator space with matrix norms defined by

$$M_m(E_0 \oplus_\infty E_1) \quad = \quad M_m(E_0) \oplus_\infty M_m(E_1)\,, \tag{1.16}$$

for all positive integers m, while the isometric embeddings

$$M_m(E_0 \oplus_1 E_1) \quad \hookrightarrow \quad CB(E_0^* \oplus_\infty E_1^*, M_m(\mathbb{C})) \;=\; M_m(E_0^* \oplus_\infty E_1^*) \;=\; M_m((E_0 \oplus_1 E_1)^{**})$$

induce an operator space structure on $E_0 \oplus_1 E_1$. For $1 < p < \infty$, $E_0 \oplus_p E_1$ is equipped with the operator space structure given by the isometric identification

$$E_0 \oplus_p E_1 \quad = \quad l_p(\{E_0, E_1\}) \,,$$

where $l_p(\{E_0, E_1\})$ is the l_p-direct sum of E_0 and E_1. Furthermore, note that $E_0 \cap E_1$ can be identified with the diagonal $\Delta = \{(x, x) \in E_0 \oplus E_1\}$ of $E_0 \oplus_\infty E_1$, while

$$E_0 + E_1 \; = \; E_0 \oplus_\infty E_1 / N \,,$$

where $N = \{\, (x_0, x_1) \in E_0 \oplus E_1 : x_0 + x_1 = 0 \,\}$. These identifications are used to equip $E_0 \cap E_1$ and, respectively, $E_0 + E_1$ with appropriate operator space matrix norms. Following the notations in [89], let $\Delta_p(E_0, E_1)$ be $\Delta(E_0, E_1)$ regarded as a subspace of $E_0 \oplus_p E_1$, and, respectively, let $N_p(E_0, E_1)$ be $N(E_0, E_1)$ regarded as a subspace of $E_0 \oplus_p E_1$. Finally, let $E_0 +_p E_1$ be identified with the quotient space $E_0 \oplus_p E_1 / N_p(E_0, E_1)$. Following Pisier [68], for $0 < \theta < 1$, we endow the interpolation space $[E_0, E_1]_\theta$ with a canonical operator space structure by defining for all positive integers m,

$$M_m([E_0, E_1]_\theta) = [M_m(E_0), M_m(E_1)]_\theta \,. \tag{1.17}$$

More generally, for all positive integers m and n, define

$$M_{m,n}([E_0, E_1]_\theta) \quad = \quad [M_{m,n}(E_0), M_{m,n}(E_1)]_\theta \,. \tag{1.18}$$

Recall that the complex method of interpolation is an exact functor of exponent θ. Thus, if (E_0, E_1) and (F_0, F_1) are two compatible couples of operator spaces, and if a map $u : E_0 + E_1 \to F_0 + F_1$ is completely bounded both from E_0 to E_1 and from F_0 to F_1, then u is completely bounded from $[E_0, E_1]_\theta$ to $[F_0, F_1]_\theta$, and, moreover,

$$\|u : [E_0, E_1]_\theta \to [F_0, F_1]_\theta\|_{cb} \quad \leq \quad \|u : E_0 \to E_1\|_{cb}^{1-\theta} \|u : F_0 \to F_1\|_{cb}^{\theta} \,. \tag{1.19}$$

If $(V, \|\cdot\|)$ is a Banach space, there are several realizations of it as an operator space. Indeed, for all positive integers m and $x = [x_{i,j}]_{i,j=1}^m \in M_m(V)$, define

$$\|x\|_{M_m(V^{\min})} = \sup\{\|[f(x_{i,j})]\|_m : f : V \to \mathbb{C} \text{ linear contraction}\}; \tag{1.20}$$

$$\|x\|_{M_m(V^{\max})} = \sup\{\|[\phi(x_{i,j})]\|_m : \phi : V \to \mathcal{B}(l_2) \text{ linear contraction}\}. \tag{1.21}$$

It follows that both $\{\|\cdot\|_{M_m(V^{\min})} : m \in \mathbb{N}\}$ and $\{\|\cdot\|_{M_m(V^{\max})} : m \in \mathbb{N}\}$ verify the Ruan axioms (1.12). We denote by $\min(V)$, respectively, $\max(V)$ the corresponding operator spaces they determin. Note that $\min(V)$ is the operator space structure on V obtained by its isometric embedding into a commutative C^*-algebra, namely, $C(V_1^*)$, where V_1^* denotes the closed unit ball of V. Furthermore, if $\{\|\cdot\|_m : m \in \mathbb{N}\}$ is another operator space matrix norm on V such that $\|\cdot\|_1 = \|\cdot\|$, then, for all positive integers m and $x = [x_{i,j}]_{i,j=1}^m \in M_m(V)$,

$$\|x\|_{M_m(V^{\min})} \leq \|x\|_m \leq \|x\|_{M_m(V^{\max})}.$$

The minimal operator space quantization was discussed in Effros-Ruan [28]. Properties of the min and max operator space matrix norms were systematically studied by Blecher-Paulsen [6].

We now briefly recall the basic definitions and discuss some important aspects of the tensor product theory between operator spaces.

1. The projective tensor product

This is the analogue, in the category of operator spaces, of the projective tensor product $\otimes^\pi$ from Banach space theory. Recall that if V and W are Banach spaces, then for $u \in V \otimes W$ we define

$$\|u\|_\pi = \inf\left\{\sum_{i=1}^m \|v_i\|\|w_i\| : u = \sum_{i=1}^m v_i \otimes w_i\right\}.$$

It follows that $\|\cdot\|_\pi$ is a cross-norm on $V \otimes W$ (in fact, the largest one). We denote by $V \otimes^\pi W$ the completion of $V \otimes W$ with respect to this norm. We have the isometry

$$(V \otimes^\pi W)^* \cong \mathcal{B}(V, W^*). \tag{1.22}$$

Another important feature of the π-tensor product is given by the following result. If (X, μ) and (Y, μ) are measure spaces, then, isometrically,

$$L_1(X, \mu) \otimes^\pi L_1(Y, \mu) \quad \cong \quad L_1(X \times Y, \mu)\,. \tag{1.23}$$

Let E and F be operator spaces and m a positive integer. If $u \in M_m(E \otimes F)$, then $u = \alpha(v \otimes w)\beta$, for some $\alpha \in M_{m,pq}$, $v \in M_p(E)$, $w \in M_q(F)$, $\beta \in M_{pq,m}$ and p, q positive integers. Define

$$\|u\|_{\wedge,m} \quad = \quad \inf\{\|\alpha\|_{\mathrm{HS}}\|v\|_p\|w\|_q\|\beta\|_{\mathrm{HS}}\}\,. \tag{1.24}$$

Then $\|\cdot\|_{\wedge,m}$ is a norm on $M_m(E \otimes F)$ and $\{\|\cdot\|_{\wedge,m} : m \in \mathbb{N}\}$ satisfy the Ruan axioms (1.12). Let $E\hat{\otimes}F$ denote the completion of $E \otimes F$ with respect to this matrix norm. It follows that $E\hat{\otimes}F$ is an operator space, called the *operator projective tensor product* of E and F. This was discovered independently by Blecher and Paulsen [6], and Effros and Ruan [26]. The following complete isometry holds

$$(E\hat{\otimes}F)^* \quad \cong \quad \mathcal{CB}(E, F^*)\,. \tag{1.25}$$

The following two propositions summarize some of the properties of the projective tensor product, namely, it is both commutative and associative, and *projective.*

Proposition 1.2.2 *Let E, F and G be operator spaces. We have completely isometric isomorphisms*

$$E\hat{\otimes}F \quad \cong \quad F\hat{\otimes}E\,; \tag{1.26}$$

$$(E\hat{\otimes}F)\hat{\otimes}G \quad \cong \quad E\hat{\otimes}(F\hat{\otimes}G) \tag{1.27}$$

Proposition 1.2.3 *Let E_0, E_1, F_0, F_1 be operator spaces. If $\phi_i : E_i \to F_i$, $i = 0, 1$ are complete contractions (respectively, complete quotient maps), then the map $\phi_0 \otimes \phi_1 : E_0 \otimes E_1 \to F_0 \otimes F_1$ extends to a complete contraction (respectively, a complete quotient map)*

$$\phi_0 \otimes \phi_1 : E_0\hat{\otimes}E_1 \to F_0\hat{\otimes}F_1\,.$$

The relationship between the operator projective tensor product norm at the Banach space level $\|\cdot\|_{\wedge,1}$ and the Banach space projective tensor product norm $\|\cdot\|_\pi$ is the following

$$\|\cdot\|_{\wedge,1} \quad \leq \quad \|\cdot\|_\pi \,. \tag{1.28}$$

Effros-Ruan [24] proved the following non-commutative analogue of (1.23). Let $R \subseteq \mathcal{B}(H)$ and $S \subseteq \mathcal{B}(K)$ be von Neumann algebras. Denote by R_*, respectively, S_* their preduals, endowed with the natural operator space structure given by completely isometric embeddings into R^*, respectively, S^*. Then, completely isometrically,

$$R_* \hat{\otimes} S_* \quad \cong \quad (R \bar{\otimes} S)_* \,, \tag{1.29}$$

where $R\bar{\otimes}S \;=\; (R \otimes S)^{-w^*} \subseteq \mathcal{B}(H \otimes K)$ is the von Neumann algebra tensor product.

2. The injective tensor product

Recall that if V and W are Banach spaces, we have canonical isometric embeddings

$$V \hookrightarrow l_\infty(V_1^*) \quad \text{and} \quad W \hookrightarrow l_\infty(W_1^*)\,.$$

The injective (or min) tensor product $V \otimes^\varepsilon W$ is defined as the completion of $V \otimes W$ in the norm induced by the embedding

$$V \otimes W \hookrightarrow l_\infty(V_1^* \times W_1^*)\,. \tag{1.30}$$

Given E and F operator spaces, by Ruan's representation theorem (Theorem 1.2.1) there exist Hilbert spaces H and K and completely isometric embeddings

$$E \hookrightarrow \mathcal{B}(H) \quad \text{and} \quad F \hookrightarrow \mathcal{B}(K)\,. \tag{1.31}$$

The embedding

$$E \otimes F \hookrightarrow \mathcal{B}(H \otimes K)$$

induces an operator space matrix norm $\|\cdot\|_\vee$ on $E \otimes F$. It is proved in [6] that this matrix norm is independent on the choice of the Hilbert spaces H and K. Namely, if m is a positive integer and $u \in M_m(E \otimes F)$, then

$$\|u\|_{\vee,m} \quad = \quad \sup\{\|(\phi \otimes \psi)_m(u)\| : \phi \in M_p(E^*)_1 , \psi \in M_q(F^*)_1 , p, q \in \mathbb{N}\} .$$

We denote by $E\check{\otimes}F$ the completion of $E \otimes F$ with respect to the operator space matrix norm $\|\cdot\|_\vee$. Then $E\check{\otimes}F$ is an operator space, called the *operator injective (or* min*) tensor product* of E and F. We have a completely isometric embedding

$$E^*\check{\otimes}F \;\hookrightarrow\; \mathcal{CB}(E, F) .$$

Moreover, if either E or F is finite dimensional, then, isometrically

$$E^*\check{\otimes}F \quad \cong \quad \mathcal{CB}(E, F) . \tag{1.32}$$

As a consequence, if m is a positive integer, we have the isometry

$$M_m\check{\otimes}E \quad \cong \quad M_m(W) . \tag{1.33}$$

The operator injective tensor product is both commutative and associative, and *injective.*

Proposition 1.2.4 (Blecher-Paulsen [6]) *Let E, F and G be operator spaces. We have completely isometric isomorphisms*

$$E\check{\otimes}F \quad \cong \quad F\check{\otimes}E ; \tag{1.34}$$

$$(E\check{\otimes}F)\check{\otimes}G \quad \cong \quad E\check{\otimes}(F\check{\otimes}G) \tag{1.35}$$

Proposition 1.2.5 (Blecher-Paulsen [6]) *Let E_0 , E_1 , F_0 , F_1 be operator spaces. If $\phi_i : E_i \to F_i , i = 0, 1$ are complete contractions, then the map $\phi_0 \otimes \phi_1 : E_0 \otimes E_1 \to F_0 \otimes F_1$ extends to a complete contraction*

$$\phi_0 \otimes \phi_1 : E_0\check{\otimes}E_1 \;\to\; F_0\check{\otimes}F_1 .$$

Moreover, if ϕ_0 and ϕ_1 are completely isometric injections, then $\phi_0 \otimes \phi_1$ is a complete isometry.

The following relation holds between the operator injective tensor product norm at the Banach space level $\|\cdot\|_{\vee,1}$ and the Banach space injective tensor product norm $\|\cdot\|_{\varepsilon}$

$$\|\cdot\|_{\varepsilon} \;\leq\; \|\cdot\|_{\vee,1}\,.$$

3. The Haagerup tensor product

This is an operator space concept. Given operator spaces E and F and $u \in M_m(E \otimes F)$, we define

$$\|u\|_{h,m} \;=\; \inf\{\|v\|\|w\| : u = v \odot w, v \in M_{m,r}(E)\,, w \in M_{r,m}(F)\,, r \in \mathbb{N}\}\,, \qquad (1.36)$$

where $v \odot w \in M_m(E \otimes F)$ is defined by

$$(v \odot w)_{ij} \;=\; \sum_{k=1}^{m} v_{ik} \otimes w_{kj}\,, \quad 1 \leq i, j \leq m\,.$$

It follows that $\|\cdot\|_{h,m}$ is a norm on $M_m(E \otimes F)$ and $\{\|\cdot\|_{h,m} : m \in \mathbb{N}\}$ verify the Ruan axioms (1.12). Let $E \otimes^h F$ denote the completion of $E \otimes F$ with respect to this operator space matrix norm. Then $E \otimes^h F$ is an operator space, called the *Haagerup tensor product* of E and F. This was originally proved by Paulsen-Smith [64], using a very difficult argument. The Haagerup tensor norm (1.36) was first introduced by Effros-Kishimoto [22] (only for $m = 1$). The following result is very useful in applications.

Lemma 1.2.6 (Paulsen-Smith [64]) *For any $u \in E \otimes_h F$ with $\|u\|_h < 1$, there exists a representation*

$$u \;=\; v \odot w \;=\; \sum_{j=1}^{r} v_j \otimes w_j$$

with $\|v\|\|w\| < 1$ for which $v_1, \ldots, v_r$ are linearly independent in E, and $w_1, \ldots, w_r$ are linearly independent in F.

The following results (from [64] and [25]) summarize some of the properties of the Haagerup tensor product, namely, it is both injective and projective, associative, self-dual (in the finite dimensional case) and, in general, not commutative.

Proposition 1.2.7 *Let E, F and G be operator spaces. We have a completely isometric isomorphism*

$$(E \otimes^h F) \otimes^h G \quad \cong \quad E \otimes^h (F \otimes^h G)\,.$$

Proposition 1.2.8 *Let $E_0\,, E_1\,, F_0\,, F_1$ be operator spaces. If $\phi_i : E_i \to F_i\,, i = 0, 1$ are complete contractions (complete isometries, respectively, complete quotients), then so is*

$$\phi_0 \otimes \phi_1 : E_0 \otimes^h E_1 \quad \to \quad F_0 \otimes^h F_1\,.$$

Proposition 1.2.9 (Effros-Ruan [25]) *Let E and F be operator spaces. We have a completely isometric inclusion*

$$E^* \otimes^h F^* \quad \hookrightarrow \quad (E \otimes^h F)^*\,.$$

Moreover, if both E and F are finite dimensional, then, completely isometrically,

$$E^* \otimes^h F^* \quad \cong \quad (E \otimes^h F)^*\,.$$

Recall that given a Hilbert space H, we use the identification $H \cong \mathcal{B}(\mathbb{C}, H)$ to obtain the *column operator space* structure on H and denote by H_c the corresponding operator space. Similarly, we define the *row operator space* H_r by the identification $H \cong \mathcal{B}(H^*, \mathbb{C})$. By Corollary 4.2 in [25], we have the complete isometry

$$(H_c)^* \quad \cong \quad (H^*)_r\,. \tag{1.37}$$

If $H = l_2$, we use the notation $H_c = C$ and $H_r = R$. Respectively, if $H = l_2^m$, where m is a positive integer, we denote H_c by C^m, and H_r by R^m.

Proposition 1.2.10 (Effros-Ruan [25]) *Let H be a Hilbert space and E an operator space. We* *have complete isometries*

$$H_c \otimes^h V \cong H_c \check{\otimes} E \quad and \quad E \otimes^h H_c \cong E \hat{\otimes} H_c \,, \tag{1.38}$$

$$H_r \otimes^h V \cong H_r \hat{\otimes} E \quad and \quad E \otimes^h H_r \cong E \check{\otimes} H_r \,, \tag{1.39}$$

$$H_c \otimes^h H_r \cong \mathcal{K}(H) \quad and \quad H_r \otimes^h H_c \cong \mathcal{T}(H) \,. \tag{1.40}$$

where $\mathcal{K}(H)$ *denotes the space of compact operators on* H *and* $\mathcal{T}(H)$ *denotes the trace class operators on* H *.*

For all positive integer m, we have the following relationship between the three operator space tensor product matrix norms discussed above:

$$\|\cdot\|_{\vee,m} \leq \quad \|\cdot\|_{h,m} \quad \leq \|\cdot\|_{\wedge,m} \,.$$

Finally, we recall the following interpolation result from [68], for which we will give a detailed proof in the Appendix A.2.

Theorem 1.2.11 (Kouba) *Let* (E_0, E_1) *and* (F_0, F_1) *be two compatible couples of operator spaces. Then* $(E_0 \otimes^h F_0, E_1 \otimes^h F_1)$ *is a compatible couple of operator spaces, and for all* $0 < \theta < 1$ *we have a complete isometry*

$$[\, E_0 \otimes^h F_0, E_1 \otimes^h F_1 \,]_\theta \quad = \quad E_\theta \otimes^h F_\theta \,, \tag{1.41}$$

where $E_\theta = [E_0, E_1]_\theta$ *, respectively,* $F_\theta = [F_0, F_1]_\theta$ *.*

1.3 Non-commutative vector-valued L_p-spaces (The discrete case)

The Schatten p-classes $S_p \,, 1 \leq p \leq \infty$, are non-commutative analogues of the Banach spaces l_p . If m is a positive integer, denote by S_∞^m the space M_m , equipped with the norm $\|\cdot\|_\infty$ determined by its identification with $\mathcal{B}(l_2^m)$. Also, we denote by S_1^m the space $\mathcal{T}_m$, where

$$\mathcal{T}_m \;=\; \{\alpha \in M_m : \|\alpha\|_1 = \mathrm{tr}((\alpha^*\alpha)^{1/2}) < \infty\}\,.$$

If $1 < p < \infty$, let

$$S_p^m \;=\; \{\alpha \in M_m : |\alpha|^p \in S_1^m\}\,.$$

It follows that, isometrically,

$$S_p^m \;=\; [S_\infty^m, S_1^m]_{\frac{1}{p}}\,. \tag{1.42}$$

Remark 1.3.1 The duality

$$M_m^* \;=\; S_1^m$$

is given by the following *parallel* duality bracket

$$\langle\, [\beta_{ij}]\,, [\alpha_{ij}]\,\rangle \;=\; \sum_{i,j=1}^{m} \beta_{ij}\alpha_{ij} \;=\; \mathrm{tr}(\beta\alpha^{\mathrm{t}})\,, \tag{1.43}$$

where α^{t} denotes the transposed matrix, i.e., $(\alpha^{\mathrm{t}})_{i,j} \;=\; \alpha_{j,i}\,, 1 \le i, j \le m$. Therefore, $S_1^m = M_m^*$ has the operator space structure of the standard dual of M_m .

Following Pisier [70], we equip S_p^m with the operator space structure (1.17) obtained by interpolation. Respectively, in the infinite dimensional case, we denote by S_∞ the space $\mathcal{K}(l_2)$ of compact operators on l_2 , equipped with the operator norm. Then S_∞ carries a natural operator space structure. Let $S_1 \;=\; S_\infty^*$, equipped with the dual operator space structure. If $1 < p < \infty$, we have isometrically,

$$S_p \;=\; [S_\infty, S_1]_{\frac{1}{p}}\,, \tag{1.44}$$

and we equip S_p with the operator space structure (1.17) obtained by interpolation.

Let E be an operator space. Pisier [70] constructed by interpolation the non-commutative vector-valued Schatten p-classes $S_p[E]$, for $1 \le p \le \infty$. For all positive integers m, define

$$S_\infty^m[E] \,=\, S_\infty^m \check{\otimes} E\,, \quad \text{respectively,} \quad S_\infty[E] \,=\, S_\infty \check{\otimes} E\,, \tag{1.45}$$

$$S_1^m[E] \,=\, S_1^m \hat{\otimes} E\,, \quad \text{respectively,} \quad S_1[E] \,=\, S_1 \hat{\otimes} E\,, \tag{1.46}$$

By (1.14), (1.25), (1.43) and (1.46), we have the completely isometric identifications

$$M_m(E^*) \,\cong\, \mathcal{CB}(E, M_m) \;\cong\; (M_m^* \hat{\otimes} E)^* \,\cong\, (S_1^m[E])^*\,. \tag{1.47}$$

Denote by $M_\infty[E]$ the space of all infinite matrices $[v_{ij}]$ with entries from E, equipped with the following topology of pointwise convergence. A sequence of matrices $[v_{ij}^{(n)}]$ converges to a matrix $[v_{ij}]$ if and only if $v_{ij}^{(n)} \to v_{ij}$ in E as $n \to \infty$, for all $i, j \ge 1$. We have continous injections

$$S_\infty^m[E]\,,\, S_1^m[E] \;\hookrightarrow\; M_m[E]\,, \tag{1.48}$$

$$S_\infty[E]\,,\, S_1[E] \;\hookrightarrow\; M_\infty[E]\,. \tag{1.49}$$

Thus $(S_\infty[E], S_1[E])$ (respectively, $(S_\infty^m[E]\,, S_1^m[E])$) is a compatible couple for interpolation.
For $1 < p < \infty$, define for all positive integers m

$$S_p^m[E] \;=\; [\,S_\infty^m[E]\,, S_1^m[E]\,]_{\frac{1}{p}}\,, \tag{1.50}$$

$$S_p[E] \;=\; [\,S_\infty[E]\,, S_1[E]\,]_{\frac{1}{p}}\,. \tag{1.51}$$

We equip $S_p^m[E]$ (respectively, $S_p[E]$) with the operator space structure (1.17) obtained by interpolation. The non-commutative vector-valued Schatten p-classes can be expressed in terms of the Haagerup tensor product. Indeed, by (1.40), for every positive integer m we have the complete isometry $M_m \,=\, C^m \otimes^h R^m$. Then, by (1.46), (1.38), together with the associativity and commutativity properties of the injective tensor product $\check{\otimes}$, we obtain the complete isometries

$$S_\infty^m[E] \,=\, M_m \check{\otimes} E \,=\, (C^m \otimes^h R^m) \check{\otimes} E \,=\, C^m \check{\otimes} (R^m \check{\otimes} E) \,=\, C^m \check{\otimes} (e \check{\otimes} R^m) \,=\, C^m \otimes^h E \otimes^h R^m\,.$$

Respectively, using (1.46) and properties of the projective tensor product $\hat{\otimes}$, we obtain a complete isometry

$$S_1^m[E] \,=\, R^m \otimes^h E \otimes^h C^m \,.$$

Similarly, in the infinite dimensional case we have complete isometries

$$S_\infty[E] \,=\, C \otimes^h E \otimes^h R \quad \text{and} \quad S_1[E] \,=\, R \otimes^h E \otimes^h C \,. \tag{1.52}$$

It is important to point out the fact that, in view of the parallel duality (1.43), the column space in S_1^m (respectively, S_1) has *row* operator space structure and, respectively, the row space in S_1^m (respectively, S_1) has *column* operator space structure. Therefore all the above completely isometric identifications are compatible with the way we identified $(S_\infty^m[E]\,, S_1^m[E])$ (respectively, $(S_\infty[E]\,, S_1[E])$) as a compatible couple (see (1.48) and (1.49)).
For $1 \le p \le \infty$ and all positive integers m, let

$$C_p \;=\; [C\,, R]_{\frac{1}{p}} \quad \text{and} \quad C_p^m \;=\; [C^m\,, R^m]_{\frac{1}{p}}\,, \tag{1.53}$$

$$R_p \;=\; [R\,, C]_{\frac{1}{p}} \quad \text{and} \quad R_p^m \;=\; [R^m\,, C^m]_{\frac{1}{p}}\,. \tag{1.54}$$

Note that we are using a different notation from the one in [70]. Namely, the space C_p is denoted there by $C\left(\frac{1}{p}\right)$ or $C[\,p\,]$ (respectively, C_p^m is denoted by $C_m\left(\frac{1}{p}\right)$ or $C_m[\,p\,]$). Similarly, the space R_p is denoted in [70] by $R\left(\frac{1}{p}\right)$ or $R[\,p\,]$ (respectively, R_p^m is denoted by $R_m\left(\frac{1}{p}\right)$ or $R_m[\,p\,]$).

Theorem 1.3.2 (Pisier [70]) *For $1 \le p \le \infty$ we have a complete isometry*

$$S_p[E] \quad = \quad C_p \otimes^h E \otimes^h R_p \,.$$

Respectively, if m is a positive integer, then, completely isometrically,

$$S_p^m[E] \quad = \quad C_p^m \otimes^h E \otimes^h R_p^m \,.$$

Kouba's interpolation result is the key ingredient in the proof of Theorem 1.3.2. As a consequence, the following duality results holds (under the parallel duality bracket (1.43)).

Corollary 1.3.3 (Pisier [70]) *If $1 < p \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then, completely isometrically,*

$$S_p[E]^* \quad = \quad S_{p'}[E^*] \, .$$

Respectively, for all positive integers m, we have the complete isometry

$$S_p^m[E]^* \quad = \quad S_{p'}^m[E^*] \, .$$

1.4 Operator space structure of non-commutative L_p-spaces

We now turn to the description of the appropriate operator space matrix norms on the non-commutative L_p-spaces. We will use Pisier's interpolation construction (1.17).
The space $L_\infty(\mathcal{M}) = \mathcal{M}$ carries a natural operator space structure, since $\mathcal{M}$ is a C^*-algebra. We want to describe the operator space structure on $L_1(\mathcal{M}, \tau)$ keeping the trace duality pairing (1.1). In order to do this, we have to consider the opposite von Neumann algebra $\mathcal{M}^{\text{op}}$, as explained in [47]. Recall that as a vector space this is simply $\mathcal{M}$, endowed instead with the reversed multiplication $x \circ y = yx$, for all $x, y \in \mathcal{M}$.

Example 1.4.1 Let $\mathcal{M} = \mathbb{B}(l_2(I))$, I an index set. Consider the map

$$\Psi : \mathbb{B}(l_2(I)) \quad \rightarrow \quad \mathbb{B}(l_2(I))^{\text{op}} \, ,$$

defined by $\Psi([x_{ij}]) = [(x^{\text{t}})_{ij}]$, where x^{t} is the transposed matrix of $x = [x_{ij}]$. Then Ψ is a C^*-isomorphism, hence a complete isometry. This induces the natural operator space structure on $\mathbb{B}(l_2(I))^{\text{op}}$.

In particular, given a von Neumann algebra $\mathcal{M} \subseteq \mathbb{B}(H)$ as above (where $H \cong l_2(I)$), we see that the opposite algebra $\mathcal{M}^{\text{op}}$ is a subalgebra of $\mathbb{B}(l_2(I))^{\text{op}}$. Therefore $\mathcal{M}^{\text{op}}$ carries a natural operator space structure, with matrix norms defined for all positive integers m by

$$\|[x_{ij}]\|_{M_m(\mathcal{M}^{\text{op}})} \quad = \quad \|[x_{j\,i}]\|_{M_m(\mathcal{M})} \, . \tag{1.55}$$

Following Junge and Ruan [47], we define the operator space structure on $L_1(\mathcal{M},\tau)$ by

$$L_1(\mathcal{M},\tau) \quad \cong \quad (\mathcal{M}^{\mathrm{op}})_* \,. \tag{1.56}$$

The identification (1.56) is given by the complete isometry

$$x \in L_1(\mathcal{M},\tau) \mapsto \tau_x \in (\mathcal{M}^{\mathrm{op}})_* \,,$$

where $\tau_x(y) = \tau(xy)$, for all $y \in \mathcal{M}^{\mathrm{op}}$. For $1 < p < \infty$ and all positive integers m define

$$M_m(L_p(\mathcal{M},\tau)) \quad = \quad [M_m(\mathcal{M}), M_m(L_1(\mathcal{M},\tau))]_{\frac{1}{p}} \tag{1.57}$$

These matrix norms verify the Ruan axioms (1.12), hence determine the natural operator space structure on $L_p(\mathcal{M},\tau)$.

Proposition 1.4.2 *If* $1 \le p \le \infty$ *and* m *is a positive integer, the following Fubini-type theorem holds isometrically*

$$S_p^m[L_p(\mathcal{M},\tau)] \quad = \quad L_p(M_m \otimes \mathcal{M}, tr \otimes \tau) \,. \tag{1.58}$$

Proof. For all $x = [x_{ij}]_{i,j=1}^m \in S_1^m[L_1(\mathcal{M},\tau)]$ we have

$$\begin{aligned}
\|[x_{ij}]\|_{S_1^m[L_1(\mathcal{M},\tau)]} &= \sup\left\{\sum_{i,j=1}^m \langle x_{ij}, y_{ij}\rangle : \|[y_{ij}]\|_{M_m(\mathcal{M}^{\mathrm{op}})} \le 1\right\} \\
&= \sup\left\{\sum_{i,j=1}^m \tau(x_{ij}y_{ij}) : \|[y_{ij}]\|_{M_m(\mathcal{M}^{\mathrm{op}})} \le 1\right\} \\
&= \sup\left\{\sum_{i,j=1}^m \tau(x_{ij}z_{j\,i}) : \|[z_{ij}]\|_{M_m(\mathcal{M})} \le 1\right\} \\
&= \sup\left\{(\mathrm{tr}\otimes\tau)(x\cdot z) : \|[z_{ij}]\|_{M_m(\mathcal{M})} \le 1\right\} \\
&= \|[x_{ij}]\|_{L_1(M_m\otimes\mathcal{M},\mathrm{tr}\otimes\tau)} \,.
\end{aligned}$$

This proves (1.58) for $p = 1$. For $p = \infty$ the statement is clearly true. By Corollary 1.4 in [70], interpolation yields (1.58) for all $1 \le p \le \infty$. $\square$

Remark 1.4.3 The identity map $I : \mathcal{M} \hookrightarrow L_1(\mathcal{M}, \tau)$ is a complete contraction.

Indeed, we have

$$\|I\|_{cb} \;=\; \sup_m \sup_{\|a\|_{S_2^m}, \|b\|_{S_2^m} \le 1} \| \, M_{ab} \otimes I : M_m(\mathcal{M}) \to S_1^m[\, L_1(\mathcal{M}, \tau) \,] \, \| \, . \tag{1.59}$$

By Proposition 1.4.2, $S_1^m[L_1(\mathcal{M}, \tau)] = L_1(M_m \otimes \mathcal{M}, \mathrm{tr} \otimes \tau)$. Thus, for $x \in M_m(\mathcal{M})$ we have

$$\begin{aligned}
\|(a \otimes 1) \cdot x \cdot (b \otimes 1)\|_{L_1(M_m \otimes \mathcal{M}, \mathrm{tr} \otimes \tau)} \;&\le\; \|a \otimes 1\|_{L_2(M_m \otimes \mathcal{M})} \|x\|_{M_m(\mathcal{M})} \|b \otimes 1\|_{L_2(M_m \otimes \mathcal{M})} \\
&=\; ((\mathrm{tr} \otimes \tau)(a^* a \otimes 1))^{\frac{1}{2}} \|x\|_{M_m(\mathcal{M})} ((\mathrm{tr} \otimes \tau)(b^* b \otimes 1))^{\frac{1}{2}} \\
&=\; (\mathrm{tr}(a^* a))^{\frac{1}{2}} \|x\|_{M_m(\mathcal{M})} (\mathrm{tr}(b^* b))^{\frac{1}{2}} \tau(1) \\
&=\; \|a\|_{S_2^m} \|x\|_{M_m(\mathcal{M})} \|b\|_{S_2^m} \tau(1) \\
&\le\; \|x\|_{M_m(\mathcal{M})} \, ,
\end{aligned}$$

for all $a, b \in S_2^m$ with $\|a\|_{S_2^m}, \|b\|_{S_2^m} \le 1$. Using (1.59) we obtain the conclusion.

Recall that if $(E, \| \cdot \|)$ is a Banach space, then we denote by $\overline{E}$ the complex conjugate of E, i.e., the vector space E equipped with the same norm $\| \cdot \|$, but with the conjugate multiplication by complex scalars. Thus $\overline{E}$ is anti-linear isometric to E, via the map

$$E \ni x \;\; \mapsto \bar{x} \in \overline{E} \, .$$

If, moreover, E is equipped with operator space matrix norms, then, as explained in [71], the conjugate $\overline{E}$ is an operator space, with matrix norms defined for all positive integers m and all matrices $[v_{ij}] \in M_m(E)$ by

$$\| \, [\, \overline{v_{ij}} \,] \, \|_{M_m(\overline{E})} \;=\; \| \, [v_{ij}] \, \|_{M_m(E)} \, . \tag{1.60}$$

Remark 1.4.4 If A is a C^*-algebra, then, as observed in [70], the involution $x \mapsto x^*$ is an anti-linear isomorphism between A and A^{op}, hence a linear one between $\overline{A}$ and A^{op}. Thus A^{op} and $\overline{A}$ are C^*-isomorphic, thus completely isometric.

Lemma 1.4.5 *Let E be an operator space. If $1 \leq p \leq \infty$ and m is a positive integer, then, completely isometrically,*

$$S_p^m[\,\overline{E}\,] \quad = \quad \overline{S_p^m[E]}\,. \tag{1.61}$$

Proof. Let $[v_{ij}] \in M_m(E)$. By Theorem 1.5 of [70] we have

$$\|\,[\,\overline{v_{ij}}\,]\,\|_{S_p^m[\,\overline{E}\,]} \quad = \quad \inf\{\|a\|_{S_{2p}^m}\|\,[\,\overline{v_{ij}}\,]\,\|_{M_m(\overline{E})}\|b\|_{S_{2p}^m}\}\,,$$

where the infimum is taken over all representations of the form $[\,\overline{v_{ij}}\,] = a \cdot [\,\overline{v_{ij}}\,] \cdot b$ with $a\,, b \in S_{2p}^m$ such that $\|a\|\,, \|b\|_{S_{2p}^m} \leq 1$, and $[\,\overline{v_{ij}}\,] \in M_m(\overline{E})$. Applying (1.60) we obtain the conclusion. □

Theorem 1.4.6 *Let $1 < p \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then, completely isometrically,*

$$L_p(\mathcal{M}, \tau) \quad = \quad \overline{L_{p'}(\mathcal{M}, \tau)}^{\;*}\,. \tag{1.62}$$

Proof. By Lemma 1.7 of [70], given operator spaces $V_0\,, V_1$, a map $u : V_0 \to V_1$ is completely bounded if and only if for some $1 \leq p \leq \infty$ we have $\sup_m \|\,\mathrm{Id}_{S_p^m} \otimes u : S_p^m[V_0] \to S_p^m[V_1]\,\| < \infty$, in which case, moreover,

$$\|u\|_{cb} \quad = \quad \sup_m \|\,\mathrm{Id}_{S_p^m} \otimes u : S_p^m[V_0] \to S_p^m[V_1]\,\|\,. \tag{1.63}$$

Futhermore, by Corollary 1.8 of [70], for all positive integers m we have the complete isometry

$$S_p^m[\,\overline{L_{p'}(\mathcal{M}, \tau)}^{\;*}] \quad = \quad \left(S_{p'}^m[\,\overline{L_{p'}(\mathcal{M}, \tau)}\,]\right)^*\,. \tag{1.64}$$

Therefore, it suffices to show that for all positive integers m we have the isometry

$$S_p^m[\,\overline{L_{p'}(\mathcal{M},\tau)}^{\;*}] \;\;=\;\; S_p^m[\,L_p(\mathcal{M},\tau)\,]\,. \tag{1.65}$$

Recall that the pairing

$$\langle x, y\rangle \;\;=\;\; \tau(y^* x)\,, \tag{1.66}$$

where $x \in L_p(\mathcal{M},\tau)$ and $y \in \overline{L_{p'}(\mathcal{M},\tau)}$ defines an (anti-linear) duality bracket between $L_p(\mathcal{M},\tau)$ and $\overline{L_{p'}(\mathcal{M},\tau)}$. Hence, by Lemma 1.4.5 and Proposition 1.4.2 we obtain the following norm estimates for $x = [x_{ij}] \in S_p^m[\,\overline{L_{p'}(\mathcal{M},\tau)}^{\;*}]$

$$\begin{aligned}
\|\,[x_{ij}]\,\|_{S_p^m[\,\overline{L_{p'}(\mathcal{M},\tau)}^{\;*}]} \;&=\; \sup\{\sum_{i,j=1}^m \langle x_{ij}\,, y_{ij}\rangle \;:\; \|\,[y_{ij}\,]\|_{S_{p'}^m[\,\overline{L_{p'}(\mathcal{M},\tau)}\,]} \;\le\; 1\}\\
&=\; \sup\{\sum_{i,j=1}^m \tau(x_{ij}y_{ij}^*) \;:\; \|\,[y_{ij}]\,\|_{S_{p'}^m[\,L_{p'}(\mathcal{M},\tau)\,]} \;\le\; 1\}\\
&=\; \sup\{\sum_{i,j=1}^m \tau(x_{ij}z_{ji}) \;:\; \|\,z\,\|_{L_{p'}(M_m\otimes\mathcal{M},\mathrm{tr}\otimes\tau)} \;\le\; 1\}\\
&=\; \sup\{(\mathrm{tr}\otimes\tau)(xz) \;:\; \|\,z\,\|_{L_{p'}(M_m\otimes\mathcal{M},\mathrm{tr}\otimes\tau)} \;\le\; 1\}\\
&=\; \|\,[x_{ij}]\,\|_{L_p(M_m\otimes\mathcal{M},\mathrm{tr}\otimes\tau)}\\
&=\; \|\,[x_{ij}]\,\|_{S_p^m[\,L_p(\mathcal{M},\tau)\,]}\,,
\end{aligned}$$

which yields the conclusion. $\square$

Proposition 1.4.7 *Let* $1 \le p \le \infty$. *We have a completely isometric isomorphism*

$$\overline{L_p(\mathcal{M},\tau)} \;\;=\;\; L_p(\mathcal{M}^{\mathrm{op}},\tau)\,. \tag{1.67}$$

Proof. The proof will be done using interpolation. Note that if (V_0, V_1) is a compatible couple of Banach spaces and $0 < \theta < 1$, then

$$[\overline{V}_0, \overline{V}_1]_\theta \;\;=\;\; \overline{[V_0, V_1]}_\theta\,. \tag{1.68}$$

The key point in the proof of (2.24) is the invariance of the strip S under conjugation, i.e., if $z \in S$, then $\bar{z} \in S$. By (1.57), (1.56) and (2.24) we have the complete isometry

$$\begin{aligned} \overline{L_p(\mathcal{M},\tau)} &= \overline{[\,\mathcal{M}, L_1(\mathcal{M},\tau)\,]_{\frac{1}{p}}} \\ &= [\,\overline{\mathcal{M}}, \overline{L_1(\mathcal{M},\tau)}\,]_{\frac{1}{p}}\,. \end{aligned} \tag{1.69}$$

On the other hand, we also have the complete isometry

$$L_p(\mathcal{M}^{\mathrm{op}},\tau) = [\,\mathcal{M}^{\mathrm{op}}, L_1(\mathcal{M}^{\mathrm{op}},\tau)\,]_{\frac{1}{p}} \tag{1.70}$$

By Remark 1.4.4, we have a complete isometry $\phi : \overline{\mathcal{M}} \to \mathcal{M}^{\mathrm{op}}$, given by

$$\overline{\mathcal{M}} \ni x \;\mapsto\; x^* \in \mathcal{M}^{\mathrm{op}}\,.$$

We will show that ϕ extends to a complete isometry $\phi : \overline{L_1(\mathcal{M},\tau)} \to L_1(\mathcal{M}^{\mathrm{op}},\tau)$. Let $x = [x_{ij}] \in S_1^m[\overline{L_1(\mathcal{M},\tau)}]$. Then, by (1.60) and Proposition 1.4.2 we obtain

$$\begin{aligned} \|\,[x_{ij}]\,\|_{S_1^m[\overline{L_1(\mathcal{M},\tau}]} &= \|\,[x_{ij}\,]\|_{S_1^m[L_1(\mathcal{M},\tau)]} \\ &= \|\,[x_{ij}]\,\|_{L_1(M_m\otimes\mathcal{M},\mathrm{tr}\otimes\tau)}\,. \end{aligned} \tag{1.71}$$

Respectively, by (1.55) and Proposition 1.4.2 we obtain

$$\begin{aligned} \|\,[x_{ij}^*]\,\|_{S_1^m[L_1(\mathcal{M}^{\mathrm{op}},\tau)]} &= \|\,[x_{ji}^*]\,\|_{S_1^m[L_1(\mathcal{M},\tau)]} \\ &= \|\,[x^*]\,\|_{L_1(M_m\otimes\mathcal{M},\mathrm{tr}\otimes\tau)}\,. \end{aligned} \tag{1.72}$$

Thus, by (1.72) and (1.73), it follows that

$$\|x\|_{S_1^m[\overline{L_1(\mathcal{M},\tau)}]} = \|x^*\|_{S_1^m[L_1(\mathcal{M}^{\mathrm{op}},\tau)]}\,.$$

Thus, we have the following diagram of complete contractions, where the top and bottom are compatible maps, i.e., they coincide on the intersection $\overline{\mathcal{M}} \cap \overline{L_1(\mathcal{M},\tau)} = \overline{\mathcal{M}}$, and, respectively,

$\mathcal{M}^{\mathrm{op}} \cap L_1(\mathcal{M}^{\mathrm{op}}, \tau) = \mathcal{M}^{\mathrm{op}}$.

$$\begin{tikzcd}
\overline{\mathcal{M}} \arrow[r, "\phi"] \arrow[d, "I_{\mathcal{M}}"] & \mathcal{M}^{\mathrm{op}} \arrow[d, "I_{\mathcal{M}^{\mathrm{op}}}"] \\
\overline{L_1(\mathcal{M}, \tau)} \arrow[r, "\phi"] & L_1(\mathcal{M}^{\mathrm{op}}, \tau)
\end{tikzcd}$$

Hence, the conclusion follows by interpolation from (1.70) and (1.70). □

1.5 Non-commutative vector-valued L_p-spaces (The continuous case)

Let $(\mathcal{M}, \tau)$ be a *hyperfinite* von Neumann algebra, i.e., $\mathcal{M}$ is the w^*-closure of an increasing net $(\mathcal{M}_n)_{n\geq 1}$ of *finite dimensional* von Neumann subalgebras, equipped with an n.f. tracial state τ . A celebrated theorem of Connes [17] establishes the connection between hyperfiniteness and another very important concept in operator algebras, namely *injectivity.* Recall that a C^*-algebra A is called *injective* if and only if given C^*-algebras B and B_1 such that $B \subseteq B_1$ and a map $\phi : B \to A$, then there exists a completely positive map $\phi_1 : B_1 \to A$ such that $\phi_1 \restriction B = \phi$.

$$\begin{tikzcd}
B_1 \arrow[dr, "\exists\ \phi_1\ \ \text{c.p.}"] & \\
B \arrow[u, hook] \arrow[r, "\phi\ \ \text{c.p.}"'] & A
\end{tikzcd}$$

A von Neumann algebra is called *injective* if it is injective as a C^*-algebra.

Theorem 1.5.1 (Connes, [17]) *A von Neumann algebra $\mathcal{M}$ is hyperfinite if and only if it is injective.*

The following characterization of injective von Neumann algebras (see Torpe [83]) is very useful in applications.

Proposition 1.5.2 *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra. The following are equivalent:*

(i) $\mathcal{M}$ *is injective.*

(ii) There exists a norm 1 *projection* $E : \mathcal{B}(H) \to \mathcal{M}$ *, which is onto.*

Let us recall a few examples of injective von Neumann algebras. We refer to [83] for proofs.

1. For every Hilbert space H , the algebra $\mathcal{B}(H)$ is injective. In particular, for all positive integers m , the matrix algebras M_m are injective.

2. $L_\infty(\Omega, \mu)$ is an injective von Neumann algebra, where (Ω, μ) is a measure space.

3. If G is an amenable group, then the group von Neumann algebra $VN(G)$ is injective.

4. If $\mathcal{M}$ is an injective von Neumann algebra and e is a projection in $\mathcal{M}$, then $e\mathcal{M}e$ is an injective von Neumann algebra.

5. If R and S are injective von Neumann algebras, then the von Neumann algebra tensor product $R\bar{\otimes}S$ is injective.

We now return to the discussion of the vector-valued non-commutative L_p-spaces associated to $(\mathcal{M}, \tau)$ and an operator space E . The construction is due to Pisier [70] and it is done by interpolation. Following Pisier [70], define

$$L_1(\mathcal{M}; E) \quad = \quad L_1(\mathcal{M}, \tau)\hat{\otimes}E \,, \tag{1.73}$$

where the operator space structure of $L_1(\mathcal{M}, \tau)$ is given by (1.56).

The key point of the construction is the fact that, by the assumptions on the von Neumann algebra $\mathcal{M}$, it follows that $(\mathcal{M}\check{\otimes}E \,, L_1(\mathcal{M}; E))$ forms a compatible couple for interpolation. Indeed, since M is a finite and hyperfinite von Neumann algebra, it follows by results of Effros and Ruan (see [27] and [28]) that the canonical inclusion

$$v : \mathcal{M} \to (\mathcal{M}^{\mathrm{op}})_* = L_1(\mathcal{M}, \tau)$$

is integral. Then, as explained in [70], the map $v \otimes \mathrm{Id}_E$ extends to a complete contraction

$$\tilde{v} : \mathcal{M}\check{\otimes}E \quad \hookrightarrow \quad L_1(\mathcal{M},\tau)\hat{\otimes}E\,. \tag{1.74}$$

For $1 < p < \infty$, define

$$L_p(\mathcal{M};E) \quad = \quad [\,\mathcal{M}\check{\otimes}E, L_1(\mathcal{M};E)\,]_{\frac{1}{p}}\,, \tag{1.75}$$

and equip $L_p(\mathcal{M};E)$ with the operator space matrix norms (1.17) obtained by interpolation. As a consequence of Proposition 1.4.2 we obtain the following Fubini theorem (see (3.6)' of [70]).

Theorem 1.5.3 *If $1 \le p \le \infty$, then, for all von Neumann algebras $\mathcal{N}$, equipped with an n.f. tracial state ϕ, we have the complete isometry*

$$L_p(\mathcal{M};L_p(\mathcal{N},\phi)) \quad = \quad L_p(\mathcal{M}\otimes\mathcal{N},\tau\otimes\phi)\,. \tag{1.76}$$

Let $(\mathcal{M}_n)_{n\geq 1}$ be a filtration of the hyperfinite (and finite) von Neumann algebra $\mathcal{M}$. For every positive integer n, the finiteness of $\mathcal{M}$ ensures the existence of a unique normal, trace-preserving conditional expectation $\mathcal{E}_n : \mathcal{M} \to \mathcal{M}_n$. This is a norm 1 projection onto $\mathcal{M}_n$. In particular, it follows by Proposition 1.5.2 and Theorem 1.5.1 that the von Neumann algebra $\mathcal{M}_n$ is hyperfinite, as well. Furthermore, as explained in the preliminaries, for $1 \le p < \infty$, the map $\mathcal{E}_n$ extends to a norm 1 projection $\mathcal{E}_n : L_p(\mathcal{M},\tau) \to L_p(\mathcal{M},\tau_n)$, where $\tau_n = \tau \restriction \mathcal{M}_n$, satisfying the modular property (1.10).

Proposition 1.5.4 *Let $1 \le p \le \infty$. Then, for all positive integers n, the conditional expectation $\mathcal{E}_n : L_p(\mathcal{M},\tau) \to L_p(\mathcal{M}_n,\tau_n)$ extends to a complete contraction*

$$\mathcal{E}_n \otimes \mathrm{Id}_E : L_p(\mathcal{M};E) \quad \to \quad L_p(\mathcal{M}_n;E)\,. \tag{1.77}$$

Proof. We first show that we have the complete contraction

$$\mathcal{E}_n \otimes \mathrm{Id}_E : \mathcal{M}\check{\otimes}E \quad \to \quad \mathcal{M}_n\check{\otimes}E \tag{1.78}$$

By Proposition 1.2.5, it suffices to prove that the map $\mathcal{E}_n : \mathcal{M} \to \mathcal{M}_n$ is a complete contraction. Since $\|\mathcal{E}_n\| = 1$, this is equivalent to showing that $\mathcal{E}_n$ is completely positive (see Paulsen [63]). This is, indeed, the case, since for all positive integers m, the map

$$\mathrm{Id}_{M_m} \otimes \mathcal{E}_n : M_m \otimes \mathcal{M} \;\to\; M_m \otimes \mathcal{M}_m$$

is the unique trace-preserving conditional expectation onto $M_m \otimes \mathcal{M}_m$, hence it is positive.
We now show that we have the complete contraction

$$\mathcal{E}_n \otimes \mathrm{Id}_E : L_1(\mathcal{M}, \tau)\hat{\otimes} E \;\;\to\;\; L_1(\mathcal{M}_n, \tau_n)\hat{\otimes} E \tag{1.79}$$

By Proposition 1.2.3, it suffices to prove that the map $\mathcal{E}_n : L_1(\mathcal{M}, \tau) \to L_1(\mathcal{M}_n, \tau_n)$ is a complete contraction. Let us denote for the moment this map by u_n, in order to avoid confusion. We will show that the dual map u_n^* is exactly $\mathcal{E}_n : \mathcal{M} \to M_n$, under the trace duality bracket (1.1). For this, we need to prove that for $x \in L_1(\mathcal{M}, \tau)$ and $y \in L_1(\mathcal{M}_n, \tau_n)$ we have

$$\langle u_n(x)\,, y\rangle \;\;=\;\; \langle x\,, \mathcal{E}_n(y)\rangle\,. \tag{1.80}$$

Indeed, using the density of $\mathcal{M}$ in $L_1(\mathcal{M}, \tau)$, and the properties of $\mathcal{E}_n$ as a conditional expectation, we obtain

$$\langle u_n(x)\,, y\rangle \;=\; \tau(u_n(x)y) \;=\; \tau(\mathcal{E}_n(x)y) \;=\; \tau(x\mathcal{E}_n(y))\,.$$

Thus (1.80) is proved. Since $\mathcal{E}_n$ is completely contractive, as shown above, it follows that

$$\|u_n\|_{cb} \;=\; \|u_n^*\|_{cb} \;=\; \|\mathcal{E}_n\|_{cb} \;\leq\; 1\,,$$

i.e., $\mathcal{E}_n : L_1(\mathcal{M}, \tau) \to L_1(\mathcal{M}_n, \tau_n)$ is a complete contraction, as wanted.
For $1 < p < \infty$ we obtain (1.77) by interpolation of exponent $\theta = \frac{1}{p}$ from (1.78) and (1.79), using the definition formula (1.75). □

Remark 1.5.5 In view of Proposition 1.5.4 we can consider vector-valued non-commutative martingales in this setting. Note that, as in the scalar-valued case, any $L_p(\mathcal{M};E)$-martingale with respect to the filtration $(\mathcal{M}_n)_{n\geq 1}$ can be approximated by finite $L_p(\mathcal{M};E)$-martingales (with respect to the same filtration). Therefore, it suffices to consider finite martingales only.

We now discuss duality in the vector-valued setting. We will show that all the duality results hold under trace duality. This is therefore consistent with the scalar-valued case.

Proposition 1.5.6 *Let $1 \leq p < \infty$ and p' be the conjugate exponent of p. If $\mathcal{M}$ is a finite dimensional von Neumann algebra equipped with an n.f. tracial state τ, then, completely isometrically,*

$$(\,L_{p'}(\mathcal{M};E)\,)^* \quad = \quad L_p(\mathcal{M}^{\mathrm{op}};E^*) \tag{1.81}$$

Proof. Since $\mathcal{M}$ is finite dimensional, it follows that

$$L_{p'}(\mathcal{M};E) \quad = \quad S_{p'}^{m_1} \oplus_{p'} \ldots \oplus_{p'} S_{p'}^{m_k}\,,$$

for some positive integers $m_1\,, \ldots m_k$. For simplicity of the argument, we will assume that

$$L_{p'}(\mathcal{M};E) \quad = \quad S_{p'}^{m}[E]\,,$$

where m is a positive integer. If $[x_{ij}]_{i,j=1}^m \in (L_{p'}(\mathcal{M};E))^*$, then

$$\begin{aligned}
\|\,[x_{ij}]\,\|_{(L_{p'}(\mathcal{M};E))^*} \quad &= \quad \sup\{\,|\,\langle\,[x_{ij}]\,,[y_{ij}]\,\rangle\,|\,:\,\|\,[y_{ij}]\,\|_{L_{p'}(\mathcal{M};E)} \leq 1\,\} \\
&= \quad \sup\left\{\,\left|\,\sum_{i,j=1}^{m} x_{ij}^*(y_{j\,i})\,\right|\,:\,\|\,[y_{j\,i}]\,\|_{L_{p'}(\mathcal{M};E)} \leq 1\,\right\} \\
&= \quad \|\,[x_{j\,i}]\,\|_{S_p^m[E^*]} \\
&= \quad \|\,[x_{ij}]\,\|_{L_p(\mathcal{M}^{\mathrm{op}};E^*)}\,,
\end{aligned} \tag{1.82}$$

where the last equality follows from the following considerations. By (1.55), the map $\psi_m : \mathcal{M} \to \mathcal{M}^{\mathrm{op}}$, defined by

$$[v_{ij}]_{i,j=1}^m \;\mapsto\; [v_{j\,i}]_{i,j=1}^m$$

is a complete isometry. Moreover, as shown in the proof of Proposition 1.4.7, the map ψ_m extends to a complete isometry $\psi_m : L_1(\mathcal{M},\tau) \to L_1(\mathcal{M}^{\mathrm{op}},\tau)$. Therefore, by Proposition 1.2.5 and Proposition 1.2.3 it follows that $\psi_m \otimes \mathrm{Id}_{E^*}$ extends, respectively, to complete isometries

$$\mathcal{M}\check{\otimes}E^* \overset{\psi_m\otimes\mathrm{Id}_{E^*}}{\longrightarrow} \mathcal{M}^{\mathrm{op}}\check{\otimes}E^*\,,$$

$$L_1(\mathcal{M},\tau)\hat{\otimes}E^* \overset{\psi_m\otimes\mathrm{Id}_{E^*}}{\longrightarrow} L_1(\mathcal{M}^{\mathrm{op}},\tau)\hat{\otimes}E^*\,.$$

By formula (1.75), interpolation with exponent $\theta = \frac{1}{p}$ shows that the map

$$\psi_m \otimes \mathrm{Id}_{E^*} : L_p(\mathcal{M};E^*) \;\to L_p(\mathcal{M}^{\mathrm{op}};E^*)$$

is a complete isometry. This completes the proof of (1.82). ☐

Furthermore, following the same argument as in Theorem 4.1 of [70] of reducing the infinite dimensional case to the finite dimensional one and using Proposition 1.5.6, we obtain the following

Theorem 1.5.7 *Let $1 \le p < \infty$ and p' be the conjugate exponent of p. We have a completely isometric embedding*

$$L_p(\mathcal{M}^{\mathrm{op}};E^*) \;\; \hookrightarrow (L_{p'}(\mathcal{M};E))^*\,. \tag{1.83}$$

Recall that the space $\Lambda_p(E)$ is defined in [70] as the completion of $\mathcal{M}\otimes E$ with respect to the norm

$$\|x\|_{\Lambda_p(E)} \;\; = \;\; \inf\{\|a\|_{L_{2p}(\mathcal{M},\tau)}\|y\|_{\mathcal{M}\check{\otimes}E}\|b\|_{L_{2p}(\mathcal{M},\tau)}\}\,,$$

where the infimum runs over all representations $x = a\cdot y\cdot b$, with $a\,,b \in L_{2p}(\mathcal{M},\tau)\,, y \in \mathcal{M}\check{\otimes}E$. If $\mathcal{M}$ is hyperfinite, then by Theorem 3.8 of [70], we have an isometric identity

$$\Lambda_p(E) \;\; = \;\; L_p(\mathcal{M};E)\,.$$

Proposition 1.5.8 *$x \in \Lambda_p(E)$ if and only if $x = a\cdot y\cdot b$, where $a\,,b \in L_{2p}(\mathcal{M},\tau)\,, y \in \mathcal{M}\check{\otimes}E$.*

The proof of this result will appear elsewhere.

Chapter 2

Interpolation Between Non-commutative BMO and Non-commutative L_p-Spaces

2.1 Some background and motivation

The space BMO, introduced by John and Nirenberg [44], and characterized by Fefferman [32] as the dual of the Hardy space H_1, plays an important role in interpolation theory. It is a natural substitute for L_∞, since many classical operators such as martingale transforms and singular integral operators (e.g., the Hilbert transform) carry L_∞ to BMO boundedly (see Garsia [35], and Garcia-Cuerva and Rubio de Francia [34]). Therefore, interpolation results involving BMO as an end-point are useful tools, and they have been considered by many authors; see, for example, Fefferman and Stein [33], Janson and Jones [43], Stroock [79], and Stroock and Varadhan [80].

This chapter is concerned with the study of non-commutative BMO and its interpolation properties. We will prove non-commutative analogues of the classical interpolation results between BMO and L_p-spaces (respectively, Hardy spaces).

Let $\mathcal{M}$ be a von Neumann algebra equipped with a faithful, normal, tracial state τ, and $(\mathcal{M}_n)_{n\geq 1}$ an increasing sequence of von Neumann subalgebras of $\mathcal{M}$, whose union generates $\mathcal{M}$ in the w*-topology. The non-commutative BMO space associated to $(\mathcal{M}, \tau)$ and to the filtration $(\mathcal{M}_n)_{n\geq 1}$

is defined in [69] as the intersection space $BMO(\mathcal{M}) = BMO_c(\mathcal{M}) \cap BMO_r(\mathcal{M})$, where

$$\begin{aligned} BMO_c(\mathcal{M}) &= \{x \in L_2(\mathcal{M}) : \|x\|_{BMO_c(\mathcal{M})} = \sup_n \|\mathcal{E}_n(|x - \mathcal{E}_{n-1}(x)|^2)\|_\infty^{1/2} < \infty\}\,, \\ BMO_r(\mathcal{M}) &= \{x \in L_2(\mathcal{M}) : \|x\|_{BMO_r(\mathcal{M})} = \|x^*\|_{BMO_c(\mathcal{M})} < \infty\}\,. \end{aligned}$$

The analogue of the classical Fefferman-Stein duality $BMO = (H_1)^*$ holds in this setting (see the Appendix in [69]).

We are now ready to state the main result of this chapter. If $1 \leq p < q < \infty$, then

$$[BMO(\mathcal{M}), L_p(\mathcal{M})]_{\frac{p}{q}} = L_q(\mathcal{M})\,. \tag{2.1}$$

We also obtain, as in the classical setting, the corresponding variations of this result, i.e., if X is either $L_\infty(\mathcal{M})$ or $BMO(\mathcal{M})$ and Y is either $L_1(\mathcal{M})$ or $H_1(\mathcal{M})$, then

$$[X, Y]_{\frac{1}{p}} = L_p(\mathcal{M})\,. \tag{2.2}$$

Based on the main inequality in the proof of (2.1) (and a preliminary version of this paper), Junge and Xu [48] discovered the following estimate for the order of constants in the noncommutative Burkholder-Gundy inequalities (1.11): namely, for $1 \leq p \leq 2$ there exists an absolute constant $\beta > 0$ such that for all $x \in H_p(\mathcal{M})$,

$$\|x\|_{L_p(\mathcal{M})} \leq \beta \|x\|_{H_p(\mathcal{M})}\,.$$

2.2 Main results

Let $(\mathcal{M}, \tau)$ be a von Neumann algebra equipped with a normal, faithful tracial state τ. Let $(\mathcal{M}_n)_{n\geq 1}$ be an increasing filtration of von Neumann subalgebras of $\mathcal{M}$ which generates $\mathcal{M}$ in the w*-topology. The von Neumann algebra $(\mathcal{M}, \tau)$ and the filtration $(\mathcal{M}_n)_{n\geq 1}$ will remain fixed throughout. Define $H_p^c(\mathcal{M})$ (respectively, $H_p^r(\mathcal{M})$) to be the space of all $L_p(\mathcal{M})$-martingales x such that $dx \in L_p(\mathcal{M}, l_2^c)$ (respectively, $dx \in L_p(\mathcal{M}, l_2^r)$) and set

$$\|x\|_{H_p^c(\mathcal{M})} = \|dx\|_{L_p(\mathcal{M}, l_2^c)} \qquad \text{and} \qquad \|x\|_{H_p^r(\mathcal{M})} = \|dx\|_{L_p(\mathcal{M}, l_2^r)} . \tag{2.3}$$

Equipped respectively with these norms, $H_p^c(\mathcal{M})$ and $H_p^r(\mathcal{M})$ are Banach spaces. As already explained in Section 1.1, the non-commutative martingale Hardy spaces associated to $(\mathcal{M}, \tau)$ and to the filtration $(\mathcal{M}_n)_{n\geq 1}$ are defined as

$$\begin{aligned} H_p(\mathcal{M}) &= H_p^c(\mathcal{M}) \cap H_p^r(\mathcal{M}) \quad \text{if} \ \ 2 \leq p < \infty , \\ H_p(\mathcal{M}) &= H_p^c(\mathcal{M}) + H_p^r(\mathcal{M}) \quad \text{if} \ \ 1 \leq p < 2 . \end{aligned} \tag{2.4}$$

Our first result in this section is concerned with complex interpolation between the column spaces $BMO_c(\mathcal{M})$ and $H_p^c(\mathcal{M})$.

Theorem 2.2.1 *Let $1 \leq p < q < \infty$. Then, the following holds with equivalent norms,*

$$[BMO_c(\mathcal{M}), H_p^c(\mathcal{M})]_{\frac{p}{q}} \quad = \quad H_q^c(\mathcal{M}) . \tag{2.5}$$

The main step in the proof of (2.5) is the case $2 < p < q < \infty$. To obtain all the other cases, we will use Wolff's interpolation theorem (see Wolff [87]). This result states that given Banach spaces A_i $(i = 1, 2, 3, 4)$ such that $A_1 \cap A_4$ is dense in both A_2 and A_3, and

$$A_2 = [A_1, A_3]_\theta \quad \text{and} \quad A_3 = [A_2, A_4]_\phi ,$$

for some $0 < \theta, \phi < 1$, then

$$A_2 = [A_1, A_4]_\zeta \quad \text{and} \quad A_3 = [A_1, A_4]_\xi , \tag{2.6}$$

where $\zeta = \frac{\theta\phi}{1-\theta+\theta\phi}$ and $\xi = \frac{\phi}{1-\theta+\theta\phi}$.

Lemma 2.2.2 *If (2.5) holds for p and q in the range $2 < p < q < \infty$, then it holds for all $1 \leq p < q < \infty$.*

Proof. The proof is based on successive applications of Wolff's interpolation result.

<u>Case 1</u>: $1 < p < q < \infty$. Let us first assume that $2 < q < \infty$. Let $0 < \theta < 1$ and $s = \frac{q}{\theta}$. Note that $p < q < s$, so there exists $0 < \phi < 1$ such that $\frac{1-\phi}{s} + \frac{\phi}{p} = \frac{1}{q}$. Recall that for $1 < v < \infty$ the space $H_v^c(\mathcal{M})$ is complemented in $L_v(\mathcal{M}; l_2^c)$ via Stein's projection (see [69],Theorem 2.3). This allows us to use (1.9) and conclude that

$$A_3 = H_q^c(\mathcal{M}) = [H_s^c(\mathcal{M}), H_p^c(\mathcal{M})]_\phi = [A_2, A_4]_\phi \,,$$

Furthermore, by the assumption (recall that $q > 2$) we get

$$A_2 = H_s^c(\mathcal{M}) = [BMO_c(\mathcal{M}), H_q^c(\mathcal{M})]_\theta = [A_1, A_3]_\theta \,.$$

By Wolff's interpolation theorem (2.6), it follows that

$$A_3 = [A_1, A_4]_\xi = [BMO_c(\mathcal{M}), H_p^c(\mathcal{M})]_\xi = H_q^c(\mathcal{M}) \,,$$

where $\xi = \frac{\phi}{1-\theta+\phi\theta}$. A simple computation shows that $\xi = \frac{p}{q}$. A further application of Wolff's theorem yields the conclusion in the full range $1 < p < q < \infty$.

<u>Case 2</u>: $1 = p < q < \infty$. Let $s > \max\{q, 2\}$. Since $1 < q < s < \infty$, there exists $0 < \theta < 1$ so that $\frac{1-\theta}{s} + \frac{\theta}{1} = \frac{1}{q}$. By Case 1 and duality ([3], Theorem 4.3.1), it follows that

$$A_3 = H_q^c(\mathcal{M}) = [H_s^c(\mathcal{M}), H_1^c(\mathcal{M})]_\theta = [A_2, A_4]_\theta \,,$$

Moreover, if $\phi = \frac{q}{s}$, then by Case 1 (recall that $s > 2$) we also get

$$A_2 = H_s^c = [BMO_c(\mathcal{M}), H_q^c(\mathcal{M})]_\phi = [A_1, A_3]_\phi \,,$$

From Wolff's result (2.6), we deduce that

$$A_3 = [A_1, A_4]_\xi = [BMO_c(\mathcal{M}), H_1^c(\mathcal{M})]_\xi = H_q^c(\mathcal{M}) \,,$$

where $\xi = \frac{\phi}{1-\theta+\phi\theta}$. A simple calculation shows that $\xi = \frac{1}{q}$.

Note that in both cases, the density of the space $A_1 \cap A_4$ in A_2 and, respectively, A_3, required in the hypothesis of Wolff's theorem, is ensured by Remark 2.2.3 below. $\square$

The proof of Theorem 2.2.1 in the range $2 < p < q < \infty$ relies on the interpolation properties of certain norms introduced in [48]. Recall that for $2 < p \leq \infty$, $L_p^c MO(\mathcal{M})$ (MO stands for *mean oscillation*) is defined in [48] as the space of all martingale difference sequences $(dx_n)_{n\geq 1} \cong x$ in $L_2(\mathcal{M}, \tau)$ such that

$$\|x\|_{L_p^c MO(\mathcal{M})} \quad = \quad \sup_m \| \sup_{n\leq m} \mathcal{E}_n(\sum_{k=n}^m dx_k^* dx_k)\|_{\frac{p}{2}}^{1/2} < \infty \,.$$

Note that this can be interpreted as a non-commutative analogue of the p-norm of the classical sharp function (see, e.g., Garsia [35]). This definition requires some explanation. In the non-commutative setting there is no obvious analogue for the pointwise supremum of a sequence of positive operators. Therefore, the above is to be understood in the sense of the suggestive notation introduced in [45]. Namely, if $1 \leq q, q' \leq \infty$ with $\frac{1}{q} + \frac{1}{q'} = 1$ then, for a sequence $(x_n)_{n\geq 1}$ of τ-measurable positive operators affiliated with $\mathcal{M}$, set

$$\| \sup_n x_n \|_q \quad = \quad \sup \left\{ \sum_{n\geq 1} \tau(x_n y_n) : \; y_n \geq 0, \left\| \sum_{n\geq 1} y_n \right\|_{q'} \leq 1 \right\} . \tag{2.7}$$

Observe that in the commutative case this coincides with the usual definition of the norm in the l_∞-valued L_q-space. It turns out that $(L_p^c MO(\mathcal{M}), \|\cdot\|_{L_p^c MO(\mathcal{M})})$ is a Banach space. Similarly, the row space $L_p^r MO(\mathcal{M})$ is defined in [48] as the space of all martingales x such that $x^* \in L_p^c MO(\mathcal{M})$, with norm given by

$$\|x\|_{L_p^r MO(\mathcal{M})} \quad = \quad \|x^*\|_{L_p^c MO(\mathcal{M})} \,.$$

Finally, let $L_p MO(\mathcal{M}) = L_p^c MO(\mathcal{M}) \cap L_p^r MO(\mathcal{M})$, equipped with the intersection norm

$$\|x\|_{L_p MO(\mathcal{M})} \quad = \quad \max\{\|x\|_{L_p^c MO(\mathcal{M})}, \|x\|_{L_p^r MO(\mathcal{M})}\} \,.$$

Note that if $p = \infty$, these spaces coincide with the usual $BMO_c(\mathcal{M}), BMO_r(\mathcal{M})$ and $BMO(\mathcal{M})$, respectively. It is immediate to see that for $2 < p < q \leq \infty$, the following contractive inclusion holds

$$L_q^c MO(\mathcal{M}) \quad \subseteq \quad L_p^c MO(\mathcal{M}) \,. \tag{2.8}$$

Also, it was proved in [48] (see also the reference therein to [45]) that if $2 < p < \infty$, then, with equivalent norms,

$$H_p^c(\mathcal{M}) \quad = \quad L_p^c MO(\mathcal{M}) \,. \tag{2.9}$$

Since we are interested in identifying the order of the constants, we give below a sketch of the proof of (2.9). The inclusion $H_p^c(\mathcal{M}) \subseteq L_p^c MO(\mathcal{M})$ is a consequence of Doob's inequality for non-commutative martingales (see [45]). Indeed,

$$\begin{aligned}
\| \sup_n \mathcal{E}_n (\sum_{k \geq n} dx_k^* dx_k) \|_{\frac{p}{2}} \quad &\leq \quad \| \sup_n \mathcal{E}_n (\sum_{k \geq 1} dx_k^* dx_k) \|_{\frac{p}{2}} \\
&\leq \quad d_{\frac{p}{2}} \| \sum_{k \geq 1} dx_k^* dx_k \|_p \\
&= \quad d_{\frac{p}{2}} \| dx \|^2_{H_p^c(\mathcal{M})} \,,
\end{aligned}$$

where $d_{\frac{p}{2}}$ denotes the best constant in the non-commutative Doob's inequality. By results of Junge and Xu [49], the estimates for the optimal order of growth of the constants are

$$d_p \approx (p-1)^{-2} \quad \text{as } p \to 1 \,, \quad \text{respectively,} \;\; d_p \approx O(1) \quad \text{as } p \to \infty \,.$$

Moreover, by the $L_p^c MO(\mathcal{M}) - H_{p'}^c(\mathcal{M})$ duality ([45], Proposition 4.2),

$$L_p^c MO(\mathcal{M}) \quad \subseteq \quad \big(H_{p'}^c(\mathcal{M}) \big)^* \quad \subseteq \quad H_p^c(\mathcal{M}) \,. \tag{2.10}$$

the last inclusion being a consequence of Stein's inequality (see [45] and [69]). Recall that the best constant in Stein's inequality is of order p as $p \to \infty$ (see [49]). While the first inclusion in (2.10) is of norm $\leq \sqrt{2}$, as proved in [45].

Remark 2.2.3 By (2.8), (2.9) and duality it follows that for $1 \leq p < q < \infty$, we have the continuous inclusion

$$BMO_c(\mathcal{M}) \subseteq H_q^c(\mathcal{M}) \subseteq H_p^c(\mathcal{M}). \tag{2.11}$$

We will prove that $H_q^c(\mathcal{M})$ is dense in $H_p^c(\mathcal{M})$. By definition, $H_p^c(\mathcal{M})$ consists of all $L_p(\mathcal{M})$-martingales x such that the difference sequence $dx = (dx_n)_{n\geq 1} \in L_p(\mathcal{M}; l_2^c)$. By [45], Corollary 2.4, the space of finite sequences is dense in $L_p(\mathcal{M}; l_2^c)$. Therefore, it suffices to prove the density for each coordinate n. Note that $L_\infty(\mathcal{M}_n)$ is dense in $L_p(\mathcal{M}_n)$. Thus, given $dx_n \in L_p(\mathcal{M}_n)$ and $\epsilon > 0$, there exists $y \in L_\infty(\mathcal{M}_n)$ such that

$$\|y - dx_n\|_p < \frac{\epsilon}{2}.$$

Since $\|\mathcal{E}_n - \mathcal{E}_{n-1}\|_p \leq 2$, it follows that $\|(\mathcal{E}_n - \mathcal{E}_{n-1})(y - dx_n)\|_p \leq \epsilon$. Or, equivalently,

$$\|dy_n - dx_n\|_p \leq \epsilon.$$

Since $dy_n \in L_\infty(\mathcal{M}_n) \subseteq L_q(\mathcal{M}_n)$, this yields the conclusion. In particular, by duality it follows also that $BMO_c(\mathcal{M})$ is dense in $H_p^c(\mathcal{M})$.

The following interpolation result between $L_p^cMO(\mathcal{M})$ spaces is the main ingredient in the proof of Theorem 2.2.1.

Proposition 2.2.4 *If $2 < p < q \leq \infty$, then, with equivalent norms,*

$$[BMO_c(\mathcal{M}), L_p^cMO(\mathcal{M})]_{\frac{p}{q}} = L_q^cMO(\mathcal{M}).$$

For the proof it will be useful to recall the definition of the following vector-valued non-commutative L_p-spaces. Let $\mathcal{N}$ be a semifinite von Neumann algebra. For $2 \leq p \leq \infty$, $L_p^c(\mathcal{N}; l_\infty)$ is defined in [45] as the space of all sequences $(x_n)_{n\geq 1}$ in $L_p(\mathcal{N}, \tau)$ such that

$$\|(x_n)\|_{L_p^c(\mathcal{N}; l_\infty)} = \|\sup_n x_n^* x_n\|_{\frac{p}{2}}^{1/2} < \infty. \tag{2.12}$$

Lemma 2.2.5 *For* $2 \leq p \leq \infty$, *let* $\frac{1}{q} + \frac{2}{p} = 1$. *Then*

$$\|(x_n)\|_{L_p^c(\mathcal{N};l_\infty)} \;=\; \sup\left\{ (\sum_{n\geq 1} \|x_n v_n\|_2^2)^{1/2} : \|\sum_{n\geq 1} v_n v_n^*\|_q \leq 1 \right\}. \tag{2.13}$$

Consequently, $L_p^c(\mathcal{N};l_\infty)$ *is a normed space.*

Proof. Denote the right hand side of (2.13) by RH. By the definition of $\|\sup_n x_n^* x_n\|_{\frac{p}{2}}$,

$$\|(x_n)\|^2_{L_p^c(\mathcal{N};l_\infty)} \;=\; \sup \sum_{n\geq 1} \tau(x_n^* x_n y_n) \;=\; \sup \sum_{n\geq 1} \|x_n y_n^{\frac{1}{2}}\|_2^2 \,,$$

where the supremum is taken over all sequences of positive elements $(y_n)_{n\geq 1}$ such that

$$\|\sum_{n\geq 1} y_n\|_q \leq 1 \,.$$

In particular, $\|\sum_{n\geq 1} ((y_n)^{\frac{1}{2}})^2\|_q \leq 1$, and hence

$$\|(x_n)\|_{L_p^c(\mathcal{M};l_\infty)} \leq RH \,.$$

Conversely, given a sequence $(v_n)_{n\geq 1}$ of τ-measurable operators such that $\|\sum_{n\geq 1} v_n v_n^*\|_q \leq 1$, define $y_n = v_n v_n^*$, for all positive integers n, and observe that

$$\sum_{n\geq 1} \|x_n v_n\|_2^2 \;=\; \sum_{n\geq 1} \tau(v_n^* x_n^* x_n v_n) \;=\; \sum_{n\geq 1} \tau(x_n^* x_n v_n v_n^*)$$

to conclude that $RH \leq \|\sup_n x_n^* x_n\|^{1/2}_{\frac{p}{2}}$. □

Remark 2.2.6 Let $e \in \mathcal{N}$ be a finite projection. Define the set $L_p^c(\mathcal{N}e;l_\infty)$ by

$$L_p^c(\mathcal{N}e;l_\infty) \;=\; \{(x_n e)_{n\geq 1} : (x_n e)_{n\geq 1} \in L_p^c(\mathcal{N};l_\infty)\} \;\subseteq\; L_p^c(\mathcal{N};l_\infty) \,.$$

Then $L_p^c(\mathcal{N}e; l_\infty)$ is a normed space under the induced norm. Moreover, the map

$$S : L_p^c(\mathcal{N}; l_\infty) \to L_p^c(\mathcal{N}e; l_\infty)$$

defined by $S((x_n)) = (x_n e)$ is a bounded projection. Indeed, if $\|(x_n)\|_{L_p^c(\mathcal{N};l_\infty)} \leq 1$, then by definitions (2.12) and (2.7),

$$\|(x_n e)\|^2_{L_p^c(\mathcal{N}e;l_\infty)} = \sup \sum_{n\geq 1} \tau(e x_n^* x_n e y_n) = \sup \sum_{n\geq 1} \tau(x_n^* x_n e y_n e),$$

where the supremum is taken over all sequences of positive elements with $\|\sum\limits_{n\geq 1} y_n\|_{\frac{p'}{2}} \leq 1$. Furthermore, for any such sequence $(y_n)_{n\geq 1}$ we have

$$\sum_{n\geq 1} \tau(x_n^* x_n e y_n e) \leq \|\sum_{n\geq 1} e y_n e\|_{\frac{p'}{2}} \leq \|\sum_{n\geq 1} y_n\|_{\frac{p'}{2}} \|e\|^2 \leq 1,$$

which completes the argument. Consequently, $L_p^c(\mathcal{N}e; l_\infty)$ is a complemented subspace of $L_p^c(\mathcal{N}; l_\infty)$. Note also that for $2 \leq p < q \leq \infty$, the following contractive inclusion holds

$$L_q^c(\mathcal{N}e; l_\infty) \subseteq L_p^c(\mathcal{N}e; l_\infty). \tag{2.14}$$

Indeed, the equality

$$\|(x_n e)\|_{L_p^c(\mathcal{N}e;l_\infty)} = \|(x_n)\|_{L_p^c(e\mathcal{N}e;l_\infty)},$$

allows us to apply Lemma 2.2.9 to the finite von Neumann algebra $e\mathcal{N}e$ and obtain (2.14). □

Note that the spaces $L_p^c(\mathcal{N}; l_\infty)$, $2 \leq p \leq \infty$ are all continuously embedded into $\prod_{n\in\mathbb{N}} L_0(\mathcal{N})$, which is a topological vector space in the product topology. This allows us to interpolate them and establish the following

Proposition 2.2.7 *If $2 \leq p, q, s \leq \infty$ and $0 < \theta < 1$ with $\frac{1}{s} = \frac{1-\theta}{p} + \frac{\theta}{q}$, then, isometrically,*

$$[L_p^c(\mathcal{N}; l_\infty), L_q^c(\mathcal{N}; l_\infty)]_\theta = L_s^c(\mathcal{N}; l_\infty).$$

Proof. From (2.13) it follows that for $\frac{1}{f(p)}+\frac{1}{p}=\frac{1}{2}$ the bilinear map

$$T: L_p^c(\mathcal{N};l_\infty)\times L_{f(p)}(\mathcal{N};l_2^r)\to l_2(L_2(\mathcal{N},\tau))$$

defined by $T((x_n),(v_n)) = (x_n v_n)$ is a contraction. By the bilinearity property of the complex interpolation method ([3], Theorem 4.4.1) , it follows that

$$\|\, T: [\, L_p^c(\mathcal{N};l_\infty), L_q^c(\mathcal{N};l_\infty)\,]_\theta \times [\, L_{f(p)}(\mathcal{N};l_2^r), L_{f(q)}(\mathcal{N};l_2^r)\,]_\theta \to l_2(L_2(\mathcal{N},\tau))\,\| \le 1\,.$$

An easy computation shows that $\frac{1-\theta}{f(p)}+\frac{\theta}{f(q)}=\frac{1}{f(s)}$. Hence, a further application of (2.13), together with (1.9) yields the contractive inclusion

$$[L_p^c(\mathcal{N};l_\infty), L_q^c(\mathcal{N};l_\infty)]_\theta \quad\subseteq\quad L_s^c(\mathcal{N};l_\infty)\,.$$

The reverse inclusion is a consequence of the following factorization property of the spaces $L_s^c(\mathcal{N};l_\infty)$, proved in [45]. Given $(x_n)_{n\ge 1}\in L_s^c(\mathcal{N};l_\infty)$ with $\|(x_n)\|_{L_s^c(\mathcal{N};l_\infty)}\le 1$, we have

$$0\;\le\; x_n^* x_n \;\in\; L_{\frac{s}{2}}(\mathcal{N};l_\infty)\,.$$

Then, by [45], Remark 3.7, there exists $0\le a\in L_s(\mathcal{N})$ and a sequence of positive contractions $y_n\in\mathcal{N}$ such that

$$x_n^* x_n = a y_n a \qquad \text{and} \qquad \|a\|_s^2 \sup_n \|y_n\|_\infty = \|\sup_n x_n^* x_n\|_{\frac{s}{2}} \le 1\,. \tag{2.15}$$

Furthermore, for all positive integers n

$$|x_n| = {y_n}^{\frac{1}{2}} a\,.$$

Hence $x_n = u_n {y_n}^{\frac{1}{2}} a$, where u_n is a partial isometry. Denoting $u_n {y_n}^{\frac{1}{2}}$ by z_n , we obtain

$$x_n \;=\; z_n a, \quad \sup_n \|z_n\|_\infty \le 1\,. \tag{2.16}$$

Using (2.16), the proof of the reverse inclusion reduces to the interpolation result (1.7). $\square$

Remark 2.2.8 Let e be a finite projection in $\mathcal{N}$, as before. Applying Proposition 2.2.7, together with the complementation result proved in Remark 2.2.6, we obtain, isometrically,

$$[L_p^c(\mathcal{N}e;l_\infty), L_q^c(\mathcal{N}e;l_\infty)]_\theta \quad = \quad L_s^c(\mathcal{N}e;l_\infty), \quad \frac{1}{s} = \frac{1-\theta}{p} + \frac{\theta}{q}\,. \tag{2.17}$$

Lemma 2.2.9 *For* $2 \le p < q < \infty$*, the following contractive inclusion holds*

$$L_q^c(\mathcal{M};l_\infty) \quad \subseteq \quad L_p^c(\mathcal{M};l_\infty)\,.$$

Proof. This is a consequence of the aforementioned factorization property. As explained in Proposition 2.2.7, given a sequence $(x_n)_{n\ge1} \in L_q^c(\mathcal{M};l_\infty)$ with $\|(x_n)\|_{L_q^c(\mathcal{M};l_\infty)} \le 1$, there exist $0 \le a \in L_q(\mathcal{M})$ and a sequence of positive contractions $y_n \in \mathcal{M}$ satisfying (2.15). Since $\mathcal{M}$ is a finite von Neumann algebra, it follows that $\|a\|_p \le \|a\|_q$. Therefore

$$\|\sup_n x_n^* x_n\|_{\frac{p}{2}}^{1/2} = \|a\|_p^2 \sup_n \|y_n\|_\infty \le \|a\|_q^2 \sup_n \|y_n\|_\infty = \|\sup_n x_n^* x_n\|_{\frac{q}{2}}^{1/2} \le 1\,.$$

which proves that $(x_n)_{n\ge1} \in L_p^c(\mathcal{M};l_\infty)$, with $\|(x_n)\|_{L_p^c(\mathcal{M};l_\infty)} \le 1$. $\square$

We are now ready to give the proof of Proposition 2.2.4. We will assume in the following that the predual of $\mathcal{M}$ is separable. This ensures the existence of a separable σ-weakly dense subalgebra X of $\mathcal{M}$. Following [56], for all positive integers n let F_n be the Hilbert C^*-module generated by $\mathcal{M}_n$ and X. Denote by $C(\mathcal{M}_n) \subset \mathcal{B}(l_2)\otimes_{\min}\mathcal{M}_n$ the space of infinite column matrices with entries from $\mathcal{M}_n$. Then, as proved in [45], there exists a right $\mathcal{M}_n$-module isomorphism $u_n : F_n \to C(\mathcal{M}_n)$ such that for all $x, y \in F_n$

$$u_n(y)^* u_n(x) \quad = \quad \mathcal{E}_n(y^* x)\,. \tag{2.18}$$

Moreover, for all $2 \le p \le \infty$ this extends to a contractive right $\mathcal{M}_n$-module map $u_n : L_p(\mathcal{M}) \to$

$L_p(\mathcal{M}_n; l_2^c)$ such that (2.18) holds for all $x \in L_p(\mathcal{M}), y \in L_q(\mathcal{M})$. An argument involving the strong topology as in [45], Proposition 2.9, shows that u_n extends furthermore to a contractive map $u_n : \mathcal{M} \to C_\infty(\mathcal{M}_n)$, where $C_\infty(\mathcal{M}_n) = C\bar{\otimes}\mathcal{M}_n$, still satisfying (2.18) for all $x, y \in \mathcal{M}$. Here C denotes the column space of $\mathcal{B}(l_2)$, so $C_\infty(\mathcal{M}_n)$ consists of sequences $(x_n)_{n\geq 1}$ with $x_n \in \mathcal{M}_n$ such that $\sum_{n\geq 1} x_n^* x_n$ converges in the σ-weak operator topology.

Proof of Proposition 2.2.4. Let $\mathcal{N} = \mathcal{M} \otimes \mathcal{B}(l_2)$. For $2 < s \leq \infty$, define a map $\Phi : L_s^c MO(\mathcal{M}) \to L_s^c(\mathcal{N}; l_\infty)$ by

$$\Phi(x) = (u_n(x - \mathcal{E}_{n-1}(x)))_{n\geq 1} .$$

The following computation shows that Φ yields an isometric embedding:

$$\begin{aligned} \|\Phi(x)\|^2_{L_s^c(\mathcal{N};l_\infty)} &= \|\sup_n u_n(x - \mathcal{E}_{n-1}(x))^* u_n(x - \mathcal{E}_{n-1}(x))\|_{\frac{s}{2}} \\ &= \|\sup_n \mathcal{E}_n([x - \mathcal{E}_{n-1}(x)]^*[x - \mathcal{E}_{n-1}(x)])\|_{\frac{s}{2}} \\ &= \|x\|^2_{L_s^c MO(\mathcal{M})} . \end{aligned} \tag{2.19}$$

Let $e = 1 \otimes e_{11} \in \mathcal{N}$, where 1 is unit of $\mathcal{M}$ and e_{11} is the matrix unit in $\mathcal{B}(l_2)$ with $e_{11}(1,1) = 1$ and $e_{11}(i,j) = 0$, if $(i,j) \neq (1,1)$. For all positive integers n and all $x \in F_n$, we have by the definition of u_n

$$u_n(x) \cdot e = u_n(x) ,$$

Together with (2.19), this yields the isometric embedding

$$\Phi(L_s^c MO(\mathcal{M})) \subseteq L_s^c(\mathcal{N}e; l_\infty) .$$

By Remark (2.2.8) it follows that $[L_\infty^c(\mathcal{N}e; l_\infty), L_p^c(\mathcal{N}e; l_\infty)]_{\frac{p}{q}} = L_q^c(\mathcal{N}e; l_\infty)$. Therefore, we obtain by interpolation that

$$\| \Phi : [BMO_c(\mathcal{M}), L_p^c MO(\mathcal{M})]_{\frac{p}{q}} \to L_q^c(\mathcal{N}e; l_\infty) \| \leq 1 .$$

Furthermore, we will prove that

$$\Phi(\,[BMO_c(\mathcal{M}), L_p^cMO(\mathcal{M})]_{\frac{p}{q}}\,) \;\subseteq\; \Phi(L_q^cMO(\mathcal{M}))\,. \tag{2.20}$$

By [3], Theorem 4.2.2, the intersection of a compatible couple of Banach spaces is dense in their interpolation space. Recall that by (2.8) we have the following contractive inclusion

$$BMO_c(\mathcal{M}) \cap L_p^cMO(\mathcal{M}) \;=\; BMO_c(\mathcal{M}) \;\subseteq\; L_q^cMO(\mathcal{M})\,. \tag{2.21}$$

Therefore, if $x \in BMO_c(\mathcal{M})$, then $\Phi(x) \in \Phi(L_q^cMO(\mathcal{M}))$. Since $\Phi : L_q^cMO(\mathcal{M}) \to L_q^c(\mathcal{N}e; l_\infty)$ is an isometry, we have

$$\begin{aligned} \|x\|_{L_q^cMO(\mathcal{M})} &= \|\Phi(x)\|_{L_q^c(\mathcal{N}e;l_\infty)} \\ &= \|\Phi(x)\|_{\Phi(\,[BMO_c(\mathcal{M}),L_p^cMO(\mathcal{M})]_{\frac{p}{q}})} \\ &\leq \|x\|_{[BMO(\mathcal{M}),L_p^cMO(\mathcal{M})]_{\frac{p}{q}}}\,. \end{aligned}$$

By density, (2.20) is proved and, moreover, we have the isometric inclusion

$$[BMO_c(\mathcal{M}), L_p^cMO(\mathcal{M})]_{\frac{p}{q}} \subseteq L_q^cMO(\mathcal{M})\,.$$

Next, observe that the following contractive inclusion holds:

$$[H_1^c(\mathcal{M}), H_{p'}^c(\mathcal{M})]_{\frac{p}{q}} \;\subseteq\; H_{q'}^c(\mathcal{M})\,. \tag{2.22}$$

By the definition of the non-commutative Hardy spaces, for $1 < s < \infty$ the map

$$d : H_s^c(\mathcal{M}) \to L_s(\mathcal{M}; l_2^c)$$

defined by $d(x) = dx$ is an isometry. Hence, using (1.9), it follows by interpolation that

$$\|\, d : [H_1^c(\mathcal{M}), H_{p'}^c(\mathcal{M})]_{\frac{p}{q}} \to L_{q'}(\mathcal{M}; l_2^c)\,\| \leq 1\,.$$

Thus, for all $x \in H^c_{p'}(\mathcal{M}) \subseteq H^c_{q'}(\mathcal{M})$ we have

$$\|x\|_{H^c_{q'}(\mathcal{M})} = \|dx\|_{L_{q'}(\mathcal{M};l^c_2)} \leq \|x\|_{[H^c_1(\mathcal{M}),H^c_{p'}(\mathcal{M})]_{\frac{p}{q}}} .$$

By the aforementioned general complex interpolation result, $H^c_{p'}(\mathcal{M}) = H^c_1(\mathcal{M}) \cap H^c_{p'}(\mathcal{M})$ is dense in the interpolation space $[\, H^c_1(\mathcal{M}), H^c_{p'}(\mathcal{M})\,]_{\frac{p}{q}}$. Hence (2.22) is proved. Since $H^c_{p'}(\mathcal{M})$ is reflexive, the equivalence theorem ([3], Theorem 4.3.1) applies. By taking duals in (2.22) and using (2.9) we then get the contractive inclusion

$$L^c_q MO(\mathcal{M}) \subseteq [BMO_c(\mathcal{M}), L^c_p MO(\mathcal{M})]_{\frac{p}{q}} \,,$$

which completes the proof. $\square$

Theorem 2.2.1 in the range $2 < p < q < \infty$ follows now by combining Proposition 2.2.4 with (2.8) and (2.9). By Lemma 2.2.2 the result extends to the full range $1 \leq p < q < \infty$.

Remark 2.2.10 The row version of Theorem 2.2.1 holds true, as well. More precisely, for $1 \leq p < q < \infty$, we have with equivalent norms

$$[BMO_r(\mathcal{M}), H^r_p(\mathcal{M})]_{\frac{p}{q}} = H^r_q(\mathcal{M})\,. \tag{2.23}$$

In order to prove this, note that by the definition of the space $BMO_r(\mathcal{M})$, the map $x \mapsto x^*$ is an anti-linear isometry between $BMO_c(\mathcal{M})$ and $BMO_r(\mathcal{M})$. Hence, we can identify isometrically $BMO_r(\mathcal{M})$ with the conjugate space $\overline{BMO_c(\mathcal{M})}$. Similarly, for $1 \leq s < \infty$, we identify isometrically $H^r_s(\mathcal{M})$ with the conjugate space $\overline{H^c_s(\mathcal{M})}$. Furthermore, note that if (A_0, A_1) is a compatible couple of Banach spaces and $0 < \theta < 1$, then

$$[\overline{A}_0, \overline{A}_1]_\theta \quad = \quad \overline{[A_0, A_1]_\theta}\,. \tag{2.24}$$

The key point in the proof of (2.24) is the invariance of the strip S under conjugation, i.e., if $z \in S$, then $\bar{z} \in S$. Combining Theorem 2.2.1 with (2.24) yields (2.23).

Theorem 2.2.11 *Let X be either $L_\infty(\mathcal{M})$ or $BMO(\mathcal{M})$, and Y be either $L_1(\mathcal{M})$ or $H_1(\mathcal{M})$. If $1 < p < \infty$, then, with equivalent norms,*

$$[X, Y]_{\frac{1}{p}} \quad = \quad L_p(\mathcal{M})\,. \tag{2.25}$$

Proof. Case 1: Let $Y = L_1(\mathcal{M})$. If $X = L_\infty(\mathcal{M})$, the conclusion follows from (1.7). Let $X = BMO(\mathcal{M})$. Assume first that $2 \le p < \infty$. Using the square function inequalities (1.11), Theorem 2.2.1 and Remark 2.2.10, we obtain

$$[BMO(\mathcal{M}), L_2(\mathcal{M})]_{\frac{2}{p}} = [BMO(\mathcal{M}), H_2(\mathcal{M})]_{\frac{2}{p}} \subseteq H_p^c(\mathcal{M}) \cap H_p^r(\mathcal{M}) = H_p(\mathcal{M}) = L_p(\mathcal{M})\,.$$

The reverse inclusion is immediate. Therefore,

$$A_2 = L_p(\mathcal{M}) = [BMO(\mathcal{M}), L_2(\mathcal{M})]_{\frac{2}{p}} = [A_1, A_3]_{\frac{2}{p}}\,.$$

Moreover, we have

$$A_3 = L_2(\mathcal{M}) = [L_p(\mathcal{M}), L_1(\mathcal{M})]_\phi = [A_2, A_4]_\phi\,,$$

where $\phi = \frac{p-2}{2(p-1)}$. An application of Wolff's theorem shows that

$$A_2 = L_p(\mathcal{M}) = [A_1, A_4]_\zeta = [BMO(\mathcal{M}), L_1(\mathcal{M})]_\zeta\,.$$

A simple computation shows that $\zeta = \frac{1}{p}$. If $1 \le p < 2$, then, by the reiteration theorem ([3], Theorem 4.6.1) it follows that

$$\begin{aligned} L_p &= [L_2, L_1]_\xi \\ &= [\,[BMO(\mathcal{M}), L_1(\mathcal{M})]_{\frac{1}{2}}, [BMO(\mathcal{M}), L_1(\mathcal{M})]_1\,]_\xi \\ &= [BMO(\mathcal{M}), L_1(\mathcal{M})]_{\frac{1+\xi}{2}}\,, \end{aligned}$$

where $\xi = \frac{2}{p} - 1$.

Case 2: $Y = H_1(\mathcal{M})$. Let $1 < q < p < \infty$. We will first show that

$$[H_1(\mathcal{M}), L_p(\mathcal{M})]_{\frac{p'}{q'}} \quad = \quad L_q(\mathcal{M})\,, \tag{2.26}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$. Indeed, by the Corollary in the Appendix of [69](see also the Remark to it in [48]), we have $H_1(\mathcal{M}) \subseteq L_1(\mathcal{M})$. Hence

$$[H_1(\mathcal{M}), L_p(\mathcal{M})]_{\frac{p'}{q'}} \quad \subseteq \quad [L_1(\mathcal{M}), L_p(\mathcal{M})]_{\frac{p'}{q'}} \quad = \quad L_q(\mathcal{M})\,. \tag{2.27}$$

Observe that $L_1(\mathcal{M}) \cap L_p(\mathcal{M}) = L_p(\mathcal{M})$ is dense in $L_q(\mathcal{M})$, since by definition m_τ is dense in both. Also, note that $L_p(\mathcal{M})$ is reflexive. The conditions in the equivalence theorem ([3], Theorem 4.3.1) are satisfied. Therefore, by passing to the duals in (2.27) we get

$$L_{q'}(\mathcal{M}) = (L_q(\mathcal{M}))^* \subseteq ([H_1(\mathcal{M}), L_p(\mathcal{M})]_{\frac{p'}{q'}})^* = [BMO(\mathcal{M}), L_{p'}(\mathcal{M})]_{\frac{p'}{q'}}\,. \tag{2.28}$$

By Case 1 and the reiteration theorem it follows that the inclusion in (2.28) is, in fact, equality. Together with (2.27), this yields (2.26). In order to prove (2.25) we will apply Wolff's interpolation theorem. Indeed, we have

$$A_2 = L_p(\mathcal{M}) = [X, L_q(\mathcal{M})]_{\frac{q}{p}} = [A_1, A_3]_{\frac{q}{p}}\,.$$

If $X = BMO(\mathcal{M})$ this follows from Case 1 and the reiteration theorem, while the case $X = L_\infty(\mathcal{M})$ follows from (1.7). Let $\theta = \frac{p'}{q'}$. By (2.26), it follows that

$$A_3 = L_q(\mathcal{M}) = [L_p(\mathcal{M}), H_1(\mathcal{M})]_{1-\theta} = [A_2, A_4]_{1-\theta}\,.$$

Note that $A_1 \cap A_4 = X \cap H_1(\mathcal{M})$ is dense in both $A_2 = L_p(\mathcal{M})$ and $A_3 = L_q(\mathcal{M})$, since $m_\tau \subseteq L_\infty(\mathcal{M}) \subseteq X \cap H_1(\mathcal{M})$, and m_τ is dense in both $L_p(\mathcal{M})$ and $L_q(\mathcal{M})$. An application of Wolff's interpolation theorem yields (2.25) and the proof is complete. $\square$

Remark 2.2.12 By the reiteration theorem it follows that for $1 \leq p < q < \infty$, we have with equivalent norms,

$$[BMO(\mathcal{M}), L_p(\mathcal{M})]_{\frac{p}{q}} \quad = \quad L_q(\mathcal{M})\,. \tag{2.29}$$

2.3 Complements

In this section, we consider another sometimes useful method of interpolation, namely the real method. We refer to [3] for details and the connection with the complex method. Recall that the non-commutative L_p-spaces associated with a semifinite von Neumann algebra form an interpolation scale with respect to the real method. More precisely (see, e.g., [72]), for $1 \leq p_0, p_1, p \leq \infty$ and $0 < \theta < 1$ with $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, we have with equivalent norms

$$L_p(\mathcal{M}, \tau) \quad = \quad [L_{p_0}(\mathcal{M}, \tau), L_{p_1}(\mathcal{M}, \tau)]_{\theta,p}\,. \tag{2.30}$$

We will first show that the main result of the previous section remains true for the real method of interpolation.

Theorem 2.3.1 *Let $1 \leq p < q < \infty$. Then, with equivalent norms,*

$$[BMO_c(\mathcal{M}), H_p^c(\mathcal{M})]_{\frac{p}{q},q} \quad = \quad H_q^c(\mathcal{M})\,.$$

Proof. For $1 < v, s, q < \infty$ and $0 < \eta < 1$ such that $\frac{1}{q} = \frac{1-\eta}{v} + \frac{\eta}{s}$, we have with equivalent norms

$$[H_v^c(\mathcal{M}), H_s^c(\mathcal{M})]_{\eta,q} \quad = \quad H_q^c(\mathcal{M})\,. \tag{2.31}$$

Indeed, for $1 < p < \infty$ the space $H_p^c(\mathcal{M})$ is complemented in $L_p(\mathcal{M}; l_2^c)$ via Stein's projection. Furthermore, as already mentioned in the preliminaries (see also [48]), $L_p(\mathcal{M}; l_2^c)$ is a 1-complemented subspace of $L_p(\mathcal{M} \otimes \mathcal{B}(l_2))$. Thus, we conclude from (2.30) that we have with equivalent norms

$$[L_v(\mathcal{M}; l_2^c), L_s(\mathcal{M}; l_2^c)]_{\eta,q} = L_q(\mathcal{M}; l_2^c)\,,$$

which in turn, by complementation, yields (2.31). Now let $1 \leq p < q < \infty$ and denote $\theta = \frac{p}{q}$. Consider $0 < \theta_0 < \frac{\theta}{2} < 1$, and set $\theta_1 = \theta - \theta_0$. There exists $0 < \eta < 1$ such that

$$\theta \quad = \quad (1-\eta)\theta_0 + \eta\theta_1\,. \tag{2.32}$$

By Theorem 4.7.2 of [3] on the connection between the real and complex method of interpolation, we obtain

$$[BMO_c(\mathcal{M}), H_p^c(\mathcal{M})]_{\theta,q} \quad = \quad [\,[BMO_c(\mathcal{M}), H_p^c(\mathcal{M})]_{\theta_0}, [BMO_c(\mathcal{M}), H_p^c(\mathcal{M})]_{\theta_1}\,]_{\eta,q}\,.$$

Applying Theorem 2.2.1, we then get

$$[H^c_{\frac{p}{\theta_0}}(\mathcal{M}), H^c_{\frac{p}{\theta_1}}(\mathcal{M})]_{\eta,q} \quad = \quad [BMO_c(\mathcal{M}), H_p^c(\mathcal{M})]_{\theta,q}\,.$$

Observe that by (2.32) we have

$$\frac{1}{q} \quad = \quad \frac{1-\eta}{\frac{p}{\theta_0}} + \frac{\eta}{\frac{p}{\theta_1}}\,.$$

An application of (2.31) yields the conclusion of the theorem. $\square$

Remark 2.3.2 Note that the row version of Theorem 2.3.1 holds true, as well. Namely, if $1 \leq p < q < \infty$, then, with equivalent norms,

$$[BMO_r(\mathcal{M}), H_p^r(\mathcal{M})]_{\frac{p}{q},q} \quad = \quad H_q^r(\mathcal{M})\,. \tag{2.33}$$

Furthermore, we can show that the non-commutative Lorentz spaces $L_{p,q}(\mathcal{M}, \tau)$ can be obtained by real interpolation between non-commutative BMO and non-commutative L_p spaces, respectively

Hardy spaces. For the corresponding results in the classical setting, we refer the reader to Weisz [85]. We first recall some definitions. Let x be a τ-measurable operator. Following Fack and Kosaki [31], the *t-th singular number of* x is

$$\mu_t(x) \quad = \quad \inf\{\|xe\| : e \text{ is a projection in } \mathcal{M}\,,\ \tau(1-e) \leq t\,\}\,.$$

Recall that if $\mathcal{M} = L_\infty((0,\infty), m)$, where m is the Lebesgue measure on $(0,\infty)$, and $x \in \mathcal{M}$ then the map $t \mapsto \mu_t(x)$ is exactly the *decreasing rearrangement* of the function $|x|$ (see Fack [30]). For $0 < p, q < \infty$, the classical Lorentz spaces are defined by

$$L_{p,q}((0,\infty), m) \quad = \quad \{x \in L_0((0,\infty), m) : \|x\|_{p,q} < \infty\}\,,$$

where

$$\|x\|_{p,q} = \begin{cases} \left(\int_0^\infty (t^{1/p}\mu_t(x))^q \frac{dt}{t}\right)^{1/q} & \text{if } q < \infty\,, \\ \sup\limits_{t>0} t^{1/p}\mu_t(x) & \text{if } q = \infty\,. \end{cases}$$

By properties of the decreasing rearrangement (see Bennett and Sharpley [2], Proposition II.1.8), it follows that for $0 < p \leq \infty$, the Lorentz space $L_{p,p}$ coincides with the Lebesgue space L_p, and for all $x \in L_p$,

$$\|x\|_{p,p} = \|x\|_p\,.$$

Note that, in general, $\|\cdot\|_{p,q}$ is a quasi-norm. However, by [2], Theorem 4.4.6, if $1 < p < \infty$ and $1 \leq q \leq \infty$, or if $p = q = \infty$, then it is possible to replace the quasi-norm $\|\cdot\|_{p,q}$ by a norm $\|\cdot\|_{(p,q)}$ under which $L_{p,q}$ becomes a rearrangement invariant (r.i.) Banach function space. The norm $\|\cdot\|_{(p,q)}$ satisfies

$$\|x\|_{p,q} \leq \|x\|_{(p,q)} \leq p'\,\|x\|_{p,q}\,,$$

for all $x \in L_0((0,\infty), m)$, as a consequence of Hardy's inequality. We refer to [2] for details. Following the general scheme of *symmetric operator spaces* associated to $(\mathcal{M}, \tau)$ and a rearrangement invariant Banach function space developed in [19] and [20], the non-commutative Lorentz spaces are defined as

$$L_{p,q}(\mathcal{M},\tau) = \{x \in L_0(\mathcal{M}) : \mu(x) \in L_{p,q}((0,\infty),m)\}\,, \quad \|x\|_{L_{p,q}(\mathcal{M},\tau)} = \|\mu(x)\|_{p,q}\,.$$

The formula describing the K-functional for the interpolation couple $(\mathcal{M}, L_1(\mathcal{M},\tau))$ (see [72], Corollary 2.3), shows that, with equivalent norms,

$$L_{p,q}(\mathcal{M},\tau) \;=\; [\mathcal{M}, L_1(\mathcal{M},\tau)]_{\frac{1}{p},q}\,. \tag{2.34}$$

We refer the reader to Randrianantoanina [73] and Xu [88] for more details on the non-commutative Lorentz spaces.

Theorem 2.3.3 *Let X be either $L_\infty(\mathcal{M})$ or $BMO(\mathcal{M})$, and Y be either $L_1(\mathcal{M})$ or $H_1(\mathcal{M})$. If $1 < p < \infty$ and $1 \leq s \leq \infty$, then, with equivalent norms,*

$$[X,Y]_{\frac{1}{p},s} \;=\; L_{p,s}(\mathcal{M})\,. \tag{2.35}$$

Proof. Let $1 \leq p_1 < p < p_2 < \infty$ and $1 \leq t \leq \infty$. There exists $0 < \eta < 1$ such that

$$\frac{1}{p} \;=\; \frac{1-\eta}{p_1} + \frac{\eta}{p_2}\,.$$

Then, by Theorem 4.7.2 of [3] and Theorem 2.2.11 we have with equivalent norms

$$\begin{aligned} [X,Y]_{\frac{1}{p},s} &= [\,[X,Y]_{\frac{1}{p_1}}, [X,Y]_{\frac{1}{p_2}}\,]_{\eta,s} \qquad (2.36)\\ &= [\,L_{p_1}(\mathcal{M})\,, L_{p_2}(\mathcal{M})\,]_{\eta,s}\,. \end{aligned}$$

An application of (2.34) together with the reiteration theorem for the real method of interpolation yields the conclusion. □

In the remainder of this section we will show that all the interpolation results between non-commutative BMO and non-commutative L_p-spaces hold in the category of operator spaces. We will first describe the operator space structure of $BMO_c(\mathcal{M})$. For all positive integers m , consider on the algebra $M_m \otimes \mathcal{M}$ the natural filtration $(M_m \otimes \mathcal{M}_n)_{n\geq 1}$. Define

$$M_m(BMO_c(\mathcal{M})) \quad = \quad BMO_c(M_m \otimes \mathcal{M})\,. \tag{2.37}$$

We note that the above matrix norms satisfy the Ruan axioms (1.12), hence determine an operator space structure on $BMO_c(\mathcal{M})$. Indeed, from the proof of Proposition 2.2.4 we will obtain a concrete realization of $BMO_c(\mathcal{M})$ as an operator space with this natural structure.

Proposition 2.3.4 *Let $\mathcal{N} = \mathcal{M} \otimes \mathcal{B}(l_2)$. The map $\Phi : BMO_c(\mathcal{M}) \to l_\infty(\mathcal{N})$, defined by*

$$\Phi(x) \quad = \quad (u_n(x - \mathcal{E}_{n-1}(x)))_{n\geq 1} \tag{2.38}$$

is a complete isometry.

Proof. We will prove that for all positive integers m and all $x = [x_{ij}] \in M_m(\mathcal{M})$ we have

$$\|(Id_{M_m} \otimes \Phi)(x)\|_{M_m(l_\infty(\mathcal{N}))} = \|x\|_{BMO_c(M_m\otimes\mathcal{M})}\,.$$

For all positive integers n, let $\mathbb{E}_n = Id_{M_m} \otimes \mathcal{E}_n$. Then clearly $\mathbb{E}_n : M_m(\mathcal{M}) \to M_m(\mathcal{M}_n)$ is the unique trace-preserving conditional expectation onto $M_m(\mathcal{M}_n) = M_m \otimes \mathcal{M}_n$.
Therefore we have

$$\begin{aligned}
\|(Id_{M_m} \otimes \Phi)(x)\|^2_{M_m(l_\infty(\mathcal{N}))} &= \|[\ \Phi(x_{ij})\ ]\|^2_{M_m(l_\infty(\mathcal{N}))} \\
&= \sup_n \left\| [\ u_n(x_{ij} - \mathcal{E}_{n-1}(x_{ij}))^* u_n(x_{ij} - \mathcal{E}_{n-1}(x_{ij}))\ ]_{ij} \right\|^2_{M_m(C_\infty(\mathcal{M}_n))} \\
&= \sup_n \left\| \left[\sum_{k=1}^m u_n(x_{ki} - \mathcal{E}_{n-1}(x_{ki}))^* u_n(x_{kj} - \mathcal{E}_{n-1}(x_{kj})) \right]_{ij} \right\|_{M_m(\mathcal{M})} \\
&= \sup_n \left\| \left[\sum_{k=1}^m \mathcal{E}_n\left([x_{ki} - \mathcal{E}_{n-1}(x_{ki})]^*[x_{kj} - \mathcal{E}_{n-1}(x_{kj})]\right) \right]_{ij} \right\|_{M_m(\mathcal{M})} \\
&= \sup_n \|\mathbb{E}_n\left([x - \mathbb{E}_{n-1}(x)]^*[x - \mathbb{E}_{n-1}(x)]\right)\|_{M_m(\mathcal{M})} \\
&= \|x\|^2_{BMO_c(M_m\otimes\mathcal{M})}\,.
\end{aligned}$$

$\square$

Remark 2.3.5 We endow $BMO_r(\mathcal{M})$ with a natural operator space structure by defining for all positive integers m

$$M_m(BMO_r(\mathcal{M})) \quad = \quad BMO_r(M_m \otimes \mathcal{M})\,. \tag{2.39}$$

With similar arguments, we obtain a concrete realization of $BMO_r(\mathcal{M})$ as an operator space with this natural structure. More precisely, it can be proved that the map $\Psi : BMO_r(\mathcal{M}) \to l_\infty(\mathcal{N})$, given by

$$\Psi(x) \quad = \quad ((u_n(x^* - \mathcal{E}_{n-1}(x^*)))^*)_{n\geq 1}$$

is a complete isometry. We equip $BMO(\mathcal{M})$ with the operator space structure of the intersection space $BMO_c(\mathcal{M}) \cap BMO_r(\mathcal{M})$. Then, by (2.37), (2.39), and (1.16) it follows that for all positive integers m we have isometrically

$$M_m(BMO(\mathcal{M})) \quad = \quad BMO(M_m \otimes \mathcal{M})\,. \tag{2.40}$$

Following definition (2.3), we endow $H_p^c(\mathcal{M})$ with the induced operator space structure of the column subspace of $L_p(\mathcal{M} \otimes \mathcal{B}(l_2))$. Similarly we endow $H_p^r(\mathcal{M})$ with the induced operator space structure of the row subspace of $L_p(\mathcal{M} \otimes \mathcal{B}(l_2))$. Then, by the definitions (2.11), according to whether $1 \leq p < 2$, or $p \geq 2$, $H_p(\mathcal{M})$ is equipped with the operator space structure of the intersection of $H_p^c(\mathcal{M})$ and $H_p^r(\mathcal{M})$, respectively of their sum. We shall also introduce the spaces

$$\begin{aligned} \tilde{H}_p(\mathcal{M}) &= H_p^c(\mathcal{M}) \cap_p H_p^r(\mathcal{M}) \quad \text{if} \;\; p \geq 2\,, \\ \tilde{H}_p(\mathcal{M}) &= H_p^c(\mathcal{M}) +_p H_p^r(\mathcal{M}) \quad \text{if} \;\; 1 \leq p < 2\,. \end{aligned}$$

Then, for all $1 \leq p < \infty$, $\tilde{H}_p(\mathcal{M})$ is an operator space completely isomorphic to $H_p(\mathcal{M})$. Indeed, since for any $a\,, b \geq 0\,,\ \max\{a, b\} \leq (a^p + b^p)^{\frac{1}{p}} \leq 2^{\frac{1}{p}} \max\{a, b\}\,,$ we obtain the following estimates

$$\begin{aligned} d_{cb}(H_p^c(\mathcal{M}) \cap H_p^r(\mathcal{M})\,, H_p^c(\mathcal{M}) \cap_p H_p^r(\mathcal{M})) &\leq 2^{\frac{1}{p}}\,, && (2.41) \\ d_{cb}(H_p^c(\mathcal{M}) + H_p^r(\mathcal{M})\,, H_p^c(\mathcal{M}) +_p H_p^r(\mathcal{M})) &\leq 2^{1+\frac{1}{p}}\,. && (2.42) \end{aligned}$$

The additional factor of 2 in (2.42) comes from the fact that the norm on the sum of spaces $H_p^c(\mathcal{M}) + H_p^r(\mathcal{M})$ was defined using (1.3) and not the equivalent norm given by (1.15).

Remark 2.3.6 Let (E, F) be a compatible couple of operator spaces. If $1 \leq p \leq \infty$ and m is a positive integer, then, as a special case of (2.9) in [70] we obtain the isometry,

$$S_p^m[E \oplus_p F] = S_p^m[E] \oplus_p S_p^m[F]\,. \tag{2.43}$$

Proposition 2.3.7 *If $1 \leq p < \infty$ and m is a positive integer, then, with equivalent norms,*

$$S_p^m[H_p(\mathcal{M})] \;=\; H_p(M_m \otimes \mathcal{M})\,. \tag{2.44}$$

Proof. We will first prove that we have isometrically

$$S_p^m[H_p^c(\mathcal{M})] \;=\; H_p^c(M_m \otimes \mathcal{M})\,. \tag{2.45}$$

By the injectivity of Pisier's non-commutative vector-valued L_p-spaces and Lemma 1.4.2, we obtain the isometric inclusion

$$\begin{aligned} S_p^m[H_p^c(\mathcal{M})] &\subseteq S_p^m[L_p(\mathcal{M} \otimes \mathcal{B}(l_2))] \\ &= L_p(M_m \otimes \mathcal{M} \otimes \mathcal{B}(l_2))\,. \end{aligned}$$

Recall that by definition $H_p^c(M_m \otimes \mathcal{M})$ consists of all $L_p(M_m \otimes \mathcal{M})$-martingales x such that

$$(\,\mathbb{E}_n(x) - \mathbb{E}_{n-1}(x))_{n\geq 1} \in L_p(M_m \otimes \mathcal{M}; l_2^c) \subseteq L_p(M_m \otimes \mathcal{M} \otimes \mathcal{B}(l_2))\,,$$

where $\mathbb{E}_n = Id_{M_m} \otimes \mathcal{E}_n\,, n \geq 1\,.$ Formula (2.45) follows then from the fact that for all $x = [x_{ij}] \in$

$M_m(\mathcal{M})$ and all $n \geq 1$,

$$[\, \mathcal{E}_n(x_{ij})\,]_{i,j=1}^m \quad = \quad \mathbb{E}_n(x)\,.$$

Similarly, the row version result holds isometrically, i.e.,

$$S_p^m[H_p^r(\mathcal{M})] \quad = \quad H_p^r(M_m \otimes \mathcal{M})\,. \tag{2.46}$$

If $2 \leq p < \infty$, then, as a consequence of (3.41) and the injectivity of the non-commutative L_p -spaces, together with (2.45) and (2.46), it follows that, isometrically,

$$\begin{aligned}
S_p^m[\tilde{H}_p(\mathcal{M})] &= S_p^m[H_p^c(\mathcal{M}) \cap_p H_p^r(\mathcal{M})] \\
&= S_p^m[H_p^c(\mathcal{M})] \cap_p S_p^m[H_p^r(\mathcal{M})] \\
&= H_p^c(M_m \otimes \mathcal{M}) \cap_p H_p^r(M_m \otimes \mathcal{M}) \\
&= \tilde{H}_p(M_m \otimes \mathcal{M})\,.
\end{aligned}$$

Moreover, by (2.41) it follows that $d_{cb}(S_p^m[H_p(\mathcal{M})]\,, H_p(M_m \otimes \mathcal{M})) \leq 2^{\frac{1}{p}}$.
If $1 \leq p < 2$, then, by (3.41), the projective property of the non-commutative L_p-spaces, together with (2.45) and (2.46) it follows that, isometrically,

$$\begin{aligned}
S_p^m[\tilde{H}_p(\mathcal{M})] &= S_p[\,[\, H_p^c(\mathcal{M}) \oplus_p H_p^r(\mathcal{M})\,]\,/N_p(H_p^c(\mathcal{M}), H_p^r(\mathcal{M}))] \\
&= S_p^m[H_p^c(\mathcal{M}) \oplus_p H_p^r(\mathcal{M})]/S_p^m[N_p(H_p^c(\mathcal{M}), H_p^r(\mathcal{M}))] \\
&= (S_p^m[H_p^c(\mathcal{M})] \oplus_p S_p^m[H_p^r(\mathcal{M})]\,)/S_p^m[N_p(H_p^c(\mathcal{M}), H_p^r(\mathcal{M}))] \\
&= (H_p^c(M_m \otimes \mathcal{M}) \oplus_p H_p^r(M_m \otimes \mathcal{M})\,)/S_p^m[N_p(H_p^c(\mathcal{M}), H_p^r(\mathcal{M}))] \\
&= (H_p^c(M_m \otimes \mathcal{M}) \oplus_p H_p^r(M_m \otimes \mathcal{M})\,)/N_p(H_p^c(M_m \otimes \mathcal{M}), H_p^r(M_m \otimes \mathcal{M})) \\
&= \tilde{H}_p(M_m \otimes \mathcal{M})\,.
\end{aligned}$$

Moreover, by (2.42) we obtain the estimate $d_{cb}(S_p^m[H_p(\mathcal{M})], H_p(M_m \otimes \mathcal{M})) \leq 2^{1+\frac{1}{p}}$. $\square$

Remark 2.3.8 We have with completely equivalent norms

$$BMO(\mathcal{M}) \;=\; \overline{H_1(\mathcal{M})}^{\,*}. \tag{2.47}$$

Note that in proving the duality between the non-commutative BMO and non-commutative Hardy space H_1, Pisier and Xu ([69], Appendix) used the duality bracket $\langle x, y\rangle = \tau(y^*x)$. By (2.40), Pisier and Xu's duality result, and Proposition 2.3.7, we have for all positive integers m

$$\begin{aligned} M_m(BMO(\mathcal{M})) &= BMO(M_m \otimes \mathcal{M}) \\ &= \overline{H_1(M_m \otimes \mathcal{M})}^{\,*} \\ &= \overline{S_1^m[\,H_1(\mathcal{M})\,]}^{\,*}. \end{aligned} \tag{2.48}$$

Using the definition of the operator space structure of the conjugate space (see [71]) and properties of the Haagerup tensor product (see [70], Theorem 1.5), we see that isometrically

$$\overline{S_1^m[\,H_1(\mathcal{M})\,]} \;=\; S_1^m[H_1(\mathcal{M})] \;=\; S_1^m[\,\overline{H_1(\mathcal{M})}\,].$$

By (1.47) we have the completely isometric identification

$$\left(S_1^m[\,\overline{H_1(\mathcal{M})}\,]\right)^* \;=\; M_m\left(\overline{H_1(\mathcal{M})}^{\,*}\right),$$

which, together with (4.60), completes the proof of (2.47).

Theorem 2.3.9 *Let X be either $L_\infty(\mathcal{M})$ or $BMO(\mathcal{M})$, and Y be either $L_1(\mathcal{M})$ or $H_1(\mathcal{M})$. If $1 < p < \infty$, then, with completely equivalent norms,*

$$[X, Y]_{\frac{1}{p}} \;=\; L_p(\mathcal{M}). \tag{2.49}$$

Proof. By [70], Lemma 1.7, it is enough to prove that for all positive integers m we have with equivalent norms

$$S_q^m[\,[X,Y]_{\frac{1}{q}}\,] \;=\; S_q^m[Y]\,.$$

By [70], Corollary 1.4, applied for $p_0 = \infty\,, p_1 = 1\,, p_\theta = q\,, \theta = \frac{1}{q}$, we have

$$S_q^m[\,[X,Y]_{\frac{1}{q}}\,] \;=\; [\,M_m(X), S_1^m[Y]\,]_{\frac{1}{q}}\,.$$

Let $X = BMO(\mathcal{M})$ and $Y = H_1(\mathcal{M})$. By (2.40), Proposition 2.3.7, Theorem 2.2.11 and Lemma 1.4.2 we obtain

$$\begin{aligned}
[M_m(BMO(\mathcal{M})), S_1^m[H_1(\mathcal{M})]]_{\frac{1}{q}} &= [BMO(M_m \otimes \mathcal{M}), H_1(M_m \otimes \mathcal{M})]_{\frac{1}{q}} \\
&= L_q(M_m \otimes \mathcal{M}) \\
&= S_q^m[L_q(\mathcal{M})]\,.
\end{aligned}$$

All the other cases follow with similar arguments. □

In [89] Xu developed the real interpolation theory for operator spaces and established connections with the complex method. Given any compatible couple of operator spaces (E_0, E_1) , four different operator space structures corresponding to the (discrete) K- and J- methods are introduced and shown that they are all equivalent. Moreover, it is proved in [89], Theorem 5.4 (ii), that for $1 \le p \le \infty$ and $0 < \theta_0\,, \theta_1, \theta, \eta < 1$ such that $\theta_0 \neq \theta_1$ and $\theta = (1-\eta)\theta_0 + \eta\theta_1$, the following holds with completely equivalent norms

$$[\,[E_0, E_1]_{\theta_0}, [E_0, E_1]_{\theta_1}\,]_{\eta,p} = [E_0, E_1]_{\theta,p}\,. \tag{2.50}$$

Remark 2.3.10 Using (2.34), we endow the non-commutative Lorentz spaces with operator space matrix norms obtained by interpolation. By the reiteration result (2.50), if $1 \le p_0, p_1, p, q \le \infty$ and $0 < \theta < 1$ with $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, then, with completely equivalent norms,

$$L_{p,q}(\mathcal{M},\tau) \;=\; [L_{p_0}(\mathcal{M},\tau), L_{p_1}(\mathcal{M},\tau)]_{\theta,q}\,. \tag{2.51}$$

Theorem 2.3.11 *Let X be either $L_\infty(\mathcal{M})$ or $BMO(\mathcal{M})$, and Y be either $L_1(\mathcal{M})$ or $H_1(\mathcal{M})$. If $1 \le p < q < \infty$ and $1 \le s \le \infty$, then, with completely equivalent norms,*

$$[X, Y]_{\frac{1}{p}, s} \quad = \quad L_{p,s}(\mathcal{M}) \,.$$

Proof. Let $1 < p_1 < p < p_2 < \infty$. There exists $0 < \eta < 1$ such that

$$\frac{1}{p} \quad = \quad \frac{1-\eta}{p_1} + \frac{\eta}{p_2} \,.$$

By (2.50), Theorem 2.3.9 and (2.51) we obtain with completely equivalent norms,

$$\begin{aligned} [X, Y]_{\frac{1}{p}, s} \quad &= \quad \big[\, [X, Y]_{\frac{1}{p_1}}, [X, Y]_{\frac{1}{p_2}} \,\big]_{\eta, s} \\ &= \quad [L_{p_1}(\mathcal{M}), L_{p_2}(\mathcal{M})]_{\eta, s} \\ &= \quad L_{p,s}(\mathcal{M}) \,. \end{aligned}$$

$\square$

Chapter 3

Operator Space UMD Property

3.1 Banach space UMD

In the classical Banach space-valued martingale theory, the notion of a UMD space (i.e., *unconditional for martingale differences*) plays an important role. We will first recall the definition.

Let B be a (real or complex) Banach space, $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space, and $(\mathcal{F}_n)_{n\geq 1}$ an increasing sequence of sub-σ-algebras of $\mathcal{F}$. The Banach space B is UMD if $\beta_p(B)$ is finite for some $1 < p < \infty$; equivalently, for all $1 < p < \infty$. Here $\beta_p(B)$ denotes the smallest $\beta > 0$ such that

$$\left\| \sum_{n=1}^{k} \varepsilon_n dx_n \right\|_p \quad \leq \quad \beta \left\| \sum_{n=1}^{k} dx_n \right\|_p , \tag{3.1}$$

for all for all positive integers k, all sequences $\varepsilon = (\varepsilon_n)_{n=1}^k$ of numbers in $\{-1, 1\}$ and all B-valued martingale difference sequences $dx = (dx_n)_{n=1}^k$ relative to the filtration $(\mathcal{F}_n)_{n\geq 1}$. In this definition, the filtration must vary, and the probability space must also vary, unless it is non-atomic.

The fact that the finiteness of $\beta_p(B)$ for some $1 < p < \infty$ implies its finiteness for all such p was first proved by Pisier; see Maurey [57].

Results of Burkholder [10] provided the first example of a UMD Banach space, namely, the real line $\mathbb{R}$. Moreover, Burkholder [14] proved that

$$\beta_p(\mathbb{R}) \quad = \quad p^* - 1 , \tag{3.2}$$

where $p^* = \max\{p, p/(p-1)\}$, and, moreover, that this equality holds for any Hilbert space, e.g.

the complex plane $\mathbb{C}$.

If B is a UMD Banach space, the same holds for subspaces and quotients of B, for its dual B^*, as well as for the vector-valued spaces l_p^B and, respectively, $L_p^B[0,1]$, for all $1 < p < \infty$. Aldous [1] showed that if $1 < p < \infty$ and $L_p^B[0,1]$ has an unconditional basis, then B is UMD. Conversely, if B is a UMD Banach space with an unconditional basis, then, for $1 < p < \infty$, the space $L_p^B[0,1]$ has an unconditional basis. The UMD property is preserved by complex, as well as real interpolation, i.e., if (B_0, B_1) is a compatible couple of UMD Banach spaces, then the interpolation spaces $[B_0, B_1]_\theta$ and $[B_0, B_1]_{\theta,p}$ are UMD, for $0 < \theta < 1$ and $1 < p < \infty$. Other examples of UMD Banach spaces include the Schatten p-classes, for $1 < p < \infty$ (Gutiérrez [37], Bourgain [8]) and the non-commutative $L_p(\mathcal{M}, \tau)$-spaces associated with a von Neumann algebra $\mathcal{M}$, equipped with an n.s.f. trace τ (Berkson, Gillespie and Muhly [4]). The spaces l_1, l_∞, $L_1[0,1]$ and $L_\infty[0,1]$ are not UMD. It was proved by Aldous [1] (see also Maurey [57]) that UMD Banach spaces are superreflexive, hence, in particular, reflexive. We refer to the Appendix A.1 for the definition of superreflexivity and connections to ultraproducts. However, superreflexivity is not a sufficient condition for a Banach space to be UMD; see the counterexamples constructed by Pisier [65], and, respectively, by Bourgain [7].

Deep connections with the boundedness of certain singular integral operators, such as the Hilbert transform, were established through the work of Burkholder, McConnell and Bourgain. Namely, Burkholder and McConnell (see [12]) proved that if a Banach space B is UMD, then the Hilbert transform is a bounded operator on $L_p^B[0,1]$, for $1 < p < \infty$. Later, Bourgain [7] showed that, conversely, the boundedness of the Hilbert transform on $L_p^B[0,1]$ ($1 < p < \infty$) implies that B is UMD.

There is a geometric property that characterizes the UMD Banach spaces. Burkholder [11], [13] proved that a Banach space B is UMD if and only if there exists a biconvex function $\zeta : B \times B \to \mathbb{R}$ such that $\zeta(0,0) > 0$ and

$$\zeta(x,y) \leq \|x+y\|$$

if $\|x\| = \|y\| = 1$. This provides a new proof for the independence of p in the UMD property. It also shows that it is enough to consider in the definition of UMD only the dyadic filtration of the probability space $([0,1], \mu)$, where μ is the Lebesgue measure on $[0,1]$.

We refer to Burkholder [15] for more properties of UMD Banach spaces, connections to other topics and further references.

3.2 Operator space UMD$_p$: definitions and properties

Let $(\mathcal{M}, \tau)$ be a von Neumann algebra equipped with a normal, faithful (abbreviated as n.f.) tracial state τ. Let $(\mathcal{M}_n)_{n\geq 1}$ be an increasing filtration of von Neumann subalgebras of $\mathcal{M}$ such that $\mathcal{M} = (\bigcup_n \mathcal{M}_n)^{-w^*}$. For all positive integers n, denote by $\mathcal{E}_n : \mathcal{M} \to \mathcal{M}_n$ the unique trace-preserving conditional expectation onto $\mathcal{M}_n$.

Proposition 3.2.1 *Let $1 < p < \infty$. There exists $c_p > 0$, depending only on p, such that*

$$\| T_\varepsilon : L_p(\mathcal{M}, \tau) \to L_p(\mathcal{M}, \tau) \|_{cb} \;\leq\; c_p \,,$$

where ε denotes a sequence $(\varepsilon_n)_{n\geq 1}$ of numbers in $\{-1, 1\}$ and T_ε is the ± 1 martingale transform generated by ε, i.e., $T_\varepsilon \left(\sum_{n=1}^{k} dx_n \right) = \sum_{n=1}^{k} \varepsilon_n dx_n$, for all positive integers k and all $L_p(\mathcal{M})$-martingale difference sequences $dx = (dx_n)_{n=1}^k$ relative to the filtration $(\mathcal{M}_n)_{n\geq 1}$.

Proof. As a consequence of the Pisier-Xu non-commutative version of the Burkholder-Gundy square function inequalities (1.11), there exists a constant $c_p > 0$, depending only on p, such that

$$\left\| \sum_{n=1}^{k} \varepsilon_n dx_n \right\|_p \;\leq\; c_p \left\| \sum_{n=1}^{k} dx_n \right\|_p , \tag{3.3}$$

for all positive integers k, all sequences $\varepsilon = (\varepsilon_n)_{n=1}^k$ of numbers in $\{-1, 1\}$ and all $L_p(\mathcal{M})$-martingale difference sequences $dx = (dx_n)_{n=1}^k$ relative to the filtration $(\mathcal{M}_n)_{n\geq 1}$. Equivalently,

$$\| T_\varepsilon : L_p(\mathcal{M}, \tau) \to L_p(\mathcal{M}, \tau) \| \;\leq\; c_p \,. \tag{3.4}$$

By Lemma 1.7 of [70], we have

$$\|T_\varepsilon\|_{cb} \;=\; \sup_m \| \mathrm{Id}_{S_p^m} \otimes T_\varepsilon : S_p^m[L_p(\mathcal{M}, \tau)] \to S_p^m[L_p(\mathcal{M}, \tau)] \| \,. \tag{3.5}$$

Let $m \geq 2$. We will prove that

$$\| \mathrm{Id}_{M_m} \otimes T_\varepsilon : S_p^m[L_p(\mathcal{M},\tau)] \to S_p^m[L_p(\mathcal{M},\tau)] \| \quad \leq \quad c_p \,. \tag{3.6}$$

By Fubini's theorem (Theorem 1.5.3) we have the isometry

$$S_p^m[L_p(\mathcal{M},\tau)] \quad = \quad L_p(M_m \otimes \mathcal{M}, \mathrm{tr} \otimes \tau) \,,$$

where tr is the standard normalized trace on M_m. Note that $(M_m \otimes \mathcal{M}_n)_{n \geq 1}$ is a filtration of the algebra $M_m(\mathcal{M}) = M_m \otimes \mathcal{M}$. For all positive integers n, denote $\mathrm{Id}_{M_m} \otimes \mathcal{E}_n$ by $\mathbb{E}_n$. Then $\mathbb{E}_n : M_m(\mathcal{M}) \to M_m(\mathcal{M}_n)$ is the unique trace preserving conditional expectation onto $M_m(\mathcal{M}_n)$. For all $x = [x_{ij}]_{i,j=1}^m \in M_m(L_p(\mathcal{M},\tau))$ and all positive integers n we have

$$[\mathcal{E}_n(x_{ij})]_{i,j=1}^m \quad = \quad \mathbb{E}_n(x) \,. \tag{3.7}$$

Therefore, by applying (3.3), together with (3.7) to the algebra $M_m(\mathcal{M})$ and its filtration $(M_m \otimes \mathcal{M}_n)_{n \geq 1}$, it follows that for all positive integers k and all sequences $\varepsilon = (\varepsilon_n)_{n=1}^k$ of numbers in $\{-1,1\}$ we have

$$\left\| \sum_{n=1}^k (\mathrm{Id}_{M_m} \otimes \varepsilon_n(\mathcal{E}_n - \mathcal{E}_{n-1}))(x) \right\|_p \quad \leq \quad c_p \left\| \sum_{n=1}^k (Id_{M_m} \otimes (\mathcal{E}_n - \mathcal{E}_{n-1}))(x) \right\|_p \,.$$

This shows that (3.6) holds and the proof is complete. □

Remark 3.2.2 For a sequence $\varepsilon = (\varepsilon_n)_{n \geq 1}$ of numbers in $\{-1,1\}$, the ± 1 martingale transform T_ε generated by ε is a self adjoint operator on $L_p(\mathcal{M},\tau)$ under the trace duality bracket (1.1), for $1 < p < \infty$. The proof of this fact is similar to the one in the classical setting, due to Burkholder [10]. The key fact is that $\left(\bigcup_n L_p(\mathcal{M}_n, \tau_n) \right)^{-\|\cdot\|_p} = L_p(\mathcal{M},\tau)$. By (3.4), T_ε is a bounded linear operator on $L_p(\mathcal{M},\tau)$. Therefore, it suffices to prove that for all positive integers n, the restriction $T_\varepsilon \restriction L_p(\mathcal{M}_n, \tau_n)$ is self adjoint. Indeed, if $x \in L_p(\mathcal{M}_n, \tau_n)$, then $x = \sum_{k=1}^n d_k(x)$, where $d_k = \mathcal{E}_k - \mathcal{E}_{k-1}$, with $\mathcal{E}_0 = 0$. With a similar argument as in the case p=1 (discussed in the proof of Proposition 1.5.4), it follows that $\mathcal{E}_j$ is a self-adjoint operator on $L_p(\mathcal{M},\tau)$, for all positive integers

j . Thus, each d_k is a self-adjoint operator with respect to the trace duality, and we have

$$T_\varepsilon(x) \quad = \quad \sum_{k=1}^{n} \varepsilon_k d_k(x)\,,$$

which yields the conclusion.

For the remainder of this section we will assume that $\mathcal{M}$ is a *hyperfinite* von Neumann algebra, equipped with an n.f. tracial state τ .

Definition 3.2.3 Let E be an operator space and $1 < p < \infty$. We say that E is UMD$_p$ *with respect to the filtration* $(\mathcal{M}_n)_{n\geq 1}$ if there exists a constant $c_p > 0$ such that

$$\left\| \sum_{n=1}^{k} \varepsilon_n dx_n \right\|_{L_p(\mathcal{M};E)} \quad \leq \quad c_p \left\| \sum_{n=1}^{k} dx_n \right\|_{L_p(\mathcal{M};E)} \tag{3.8}$$

for all positive integers k , all sequences $\varepsilon = (\varepsilon_n)_{n=1}^k$ of numbers in $\{-1,1\}$ and all martingale difference sequences $dx = (d_n)_{n=1}^k \subset L_p(\mathcal{M};E)$, relative to the filtration $(\mathcal{M}_n)_{n\geq 1}$.

If this holds for all hyperfinite von Neumann algebras $\mathcal{M}$, equipped with an n.f. tracial state τ , and all filtrations of $\mathcal{M}$, we say that E is UMD$_p$.

Remark 3.2.4 By Definition 3.2.3, E is UMD$_p$ *with respect to the filtration* $(\mathcal{M}_n)_{n\geq 1}$ if and only if there exists a constant $c_p > 0$, depending on p and E, such that

$$\| \, T_\epsilon \otimes \mathrm{Id}_E : L_p(\mathcal{M};E) \to L_p(\mathcal{M};E) \, \| \quad \leq \quad c_p\,, \tag{3.9}$$

for all finite sequences $\varepsilon = (\varepsilon_n)_{n\geq 1}$ of numbers in $\{-1,1\}$. Martingale differences are considered with respect to the filtration $(\mathcal{M}_n)_{n\geq 1}$. Note that a priori the constant c_p might also depend on the algebra $\mathcal{M}$ and on the filtration.

Lemma 3.2.5 *If E is UMD$_p$, there exists $c_p > 0$, depending only on p and E, such that*

$$\| \, T_\epsilon \otimes Id_E : L_p(\mathcal{M};E) \to L_p(\mathcal{M};E) \, \| \quad \leq \quad c_p\,, \tag{3.10}$$

for all hyperfinite von Neumann algebras $(\mathcal{M},\tau)$, equipped with an n.f. tracial state τ , all filtrations *of $\mathcal{M}$ and all finite sequences $\varepsilon = (\varepsilon_n)_{n\geq 1}$ of numbers in $\{-1,1\}$.*

Proof. Assume that there exists a sequence $(\mathcal{M}_k, \tau_k)_{k\geq 1}$ of hyperfinite von Neumann algebras, equipped with n.f. tracial states τ_k, with the property that the sequence of corresponding constants $c_p^{(k)}$ converges to ∞. We will show that this leads to a contradiction. Let $\mathcal{M} = \oplus_k M_k$ be the direct sum of the algebras $\mathcal{M}_k$. Note that $\mathcal{M}$ is a hyperfinite von Neumann algebra and that

$$\tau((x_k)_{k\geq 1}) \;=\; \sum_{k\geq 1} \frac{1}{2^k}\, \tau_k(x_k)$$

defines a tracial state on $\mathcal{M}$.

For each positive integer k, since the algebra $\mathcal{M}_k$ is hyperfinite, it follows that $\mathcal{M}_k = \left(\bigcup_n \mathcal{M}_k^{(n)}\right)^{-w^*}$, where $(\mathcal{M}_k^{(n)})_{n\geq 1}$ is an increasing net of finite dimensional algebras. For a finite sequence $\varepsilon = (\varepsilon_n)_{n\geq 1}$ of numbers in $\{-1,1\}$, let $T_\varepsilon^{(k)}$ denote the vector-valued ± 1 martingale transform generated by ε, where martingale differences are considered with respect to the algebra $\mathcal{M}_k$ and its filtration $(\mathcal{M}_k^{(n)})_{n\geq 1}$. For all positive integers n, let

$$\mathcal{M}^{(n)} \;=\; \oplus_k \mathcal{M}_k^{(n)}\,.$$

Then $(\mathcal{M}^{(n)})_{n\geq 1}$ is a filtration of $\mathcal{M}$. Let T_ε denote the ± 1 martingale transform generated by ε, where martingales are considered with respect to the algebra $\mathcal{M}$ and its filtration $(\mathcal{M}^{(n)})_{n\geq 1}$. By our assumption it follows that for all positive integers k, there exists $y_k \in L_p(\mathcal{M}_k; E)$ such that $\|y_k\|_{L_p(\mathcal{M}_k;E)} = 1$ and

$$\|\,(T_\varepsilon^{(k)} \otimes \mathrm{Id}_E)(y_k)\,\|_{L_p(\mathcal{M}_k;E)} \geq c_p^{(k)}\,.$$

Let $Y_k = (0,\ldots,0,\underbrace{y_k}_{k^{\text{th}}},0,\ldots)$. It follows that

$$\begin{aligned}
\|\,(T_\varepsilon \otimes \mathrm{Id}_E)(Y_k)\,\|_{L_p(\mathcal{M}_k;E)} \;&=\; \left(\frac{1}{2^k}\tau_k(\,|\,(T_\varepsilon^{(k)} \otimes \mathrm{Id}_E)(y_k)\,|^p\,)\right)^{\frac{1}{p}} \\
&=\; \frac{1}{2^{k/p}}\;\|\,(T_\varepsilon^{(k)} \otimes \mathrm{Id}_E)(y_k)\,\|_{L_p(\mathcal{M}_k;E)} \\
&\geq\; \frac{1}{2^{k/p}}\;c_p^{(k)}\,.
\end{aligned}$$

Moreover, we have

$$
\begin{aligned}
\| Y_k \|_{L_p(\mathcal{M};E)} &= \left(\frac{1}{2^k} \, \tau_k(\, |y_k|^p \,) \right)^{\frac{1}{p}} \\
&= \frac{1}{2^{k/p}} \, \|y_k\|_{L_p(\mathcal{M}_k;E)} \\
&= \frac{1}{2^{k/p}} \, .
\end{aligned}
$$

This shows that

$$\| T_\varepsilon \otimes \mathrm{Id}_E : L_p(\mathcal{M};E) \to L_p(\mathcal{M};E) \| \geq c_p^{(k)} \, ,$$

for all positive integers k . Thus, $\| T_\varepsilon \otimes \mathrm{Id}_E : L_p(\mathcal{M};E) \to L_p(\mathcal{M};E) \| = \infty$. Hence, if the assertion made in Lemma 3.2.5 is not true, then there is a hyperfinite von Neumann algebra $(\mathcal{M}, \tau)$ equipped with an n.f. tracial state τ , and a filtration $(\mathcal{M}^{(n)})_{n\geq 1}$ of $\mathcal{M}$ for which the corresponding constant is ∞. This contradicts the assumption that E is UMD_p . □

We denote by $c_p(V)$ the smallest constant $c_p > 0$ satisfying (3.10).

Remark 3.2.6 With a similar argument as in the proof of Proposition 3.2.1 it follows that an operator space E is UMD_p , for some $1 < p < \infty$ if and only if

$$\| T_\varepsilon \otimes \mathrm{Id}_E : L_p(\mathcal{M};E) \to L_p(\mathcal{M};E) \|_{cb} \;\; \leq \;\; c_p(E) \, , \tag{3.11}$$

for all hyperfinite von Neumann algebras $(\mathcal{M}, \tau)$, equipped with an n.f. tracial state τ, all filtrations of $\mathcal{M}$ and all finite sequences $\varepsilon = (\varepsilon_n)_{n\geq 1}$ of numbers in $\{-1, 1\}$.

Remark 3.2.7 Note that if in the Definition 3.2.3 we assume that $\mathcal{M}$ is a commutative von Neumann algebra, then we are back to the classical notion of a UMD Banach space. In particular, if an operator space E is UMD_p , for some $1 < p < \infty$, then E is UMD (as a Banach space) and, moreover, we have (see Section 2.1)

$$p^* - 1 \, = \, \beta_p(\mathbb{C}) \, \leq \, \beta_p(E) \, \leq \, c_p(E) \, .$$

Remark 3.2.8 By Proposition 3.2.1 and Remark 3.2.6, it follows that the operator space $E = \mathbb{C}$

is UMD$_p$, for $1 < p < \infty$. Moreover, as proved by Randrianantoanina [74], the corresponding constant $c_p(\mathbb{C})$ has the same optimal order of growth as $\beta_p(\mathbb{C})$, namely,

$$c_p(\mathbb{C}) \approx O(p) \quad \text{as} \quad p \to \infty\,.$$

The estimate follows directly by interpolation from a striking result proved in [74], namely the fact that the non-commutative ± 1 martingale transforms are of weak type $(1,1)$.

Remark 3.2.9 By Definition 3.2.3 it follows that UMD$_p$ is a local property, i.e., if for an increasing sequence of closed subspaces $(E_k)_{k\geq 1}$ of E , whose union generates E , we know that each E_k is UMD$_p$, and the corresponding constants satisfy $\sup_k c_p(E_k) < \infty$, then E is UMD$_p$.

The following proposition summarizes some of the properties of UMD$_p$ operator spaces. We refer to the Appendix A.1 for background on the ultraproduct theory for operator spaces.

Proposition 3.2.10 *Let $1 < p < \infty$ and E be an operator space.*

1. *If E is UMD$_p$ and F is an operator space completely isomorphic to E , then F is UMD$_p$.*

2. *If E is UMD$_p$ and F is a closed subspace of E , then both F and E/F are UMD$_p$.*

3. *If E is UMD$_p$, then $L_p(\mathcal{M}; E)$ is UMD$_p$, for all hyperfinite von Neumann algebras $\mathcal{M}$, equipped with an n.f. tracial state τ .*

4. *If E is UMD$_p$, then $M_m(E)$ is UMD$_p$, for all positive integers m .*

5. *If E is UMD$_p$, then its dual E^* is UMD$_{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$.*

6. *Let $1 < q, r < \infty$ and $0 < \theta < 1$ be such that $\frac{1-\theta}{q} + \frac{\theta}{r} = \frac{1}{p}$. If (E_0, E_1) is a compatible couple of operator spaces, where E_0 is UMD$_q$ and E_1 is UMD$_r$, then $E_\theta = [E_0, E_1]_\theta$ is UMD$_p$.*

7. *Let $(E_i)_{i\in I}$ be a family of UMD$_p$ operator spaces, where I is an index set. Assume that the corresponding UMD$_p$ constants satisfy $\sup_i c_p(E_i) < \infty$. Let $\mathcal{U}$ be an ultrafilter on I. Then the ultraproduct $\hat{E} = (E_i)_{\mathcal{U}}$ is UMD$_p$.*

8. *If E and F are UMD$_p$, then $E \oplus_q F$ is UMD$_p$, for $1 < q < \infty$.*

Proof. (1) Let $(\mathcal{M}, \tau)$ be a hyperfinite von Neumann algebra equipped with an n.f. tracial state τ, and $(\mathcal{M}_n)_{n\geq 1}$ a filtration of $\mathcal{M}$. Since E is UMD$_p$, it follows by Remark 3.2.6 that there exists a constant $c_p(E) > 0$, such that

$$\| T_\varepsilon \otimes \mathrm{Id}_E : L_p(\mathcal{M}; E) \to L_p(\mathcal{M}; E) \|_{cb} \quad \leq \quad c_p(E) . \tag{3.12}$$

By the assumption, there is a complete isomorphism $u : E \to F$. As observed in (3.1) of [70], the map $\mathrm{Id}_{L_p(\mathcal{M},\tau)} \otimes u$ extends to a complete isomorphism

$$\tilde{u} : L_p(\mathcal{M}; E) \quad \to \quad L_p(\mathcal{M}; F) . \tag{3.13}$$

Hence, by (3.12) and (3.13), there exists a constant $c_p'(F)$, depending only on p and F, such that

$$\| T_\varepsilon \otimes \mathrm{Id}_F : L_p(\mathcal{M}; F) \to L_p(\mathcal{M}; F) \|_{cb} \quad \leq \quad {c'}_p(F) . \tag{3.14}$$

By Remark 3.2.6, this implies that F is UMD$_p$.

(2) Let $(\mathcal{M}, \tau)$ be a hyperfinite von Neumann algebra equipped with an n.f. tracial state τ, and $(\mathcal{M}_n)_{n\geq 1}$ a filtration of $\mathcal{M}$. Since E is UMD$_p$, it follows by Remark 3.2.6 that there exists a constant $c_p(E) > 0$, such that (3.12) holds. Since F is a closed subspace of E, then $L_p(\mathcal{M}; F)$ can be identified, completely isometrically, with a closed subspace of $L_p(\mathcal{M}; E)$. Thus, we have

$$\| T_\varepsilon \otimes \mathrm{Id}_F : L_p(\mathcal{M}; F) \to L_p(\mathcal{M}; F) \|_{cb} \quad \leq \quad c_p(E) . \tag{3.15}$$

Moreover, as observed in [70] (see (3.4)), we have the complete isometry

$$L_p(\mathcal{M}; E/F) \;\cong\; L_p(\mathcal{M}; E)/L_p(\mathcal{M}; F)\,.$$

Together with (3.12), this shows that

$$\| T_\varepsilon \otimes \mathrm{Id}_{E/F} : L_p(\mathcal{M}; E/F) \to L_p(\mathcal{M}; E/F) \|_{cb} \;\le\; c_p(E)\,. \tag{3.16}$$

By (3.15) and, respectively, (3.16) it follows from Remark 3.2.6 that F and E/F are UMD_p , and, moreover, we have following estimates for the corresponding UMD_p constants

$$c_p(F) \le c_p(E) \quad \text{and} \quad c_p(E/F) \le c_p(E)\,. \tag{3.17}$$

(3) This is an application of Fubini's theorem (see Theorem 1.5.3)). Indeed, let $(\mathcal{N}, \phi)$ be a hyperfinite von Neumann algebra, equipped with an n.f. tracial state ϕ , and $(\mathcal{N}_n)_{n\ge1}$ a filtration of $\mathcal{N}$. We will show that $L_p(\mathcal{M}; E)$ is UMD_p with respect to the algebra $\mathcal{N}$ and its filtration $(\mathcal{N}_n)_{n\ge1}$. By Fubini's theorem, we have a complete isometry

$$L_p(\mathcal{N}; L_p(\mathcal{M}; E)) \;\cong\; L_p(\mathcal{N} \otimes \mathcal{M}; E)\,. \tag{3.18}$$

The fact that E is UMD_p with respect to the hyperfinite algebra $\mathcal{N} \otimes \mathcal{M}$ and its filtration $(\mathcal{N}_n \otimes \mathcal{M})_{n\ge1}$, together with (3.18), yields the conclusion. Moreover, from the proof we obtain the following estimate for the corresponding UMD_p constants

$$c_p(L_p(\mathcal{N}; E)) \;\le\; c_p(E)\,. \tag{3.19}$$

(4) We will prove that $M_m(E) = M_m \check{\otimes} E$ is completely isomorphic to $S_p^m[E] = L_p(M_m; E)$. Equivalently,

$$d_{cb}(\, M_m(E), L_p(M_m; E)\,) \;<\; \infty\,,$$

where, given operator spaces E_0 and E_1, their *c.b.-Banach-Mazur distance* $d_{cb}(E_0, E_1)$ is defined by

$$d_{cb}(E_0, E_1) \;=\; \inf\{\|\phi\|_{cb}\|\phi^{-1}\|_{cb} : \phi : E_0 \to E_1 \text{ is an isomorphism}\}\,.$$

As explained in Section 1.5, we have a complete contraction

$$\tilde{v} : M_m(E) \;\hookrightarrow L_1(M_m)\hat{\otimes} E\,. \tag{3.20}$$

Moreover, the map $\tilde{v}$ is $1-1$ (see [70]). From (1.75) and (3.20) it follows by interpolation that

$$\|\tilde{v} : M_m(E) \to L_p(M_m; E)\|_{cb} \;\leq\; 1\,. \tag{3.21}$$

It remains to show that $\tilde{v}^{-1} \in \mathcal{CB}(L_p(M_m; E)\,, M_m(E))$. By results of Effros and Ruan [29], the map $\tilde{v}^{-1} : L_1(M_m)\hat{\otimes} E \to M_m \check{\otimes} E$ is completely bounded if and only if the map $v^{-1} : L_1(M_m) \to M_m$ (see the notation in Section 1.5) is nuclear, and, moreover,

$$\|\, \tilde{v}^{-1} : L_1(M_m)\hat{\otimes} E \to M_m \check{\otimes} E \,\|_{cb} \;\leq\; \nu(v^{-1})\,. \tag{3.22}$$

Indeed, let $\{e_{i,j}, e^*_{i,j}\}_{1\leq i,j\leq m} \in M_m \times M^*_m$ be an Auerbach basis for M_m. Then

$$\begin{aligned}
\nu(v^{-1}) \;&\leq\; \|\sum_{i,j=1}^{m} e^*_{i,j}(e_{i,j})\| \\
&\leq\; \sum_{i,j=1}^{m} \|e^*_{i,j}(e_{i,j})\| \;\leq\; m^2\,.
\end{aligned}$$

From (3.22) and the fact that $\|\, \tilde{v}^{-1} : M_m(E) \to M_m(E) \,\|_{cb} \leq 1$, it follows by interpolation that

$$\|\, \tilde{v}^{-1} : L_p(M_m; E) \to M_m(E) \,\|_{cb} \;\leq\; m^{\frac{2}{p}}\,. \tag{3.23}$$

Together with (3.20), this implies that

$$d_{cb}(\, M_m(E), L_p(M_m; E)\,) \;\leq\; m^{\frac{2}{p}} \;<\; \infty\,.$$

By properties (1) and (3) it follows that $M_m(E)$ is UMD_p.

(5) Let $(\mathcal{M},\tau)$ be a hyperfinite von Neumann algebra equipped with an n.f. tracial state τ, and $(\mathcal{M}_n)_{n\geq 1}$ a filtration of $\mathcal{M}$. Let $c_p(E)$ be the UMD_p constant of V. We will show that

$$\| T_\varepsilon \otimes \mathrm{Id}_{E^*} : L_{p'}(\mathcal{M}; E^*) \to L_{p'}(\mathcal{M}; E^*) \|_{cb} \quad \leq \quad c_p(E)\,. \tag{3.24}$$

Since $\mathcal{M}$ is a hyperfinite von Neumann algebra, it follows by the definition that there exists a filtration $(\mathcal{N}_\alpha)_{\alpha\geq 1}$ of *finite dimensional* von Neumann subalgebras of $\mathcal{M}$. For all $\alpha \geq 1$, denote by $\tilde{\mathcal{E}_\alpha}$ the unique trace-preserving conditional expectation from $\mathcal{M}$ onto $\mathcal{N}_\alpha$. By Remark 3.2 in [70] (see also Theorem 3.4. in [70]) it follows that $\left(\bigcup_\alpha L_p(\mathcal{N}_\alpha; E^*)\right)^{-\|\cdot\|_p} = L_p(\mathcal{M}; E^*)$. This implies that

$$\| T_\varepsilon \otimes \mathrm{Id}_{E^*} \|_{cb} \quad \leq \quad \sup_\alpha \| \tilde{\mathcal{E}_\alpha}(T_\varepsilon \otimes \mathrm{Id}_{E^*})\tilde{\mathcal{E}_\alpha} \|_{cb}\,. \tag{3.25}$$

Note that, for all $\alpha \geq 1$,

$$\tilde{\mathcal{E}_\alpha}(T_\varepsilon \otimes \mathrm{Id}_{E^*})\tilde{\mathcal{E}_\alpha} : L_{p'}(\mathcal{N}_\alpha; E^*) \quad \to \quad L_{p'}(\mathcal{N}_\alpha; E^*)\,, \tag{3.26}$$

and, moreover, by the assumption on the von Neumann algebras $\mathcal{N}_\alpha$ and Proposition 1.5.6, we have the following complete isometry, which holds under trace duality

$$L_{p'}(\mathcal{N}_\alpha; E^*) \quad = \quad (L_p(\mathcal{N}_\alpha^{\mathrm{op}}; E))^*\,. \tag{3.27}$$

Since $\left(\bigcup_\alpha L_p(\mathcal{N}_\alpha^{\mathrm{op}}; E)\right)^{-\|\cdot\|_p} = L_p(\mathcal{M}^{\mathrm{op}}; E)$ and E is UMD_p (so, in particular E is UMD_p with respect to $\mathcal{M}^{\mathrm{op}}$), it follows that

$$\| T_\varepsilon \otimes \mathrm{Id}_E \|_{cb} \quad \leq \quad \sup_\alpha \| \tilde{\mathcal{E}_\alpha}(T_\varepsilon \otimes \mathrm{Id}_E)\tilde{\mathcal{E}_\alpha} \|_{cb} \;\leq\; c_p(E)\,, \tag{3.28}$$

where, for all $\alpha \geq 1$,

$$\tilde{\mathcal{E}_\alpha}(T_\varepsilon \otimes \mathrm{Id}_E)\tilde{\mathcal{E}_\alpha} : L_p(\mathcal{N}_\alpha^{\mathrm{op}}; E) \quad \to \quad L_p(\mathcal{N}_\alpha^{\mathrm{op}}; E)\,. \tag{3.29}$$

By passing to the dual and using the fact that $\tilde{\mathcal{E}_\alpha}$ are self-dual operators (under trace duality), we obtain

$$\| \tilde{\mathcal{E}_\alpha}(T_\varepsilon \otimes \mathrm{Id}_E)^* \tilde{\mathcal{E}_\alpha} : L_p(\mathcal{N}_\alpha^{\mathrm{op}}; E)^* \to L_p(\mathcal{N}_\alpha^{\mathrm{op}}; E)^* \|_{cb} \quad \leq \quad c_p(E)\,. \tag{3.30}$$

Recall that, by Remark 3.2.2, T_ε is a self-dual operator., with respect to trace duality. Therefore, we conclude from (3.26), (3.27) and (3.30) that

$$\| \tilde{\mathcal{E}_\alpha}(T_\varepsilon \otimes \mathrm{Id}_{E^*}) \tilde{\mathcal{E}_\alpha} : L_{p'}(\mathcal{N}_\alpha; E^*) \;\to\; L_{p'}(\mathcal{N}_\alpha; E^*) \|_{cb} \;\leq\; c_p(E)\,.$$

Together with (3.25), this yields (3.24). By Remark 3.2.6, this implies that E^* is $\mathrm{UMD}_{p'}$, and, moreover, we have the following estimate for the corresponding $\mathrm{UMD}_{p'}$ constant

$$c_{p'}(E^*) \quad \leq \quad c_p(E)\,. \tag{3.31}$$

(6) This follows immediately from the characterization of UMD_p in Remark 3.2.6, together with the following completely isometric identity (cf. (3.5) in [70]) :

$$[L_q(\mathcal{M}; E_0), L_r(\mathcal{M}; E_1)]_\theta \quad = \quad L_p(\mathcal{M}; E_\theta)\,.$$

Moreover, by (1.19), we have the following estimate for the corresponding constants

$$c_p(E_\theta) \quad \leq \quad [c_q(E_0)]^{1-\theta}[c_r(E_1)]^\theta\,. \tag{3.32}$$

(7) Let $(\mathcal{M}, \tau)$ be a hyperfinite von Neumann algebra equipped with an n.f. tracial state τ, and $(\mathcal{M}_n)_{n\geq 1}$ a filtration of $\mathcal{M}$. We will show that there exists a constant $c_p(\hat{E}) > 0$ such that

$$\| T_\varepsilon \otimes \mathrm{Id}_{\hat{E}} : L_p(\mathcal{M}; \hat{E}) \to L_p(\mathcal{M}; \hat{E}) \|_{cb} \quad \leq \quad c_p(\hat{E})\,, \tag{3.33}$$

for all finite sequences ε of numbers in $\{-1, 1\}$. Martingale differences are considered with respect to the filtration $(\mathcal{M}_n)_{n\geq 1}$. We will use a similar idea as in the proof of (5) above.

Since $\mathcal{M}$ is a hyperfinite von Neumann algebra, it follows by the definition that there exists a filtration $(\mathcal{N}_\alpha)_{\alpha\geq 1}$ of *finite dimensional* von Neumann subalgebras of $\mathcal{M}$. For all $\alpha \geq 1$, let $m_\alpha = \dim \mathcal{N}_\alpha$, and denote by $\tilde{\mathcal{E}_\alpha}$ the unique trace-preserving conditional expectation from $\mathcal{M}$ onto $\mathcal{N}_\alpha$. By Remark 3.2 in [70] (see also Theorem 3.4. in [70]) it follows that $\left(\bigcup_\alpha L_p(\mathcal{N}_\alpha;\hat{E})\right)^{-\|\cdot\|_p} = L_p(\mathcal{M};\hat{E})$. This implies that

$$\| T_\varepsilon \otimes \mathrm{Id}_{\hat{E}} \|_{cb} \quad \leq \quad \sup_\alpha \| \tilde{\mathcal{E}_\alpha}(T_\varepsilon \otimes \mathrm{Id}_{\hat{E}})\tilde{\mathcal{E}_\alpha} \|_{cb} \,. \tag{3.34}$$

Note that, for all $\alpha \geq 1$,

$$\tilde{\mathcal{E}_\alpha}(T_\varepsilon \otimes \mathrm{Id}_{\hat{E}})\tilde{\mathcal{E}_\alpha} : L_p(\mathcal{N}_\alpha;\hat{E}) \quad \to \quad L_p(\mathcal{N}_\alpha;\hat{E})\,,$$

and, moreover, by the assumption on the von Neumann algebras $\mathcal{N}_\alpha$, we have

$$L_p(\mathcal{N}_\alpha;E_i) \quad = \quad S_p^{m_\alpha}[\,E_i\,]\,, \quad i \in I\,, \tag{3.35}$$

$$L_p(\mathcal{N}_\alpha;\hat{E}) \quad = \quad S_p^{m_\alpha}[\,\hat{E}\,]\,. \tag{3.36}$$

By Lemma A.1.6, there is a complete isometry

$$\phi_\alpha : S_p^{m_\alpha}[\,\hat{E}\,] \quad \to \quad (S_p^{m_\alpha}[E_i])_{\mathcal{U}}\,.$$

By the assumption, for all $i \in I$ the operator space E_i is UMD_p. This implies by (3.35) that the map $\tilde{\mathcal{E}_\alpha}(T_\varepsilon \otimes \mathrm{Id}_{E_i})\tilde{\mathcal{E}_\alpha} : S_p^{m_\alpha}[\,E_i\,] \to S_p^{m_\alpha}[\,E_i\,]$ is completely bounded. Moreover, since $\tilde{\mathcal{E}_\alpha}$ are complete contractions, we obtain

$$\| \tilde{\mathcal{E}_\alpha}(T_\varepsilon \otimes \mathrm{Id}_{E_i})\tilde{\mathcal{E}_\alpha} \|_{cb} \;\leq\; \| T_\varepsilon \otimes \mathrm{Id}_{E_i} \|_{cb} \;\leq\; c_p(E_i)\,. \tag{3.37}$$

Define a map ψ_α by

$$\psi_\alpha \quad = \quad \Big(\tilde{\mathcal{E}_\alpha}(T_\varepsilon \otimes \mathrm{Id}_{E_i})\tilde{\mathcal{E}_\alpha}\Big)_{\mathcal{U}}\,.$$

From (3.37) and the fact that $\sup_i c_p(E_i) < \infty$, it follows by (A.3) that

$$\psi_\alpha \in \mathcal{CB}\big((S_p^{m_\alpha}[\,E_i\,])_{\mathcal{U}}, (S_p^{m_\alpha}[\,E_i\,])_{\mathcal{U}} \big),$$

and, moreover,

$$\|\,\psi_\alpha\,\|_{cb} \leq \sup_i c_p(E_i)\,.$$

Therefore, we obtain the following commuting diagram of completely bounded maps

$$\begin{tikzcd}
S_p^{m_\alpha}[\,\hat{E}\,] \arrow[r, "\tilde{\mathcal{E}_\alpha}(T_\varepsilon \otimes \mathrm{Id}_{\hat{E}})\tilde{\mathcal{E}_\alpha}"] \arrow[d, "\phi_\alpha"] & S_p^{m_\alpha}[\,\hat{E}\,] \\
(S_p^{m_\alpha}[\,E_i\,])_{\mathcal{U}} \arrow[r, "\psi_\alpha"] & (S_p^{m_\alpha}[\,E_i\,])_{\mathcal{U}} \arrow[u, "\phi_\alpha^{-1}"']
\end{tikzcd}$$

It follows that

$$\begin{aligned}
\|\,\tilde{\mathcal{E}_\alpha}(T_\varepsilon \otimes \mathrm{Id}_{\hat{E}})\tilde{\mathcal{E}_\alpha}\,\|_{cb} &\leq \|\,\phi_\alpha^{-1} \circ \psi_\alpha \circ \phi_\alpha\,\|_{cb} \\
&\leq \|\,\phi_\alpha^{-1}\,\|_{cb}\|\,\psi_\alpha\,\|_{cb}\|\,\phi_\alpha\,\|_{cb} \\
&\leq \sup_i c_p(E_i)\,.
\end{aligned} \tag{3.38}$$

By (3.34), this yields the conclusion. Moreover, we obtain the following estimate for the UMD_p constant of the ultraproduct $\hat{E}$

$$c_p(\hat{E}) \leq \sup_i c_p(E_i)\,.$$

By general results on ultraproducts, it follows that

$$c_p(\hat{E}) \leq \lim_{\mathcal{U}} c_p(E_i)\,.$$

(8) We first show that $E\oplus_p F$ is UMD$_p$. Let $(\mathcal{M},\tau)$ be a hyperfinite von Neumann algebra equipped with an n.f. tracial state τ, and $(\mathcal{M}_n)_{n\geq 1}$ a filtration of $\mathcal{M}$. We will show that there exists a constant $c_p(E,F)>0$ such that

$$\| T_\varepsilon \otimes \mathrm{Id}_{E\oplus_p F} : L_p(\mathcal{M};E\oplus_p F) \to L_p(\mathcal{M};E\oplus_p F)\|_{cb} \quad \leq \quad c_p(E,F)\,, \tag{3.39}$$

for all finite sequences ε of numbers in $\{-1,1\}$. Martingale differences are considered with respect to the filtration $(\mathcal{M}_n)_{n\geq 1}$. It is enough to prove that for all positive integers m we have

$$\| T_\varepsilon \otimes \mathrm{Id}_{E\oplus_p F} : S_p^m(\mathcal{M};E\oplus_p F) \to S_p^m(\mathcal{M};E\oplus_p F)\|_{cb} \quad \leq \quad c_p(E,F)\,, \tag{3.40}$$

Recall that by Lemma 2.3.6 we have the complete isometry

$$S_p^m[E\oplus_p F] = S_p^m[E]\oplus_p S_p^m[F]\,.$$

It follows that

$$\begin{aligned} \| T_\varepsilon \otimes \mathrm{Id}_{E\oplus_p F} : S_p^m[E\oplus_p F] \to S_p^m[E\oplus_p F]\|_{cb} \quad &= \\ \| T_\varepsilon \otimes \mathrm{Id}_{E\oplus_p F} : S_p^m[E]\oplus_p S_p^m[F] \to S_p^m[E]\oplus_p S_p^m[F]\|_{cb} \quad &\leq \\ [(c_p(E))^p + (c_p(F))^p]^{\frac{1}{p}}\,, \end{aligned}$$

where $c_p(E)$ and $c_p(F)$ are the UMD$_p$ constants of E, respectively F. Thus $E\oplus_p F$ is UMD$_p$. Furthermore, note that if $1<q<\infty$,

$$d_{cb}(E\oplus_p F\,, E\oplus_q F) \quad \leq \quad 2\,. \tag{3.41}$$

Hence, by (1) it follows that $E\oplus_q F$ is UMD$_p$ and the proof ends. □

Example 3.2.11 If $1<p<\infty$, then the matrix algebras M_m are UMD$_p$, for all positive integers m. This follows immediately from Remark 3.2.8 and (3) in Proposition 3.2.10.

The following lemma gives a necessary condition for an operator space E to be UMD_p.

Lemma 3.2.12 *Let* $1 < p < \infty$, *and* E *an operator space. If* E *is* UMD_p, *then* $S_p[E]$ *is UMD (as a Banach space).*

Proof. Since UMD is a local property, it suffices to show that $S_p^m[E]$ is UMD (as a Banach space), for all positive integers m. For this, we prove that there exists a constant $k_p > 0$, independent of m, such that

$$\| T_\varepsilon \otimes \mathrm{Id}_{S_p^m[E]} : L_p(([0,1], \mathcal{F}, \mu); S_p^m[E]) \to L_p(([0,1], \mathcal{F}, \mu); S_p^m[E]) \| \quad \leq \quad k_p , \tag{3.42}$$

where μ is the Lebesgue measure on $[0,1]$, $\mathcal{F}$ a σ-algebra of parts of $[0,1]$, and $(\mathcal{F}_n)_{n\geq 1}$ a filtration of $([0,1], \mathcal{F})$. Martingale differences are considered with respect to the above filtration. By Fubini's theorem, we have the complete isometry

$$\begin{aligned} L_p([0,1]; S_p^m[E]) &= L_p([0,1]; L_p(M_m; E)) \\ &= L_p(L_\infty([0,1]); L_p(M_m; E)) \\ &= L_p(L_\infty([0,1]) \otimes M_m; E) \end{aligned} \tag{3.43}$$

Note that $(L_\infty([0,1], \mathcal{F}_n, \mu) \otimes M_m)_{n\geq 1}$ is a filtration of the hyperfinite von Neumann algebra $L_\infty([0,1]) \otimes M_m$. For all positive integers n, the map $\mathcal{E}_n = \mathbb{E}_n \otimes \mathrm{Id}_{M_m}$, where $\mathbb{E}_n = \mathbb{E}(\,\cdot\,|\mathcal{F}_n)$, is the unique trace-preserving conditional expectation onto $L_\infty([0,1], \mathcal{F}_n) \otimes M_m$.
Let $k_p = c_p(E)$, where $c_p(E)$ is the corresponding UMD_p constant for E. Then, the fact that E is UMD_p with respect to $L_\infty([0,1]) \otimes M_m$ and its filtration $(L_\infty([0,1], \mathcal{F}_n, \mu) \otimes M_m)_{n\geq 1}$, together with (2.19) yields (3.42). $\square$

Example 3.2.13 There exists a Hilbert space, subspace of some commutative C^*-algebra which is UMD (as a Banach space), but not UMD_p, for any $1 < p < \infty$.

Let $E = \min(l_2)$. Then E is a Hilbert space, hence it is UMD (as a Banach space). Assume that E is UMD$_p$, for some $1 < p < \infty$. We will show that this leads to a contradiction. Following Pisier's argument in [70], let $(\mathcal{M}, \tau)$ be the hyperfinite II_1 factor equipped with the canonical "dyadic filtration" $(\mathcal{M}_n)_{n \geq 1}$, where $\mathcal{M}_n = M_2 \otimes \ldots \otimes M_2$ (n times). By (3) in Proposition 3.2.10, it follows that $L_p(\mathcal{M}; E)$ is UMD$_p$. Hence $L_p(\mathcal{M}; E)$ is UMD (as a Banach space). However, by Proposition 4.3 in [70], $L_p(\mathcal{M}; E)$ contains a subspace isomorphic to c_0 . This contradicts the fact that $L_p(\mathcal{M}; E)$ is UMD (as a Banach space), since c_0 is not UMD. Therefore E is not UMD$_p$, for any $1 < p < \infty$.

3.3 On the operator space UMD$_p$ for non-commutative L_p-spaces

Proposition 3.3.1 *If $1 < p < \infty$, then $L_p(\mathcal{N}, \phi)$ is UMD$_p$, for every von Neumann algebra $\mathcal{N}$ equipped with an n.f. tracial state ϕ .*

Proof. Let $(\mathcal{M}, \tau)$ be a hyperfinite von Neumann algebra equipped with an n.f. tracial state τ , and $(\mathcal{M}_n)_{n \geq 1}$ a filtration of $\mathcal{M}$. We will show that $L_p(\mathcal{N}, \phi)$ is UMD$_p$ with respect to the algebra $(\mathcal{M}, \tau)$ and its filtration $(\mathcal{M}_n)_{n \geq 1}$.
By Fubini's theorem (Theorem 1.5.3), we have the complete isometry

$$L_p(\mathcal{M}; L_p(\mathcal{N}, \phi)) \quad = \quad L_p(\mathcal{M} \otimes \mathcal{N}, \tau \otimes \phi) \,. \tag{3.44}$$

By Proposition 3.2.1, there exists a constant $c_p > 0$, depending only on p , such that

$$\| T_\varepsilon : L_p(\mathcal{M} \otimes \mathcal{N}, \tau \otimes \phi) \to L_p(\mathcal{M} \otimes \mathcal{N}, \tau \otimes \phi) \|_{cb} \quad \leq \quad c_p \,.$$

for all finite sequences ε of numbers in $\{-1, 1\}$ and all martingale difference sequences with respect to the filtration $(\mathcal{M}_n \otimes \mathcal{N})_{n \geq 1}$ of the algebra (not necessarely hyperfinite) $\mathcal{M} \otimes \mathcal{N}$. Together with (3.44), this yields the conclusion. □

Remark 3.3.2 As a consequence, it follows that for $1 < p < \infty$, $S_p = L_p(\mathcal{B}(l_2), \mathrm{tr})$, and, respectively, $S_p^m = L_p(M_m, \mathrm{Tr})$, where m is a positive integer and Tr denotes the non-normalized trace on M_m , are all UMD_p . Moreover, we have $\sup_m c_p(S_p^m) \le c_p(S_p) < \infty$.

Proposition 3.3.3 *The operator Hilbert space OH is UMD_p , for $1 < p < \infty$.*

Proof. Recall that by Corollary 2.6 in [68] , OH can be obtained by complex interpolation between the column Hilbert space C and the row Hilbert space R, namely,

$$OH = [C, R]_{\frac{1}{2}} . \tag{3.45}$$

By the reiteration theorem for the complex method ([3], Theorem 4.6.1), it follows from (1.53) and (1.54) that for $1 < p < \infty$ we have

$$OH = [C_p, R_p]_{\frac{1}{2}} . \tag{3.46}$$

Note that we can view C_p as the column space of S_p and, respectively , we can view R_p as the row space of S_p . Hence, by (2) in Proposition 3.2.10, and Remark 3.3.2 it follows that both C_p and R_p are UMD_p . Therefore, by (3.46), a further application of (6) in Proposition 3.2.10 yields the conclusion. □

Corollary 3.3.4 *Every finite dimensional operator space E is UMD_p , for $1 < p < \infty$.*

Proof. Let E be an n-dimensional operator space. By Corollary 9.3 of [68], there is an isomorphism

$$\phi : OH_n \to E ,$$

such that $\|\phi\|_{cb}\|\phi^{-1}\|_{cb} \le \sqrt{n}$. Thus the c.b.-Banach Mazur distance between E and OH_n satisfies

$$d_{cb}(E, OH_n) \le \sqrt{n} .$$

Hence E is completely isomorphic to OH_n . By Proposition 3.3.3 (applied in the finite dimensional case) and (1) in Proposition 3.2.10, it follows that E is UMD_p , for $1 < p < \infty$. □

Theorem 3.3.5 *If $1 < u, p < \infty$, then C_u is UMD_p.*

Proof. Since UMD_p is a local property (see Remark 3.2.9), by the density of $(C_u^m)_{m\geq 1}$ in C_u, it suffices to show that C_u^m is UMD_p, for all positive integers m, and that $\sup_m c_p(C_u^m) < \infty$.

Let $(\mathcal{M}, \tau)$ be a hyperfinite von Neumann algebra equipped with an n.f. tracial state τ, and $(\mathcal{M}_n)_{n\geq 1}$ a filtration of $\mathcal{M}$. We will show that C_u^m is UMD_p with respect to $(\mathcal{M}, \tau)$ and the filtration $(\mathcal{M}_n)_{n\geq 1}$.

By Remark 3.3.2, S_p^m is UMD_p with respect to $(\mathcal{M}, \tau)$ and the filtration $(\mathcal{M}_n)_{n\geq 1}$. By (2) in Proposition 3.2.10, the same holds for C_p^m and R_p^m, as subspaces of S_p^m.

We will prove that there exist $0 < \theta < 1$ and $1 < q, s < \infty$ such that we have the complete isometry

$$[L_q(\mathcal{M}; C_q^m), L_s(\mathcal{M}; R_s^m)]_\theta \quad = \quad L_p(\mathcal{M}; C_u^m). \tag{3.47}$$

By applying the result (3.5) in [70], this reduces to showing the existence of $0 < \theta < 1$ and $1 < q, s < \infty$ such that the following two relations hold

$$\frac{1}{p} \quad = \quad \frac{1-\theta}{q} + \frac{\theta}{s}, \tag{3.48}$$

$$C_u^m \quad = \quad [C_q^m, R_s^m]_\theta. \tag{3.49}$$

Note that by (1.53) and (1.54) we have

$$R_s^m \; = \; [R^m, C^m]_{\frac{1}{s}} \; = \; [C^m, R^m]_{1-\frac{1}{s}} \; = \; C_{s'}^m,$$

where $\frac{1}{s} + \frac{1}{s'} = 1$. Therefore, relations (3.48) and (3.49) are equivalent to

$$\begin{cases} \frac{1}{p} \; = \; \frac{1-\theta}{q} + \frac{\theta}{s}, \\ \\ \frac{1}{u} \; = \; \frac{1-\theta}{q} + \frac{\theta}{s'}. \end{cases}$$

Equivalently,

$$\begin{cases} \frac{1}{p} + \frac{1}{u} \; = \; \frac{2(1-\theta)}{q} + \theta, \\ \\ \frac{1}{p} - \frac{1}{u} \; = \; \frac{2\theta}{s} - \theta, \end{cases}$$

or

$$\begin{cases} \frac{1}{p} + \frac{1}{u} - \theta & = & \frac{2(1-\theta)}{q}\,, \\ \frac{1}{p} - \frac{1}{u} + \theta & = & \frac{2\theta}{s}\,. \end{cases}$$

Thus, we have to show that there exists $0 < \theta < 1$ such that the following conditions are satisfied

$$\begin{cases} \frac{1}{p} + \frac{1}{u} - \theta > 0\,, \\ 2(1-\theta) > \frac{1}{p} + \frac{1}{u} - \theta\,, \\ \frac{1}{p} - \frac{1}{u} + \theta > 0\,, \\ 2\,\theta > \frac{1}{p} - \frac{1}{u} + \theta\,. \end{cases}$$

Equivalently, we have to show that there exists $0 < \theta < 1$ such that all four conditions

$$\theta < \frac{1}{p} + \frac{1}{u}\,, \quad \theta < 2 - (\frac{1}{p} + \frac{1}{u})\,, \quad \theta > \frac{1}{u} - \frac{1}{p}\,, \quad \theta > \frac{1}{p} - \frac{1}{u}\,.$$

are satisfied. They are equivalent to

$$\left|\frac{1}{p} - \frac{1}{u}\right| < \quad \theta \quad < \min\left\{\frac{1}{p} + \frac{1}{u}, 2 - \left(\frac{1}{p} + \frac{1}{u}\right)\right\}\,. \tag{3.50}$$

Since $1 < p, u < \infty$, the following relations hold

$$\left|\frac{1}{p} - \frac{1}{u}\right| < \max\left\{\frac{1}{p}, \frac{1}{u}\right\} \quad < \quad 1\,, \tag{3.51}$$

$$\min\left\{\frac{1}{p} + \frac{1}{u}, 2 - \left(\frac{1}{p} + \frac{1}{u}\right)\right\} \quad > \quad 0\,. \tag{3.52}$$

Since $1 < p, u$ we also have

$$\frac{1}{p} - \frac{1}{u} < 2 - \left(\frac{1}{p} + \frac{1}{u}\right) \quad \text{and} \quad \frac{1}{u} - \frac{1}{p} < 2 - \left(\frac{1}{p} + \frac{1}{u}\right)\,.$$

Therefore, we see that

$$0 \;<\; \left|\frac{1}{p} - \frac{1}{u}\right| \;<\; \min\left\{\frac{1}{p} + \frac{1}{u}, 2 - \left(\frac{1}{p} + \frac{1}{u}\right)\right\}\,,$$

which implies the existence of $0<\theta<1$ satisfying (3.50). Set

$$q \;=\; \frac{2(1-\theta)}{\frac{1}{p}+\frac{1}{u}-\theta} \quad \text{and} \quad s \;=\; \frac{2\,\theta}{\frac{1}{p}-\frac{1}{u}+\theta}\,.$$

The above relations ensure that $1<q,s<\infty$ and therefore (3.47) is proved. An application of (6) in Proposition 3.2.10 shows that C_u^m is UMD_p. Moreover, by (3.32) and (3.17), we have the folowing estimates for the corresponding UMD_p constants :

$$\begin{aligned} c_p(C_u^m) &\;\le\; [c_q(C_q^m)]^{1-\theta}[c_s(R_s^m)]^{\theta} \\ &\;\le\; [c_q(S_q)]^{1-\theta}[c_s(S_s)]^{\theta}\,. \end{aligned}$$

This implies that $\sup_m c_p(C_u^m)<\infty$, and the conclusion follows. □

Remark 3.3.6 Let $1<u,p<\infty$. By (1.53) it follows from the equivalence theorem for the complex method (Theorem 4.3.1 in [3]) that

$$(C_u)^* \;=\; R_{u'} \quad \text{and} \quad (R_u)^* \;=\; C_{u'}\,, \tag{3.53}$$

where $\frac{1}{u}+\frac{1}{u'}=1$. Therefore, by Theorem 3.3.5 and (5) in Proposition 3.2.10 it follows that R_u is UMD_p.

Proposition 3.3.7 *Let $1<u,p<\infty$. Then the spaces $C_p\otimes^h C_u$, $C_u\otimes^h R_p$, $C_p\otimes^h R_u$, $R_u\otimes^h R_p$ are all UMD_p.*

Proof. Let $(\mathcal{M},\tau)$ be a hyperfinite von Neumann algebra equipped with an n.f. tracial state τ, and $(\mathcal{M}_n)_{n\ge1}$ a filtration of $\mathcal{M}$. Let m be a positive integer. By Theorem 3.3.5, C_u^m is UMD_p with respect to $(\mathcal{M},\tau)$ and the filtration $(\mathcal{M}_n)_{n\ge1}$. Therefore,

$$\|\,T_\varepsilon\otimes \mathrm{Id}_{C_u^m} : L_p(\mathcal{M}\otimes M_m; C_u^m)\to L_p(\mathcal{M}\otimes M_m; C_u^m)\,\|_{cb} \;\le\; c_p(C_u^m) \;\le\; c_p(C_u) \;<\; \infty\,.$$

By Fubini's theorem (Theorem 1.5.3), we have the complete isometry

$$\begin{aligned} L_p(\mathcal{M} \otimes M_m; C_u^m) &= L_p(\mathcal{M}; L_p(M_m; C_u^m)) \\ &= L_p(\mathcal{M}; S_p^m[C_u^m]) \\ &= L_p(\mathcal{M}; C_p^m \otimes^h C_u^m \otimes^h R_p^m)\,. \end{aligned}$$

By the injectivity of the Haagerup tensor product it follows that

$$\begin{aligned} L_p(\mathcal{M}; C_p^m \otimes^h C_u^m) &\subseteq L_p(\mathcal{M}; C_p^m \otimes^h C_u^m \otimes^h R_p^m)\,; \\ L_p(\mathcal{M}; C_u^m \otimes^h R_p^m) &\subseteq L_p(\mathcal{M}; C_p^m \otimes^h C_u^m \otimes^h R_p^m)\,. \end{aligned}$$

This shows that $C_p^m \otimes^h C_u^m$ and $C_u^m \otimes^h R_p^m$ are both UMD$_p$ with respect to $(\mathcal{M}, \tau)$ and the filtration $(\mathcal{M}_n)_{n\geq 1}$. Since the algebra $(\mathcal{M}, \tau)$ and its filtration $(\mathcal{M}_n)_{n\geq 1}$ are arbitrary, it follows that the operator spaces $C_p^m \otimes^h C_u^m$ and $C_u^m \otimes^h R_p^m$ are both UMD$_p$. Moreover, from the proof we obtain the following estimates for the correspondent UMD$_p$ constants

$$c_p(C_p^m \otimes^h C_u^m)\,,\ c_p(C_u^m \otimes^h R_p^m) \ \leq\ c_p(C_u^m) \ \leq\ c_p(C_u) \ <\ \infty\,. \tag{3.54}$$

A similar argument applied to the space R_u^m, which is UMD$_p$ as a subspace of R_u (see Remark 3.3.6), shows that the operator spaces $C_p^m \otimes^h R_u^m$ and $R_u^m \otimes^h R_p^m$ are UMD$_p$. Moreover,

$$c_p(C_p^m \otimes^h R_u^m)\,,\ c_p(R_u^m \otimes^h R_p^m) \ \leq\ c_p(R_u^m) \ \leq\ c_p(R_u) \ <\ \infty\,. \tag{3.55}$$

Since m is an arbitrarily chosen positive integer, the conclusion follows by density and the fact that UMD$_p$ is a local property. □

Proposition 3.3.8 *Let $1 < u, p < \infty$. Then the spaces $R_p \otimes^h C_u$, $C_u \otimes^h C_p$, $R_p \otimes^h R_u$ and $R_u \otimes^h C_p$ are all UMD$_p$.*

Proof. This follows by duality from the above result. Indeed, for all positive integers m, by Proposition 3.3.7, the operator spaces $C_{p'}^m \otimes^h C_{u'}^m, C_{u'}^m \otimes^h R_{p'}^m, C_{p'}^m \otimes^h R_{u'}^m$ and $R_{u'}^m \otimes^h R_{p'}^m$ are all

$\mathrm{UMD}_{p'}$. By (5) in Proposition 3.2.10 it follows that the dual spaces are UMD_p. By (3.53) and the fact that the Haagerup tensor product is self-dual in the finite dimensional case (see Proposition 1.2.9), we obtain

$$(C_{p'}^m \otimes^h C_{u'}^m)^* \;=\; R_p^m \otimes^h R_u^m \quad \text{and} \quad (C_{u'}^m \otimes^h R_{p'}^m)^* \;=\; R_u^m \otimes^h C_p^m\,,$$

respectively,

$$(C_{p'}^m \otimes^h R_{u'}^m)^* \;=\; R_p^m \otimes^h R_u^m \quad \text{and} \quad (R_{u'}^m \otimes^h R_{p'}^m)^* \;=\; C_u^m \otimes^h C_p^m\,.$$

Moreover, from (3.54), (3.55) and (3.31) we obtain the following estimates for the corresponding UMD_p constants

$$\begin{aligned} c_p(R_p^m \otimes^h R_u^m)\,,\; c_p(R_u^m \otimes^h C_p^m) \;&\le\; c_{p'}(C_u^m) \;\le\; c_{p'}(C_u) \;<\; \infty\,,\\ c_p(R_p^m \otimes^h R_u^m)\,,\; c_p(C_u^m \otimes^h C_p^m) \;&\le\; c_{p'}(R_u^m) \;\le\; c_{p'}(R_u) \;<\; \infty\,. \end{aligned}$$

Since m is an arbitrarily chosen positive integer, and UMD_p is a local property, the conclusion follows. □

Proposition 3.3.9 *Let $1<p<\infty$. If $1<q<\infty$ satisfies $\frac{2p}{p+1}<q<2p$, then S_q is UMD_p.*

Proof. By Kouba's interpolation result (Theorem 1.2.11), if $0<\theta<1$ and $1<u,v<\infty$, we have the complete isometry

$$[C_u \otimes^h R_p\,, C_p \otimes^h R_v]_\theta \quad = \quad [C_u\,, C_p]_\theta \otimes^h [R_p\,, R_v]_\theta\,. \tag{3.56}$$

We will show that if $1<q<\infty$ satisfies $\frac{2p}{p+1}<q<2p$, then there exist $0<\theta<1$ and $1<u,v<\infty$ such that

$$[C_u\,, C_p]_\theta = C_q \quad \text{and} \quad [R_p\,, R_v]_\theta = R_q\,. \tag{3.57}$$

This is equivalent to showing that there exist $0 < \theta < 1$ and $1 < u, v < \infty$ such that the following relations hold

$$\frac{1}{q} = \frac{1-\theta}{u} + \frac{\theta}{p}, \tag{3.58}$$

$$\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{v}. \tag{3.59}$$

By the assumption on q we have $\frac{2}{q} < 1 + \frac{1}{p}$. An easy computation shows that

$$p'\left(\frac{1}{q} - \frac{1}{p}\right) < \frac{p'}{q'},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. This ensures the existence of some $0 < \theta < 1$ such that

$$\max\left\{0, p'\left(\frac{1}{q} - \frac{1}{p}\right)\right\} < \theta < \min\left\{1, \frac{p'}{q'}\right\}. \tag{3.60}$$

By (3.60) it follows that $\frac{1}{q} - \frac{\theta}{p} < 1 - \theta$. This implies the existence of some $1 < u < \infty$ such that (3.58) holds. Furthermore, (3.60) also shows that $\frac{1}{q} < \frac{1-\theta}{p} + \theta$. This implies the existence of some $1 < v < \infty$ such that (3.59) holds. Thus, with $0 < \theta < 1$ and $1 < u, v < \infty$ found above, we have

$$[C_u \otimes^h R_p, C_p \otimes^h R_v]_\theta = C_q \otimes^h R_q = S_q. \tag{3.61}$$

By Proposition 3.3.7, both spaces $C_u \otimes^h R_p$ and $C_p \otimes^h R_v$ are UMD_p. Therefore, by (6) in Proposition 3.2.10, the interpolation result (3.61) shows that S_q is UMD_p. □

Proposition 3.3.10 *If* $1 < p < \infty$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$, *then* $S_{p'}$ *is* UMD_p.

Proof. By Kouba's interpolation result (Theorem 1.2.11), if $0 < \theta < 1$ and $1 < u, v < \infty$, we have the complete isometry

$$[C_u \otimes^h R_p, C_p \otimes^h R_v]_\theta = [C_u, C_p]_\theta \otimes^h [R_p, R_v]_\theta. \tag{3.62}$$

We will show that if $2 < p < \infty$, then there exist $0 < \theta < 1$ and $1 < u, v < \infty$ such that

$$[C_u\,, C_p]_\theta = C_{p'} \quad \text{and} \quad [R_p\,, R_v]_\theta = R_{p'}\,. \tag{3.63}$$

This is equivalent to showing that there exist $0 < \theta < 1$ and $1 < u, v < \infty$ such that

$$\frac{1}{p'} = \frac{1-\theta}{u} + \frac{\theta}{p}\,, \tag{3.64}$$

$$\frac{1}{p'} = \frac{1-\theta}{p} + \frac{\theta}{v}\,. \tag{3.65}$$

Note that (3.64) is equivalent to

$$\begin{aligned} \frac{1}{p} = \frac{1-\theta}{u'} + \frac{\theta}{p'} \quad &\Leftrightarrow \quad \frac{1}{p} = \frac{1-\theta}{u'} + \theta\left(1 - \frac{1}{p}\right) \\ &\Leftrightarrow \quad \frac{1+\theta}{p} = \frac{1-\theta}{u'} + \theta\,. \end{aligned} \tag{3.66}$$

This yields a restriction upon θ as follows

$$\begin{aligned} \theta < \frac{1+\theta}{p} < 1 \quad &\Leftrightarrow \quad 1+\theta < p < \frac{1+\theta}{\theta} \\ &\Leftrightarrow \quad 1+\theta < p < 1 + \frac{1}{\theta} \\ &\Leftrightarrow \quad \frac{1}{p-1} < \theta < p-1\,. \end{aligned} \tag{3.67}$$

Note that since $2 < p < \infty$, we have $0 < \frac{1}{p-1} < 1 < p-1$. Therefore, the above relations yield the restriction

$$\frac{1}{p-1} < \quad \theta \quad < 1\,. \tag{3.68}$$

Choose $0 < \theta < 1$ such that (3.68) holds. By solving (3.66), we can find $1 < u' < \infty$, and, therefore, its conjugate exponent u. Furthermore, with the choice of θ from above, (3.65) is equivalent to

$$1 - \frac{1}{p} = \frac{1-\theta}{p} + \frac{\theta}{v} \quad \Leftrightarrow \quad 1 - \frac{2-\theta}{p} = \frac{\theta}{v}\,. \tag{3.69}$$

This implies that

$$0 < \frac{2-\theta}{p} < 1 \quad \Leftrightarrow \quad p > 2-\theta\,, \tag{3.70}$$

which is obviously true, since $p > 2 > 2-\theta$. Therefore, we can solve (3.69) for v. Our claim is completely proved. We have

$$[C_u \otimes^h R_p\,, C_p \otimes^h R_v] \;=\; C_{p'} \otimes^h R_{p'} \;=\; S_{p'}\,. \tag{3.71}$$

By Proposition 3.3.7, both spaces $C_u \otimes^h R_p$ and $C_p \otimes^h R_v$ are UMD_p. Therefore, by (6) in Proposition 3.2.10, the interpolation result (3.71) shows that $S_{p'}$ is UMD_p.

The case $1 < p < 2$ follows by duality. Indeed, in this range we have $2 < p' < \infty$. By the result proved above, it follows that $S_{(p')'} = S_p$ is $\mathrm{UMD}_{p'}$. By (5) in Proposition 3.2.6, this implies that $S_{p'}$ is UMD_p. □

Lemma 3.3.11 *If* $1 < q < \infty$, *then* S_q *is* UMD_2.

Proof. By Remark 3.3.2, S_q is UMD_q. Furthermore, by Proposition 3.3.10, S_q is $\mathrm{UMD}_{q'}$. By (6) in Proposition 3.2.10, interpolation with exponent $\theta = \frac{1}{2}$ yields the conclusion. □

Theorem 3.3.12 *If* $1 < p, q < \infty$, *then* S_q *is* UMD_p.

Proof. Case 1: $2 \le q < \infty$. We will first show that if $p \ge 2$, then S_q is UMD_p. Indeed, if $p \ge q$, then we have $\frac{2p}{p+1} < 2 < q < 2p$. Thus, by Proposition 3.3.9, it follows that S_q is UMD_p. If $2 < p < q$, then there exists $0 < \theta < 1$ such that

$$\frac{1}{p} \;=\; \frac{1-\theta}{2} + \frac{\theta}{q}\,. \tag{3.72}$$

By Lemma 3.3.11, S_q is UMD_2. Also, by Remark 3.3.2, S_q is UMD_q. Thus, from (6) in Proposition 3.2.10 it follows by interpolation with exponent θ given by (3.72) that S_q is UMD_p.

Furthermore, note that $1 < q' < 2$, where $\frac{1}{q} + \frac{1}{q'} = 1$. Assume that $q' < p < 2$. There exists $0 < \theta < 1$ such that

$$\frac{1}{p} \quad = \quad \frac{1-\theta}{q'} + \frac{\theta}{2}\,. \tag{3.73}$$

By Proposition 3.3.10, S_q is UMD$_{q'}$. Also, by Lemma 3.3.11, S_q is UMD$_2$. Thus, by (6) in Proposition 3.2.10, interpolation with exponent θ given by (3.73) shows that S_q is UMD$_p$. Combining this with the conclusion of first paragraph, we see that if $p \geq q'$, then S_q is UMD$_p$.
It remains to prove that if $1 < p < q'$, then S_q is UMD$_p$. We have $q' < 2 < p'$, where $\frac{1}{p} + \frac{1}{p'} = 1$. By what we proved above, it follows that S_q is UMD$_{p'}$. By Proposition 3.3.10, S_p is UMD$_{p'}$. Note that $p < q' < q$. There exists $0 < \theta < 1$ such that

$$\frac{1}{q'} \quad = \quad \frac{1}{p} + \frac{1}{q}\,. \tag{3.74}$$

By (5) in Proposition 3.2.10, interpolation with exponent θ given by (3.74) implies that $S_{q'}$ is UMD$_{p'}$. By duality, an application of (5) in Proposition 3.2.10 shows that S_q is UMD$_p$.

<u>Case 2</u>: $1 < q < 2$. Note that $2 \leq q' < \infty$. Therefore, if $1 < p < \infty$, it follows by Case 1 that $S_{q'}$ is UMD$_{p'}$. By duality, (5) in Proposition 3.2.10 implies that S_q is UMD$_p$. This completes the proof. □

Proposition 3.3.13 *If* $1 < p, q, u < \infty$, *then* $S_q[S_u]$ *is UMD*$_p$.

Proof. We first show that $S_q[S_u]$ is UMD$_{u'}$, where u' is the conjugate exponent of u. As showed in the proof of Theorem 3.3.5, there exist $0 < \theta < 1$ and $1 < v, s < \infty$, such that

$$\begin{cases} \frac{1}{q} & = & \frac{1-\theta}{v} + \frac{\theta}{s}\,, \\ \frac{1}{u} & = & \frac{1-\theta}{v} + \frac{\theta}{s'}\,, \end{cases}$$

where s' is the conjugate exponent of s. Then, the following relations hold

$$S_q = [S_v, S_s]_\theta \tag{3.75}$$

$$S_u = [S_v, S_{s'}]_\theta \tag{3.76}$$

$$\frac{1}{u'} = \frac{1-\theta}{v'} + \frac{\theta}{s}, \tag{3.77}$$

where v' is the conjugate exponent of v. From (3.75) and (3.76) it follows by Corollary 1.4 of [70] that

$$S_q[S_u] = [\, S_v[S_v]\,, S_s[S_{s'}]\,]_\theta\,. \tag{3.78}$$

By Proposition 3.3.12 it follows that $S_v[S_v]$ is $\mathrm{UMD}_{v'}$, and $S_s[S_{s'}]$ is UMD_s. Therefore, by (6) of Proposition 3.2.10, interpolation with exponent θ satisfying (3.77) shows that $S_q[S_u]$ is $\mathrm{UMD}_{u'}$. Moreover, since by Proposition 3.3.12, S_v is UMD_q, it follows by (3) of Proposition 3.2.10 that $S_q[s_v]$ is UMD_q. Using the result proved above, it follows by interpolation that $S_q[S_v]$ is UMD_p, for $\min\{u', q\} \leq p \leq \max\{u', q\}$.

Let $\max\{u', q, 2\} < p < \infty$. Let $v < q < p < \tilde{v}$. There exists $0 < \theta < 1$, such that

$$\frac{1}{q} = \frac{1-\theta}{v} + \frac{\theta}{\tilde{v}}\,. \tag{3.79}$$

Let $2 < s < \infty$ be such that

$$\frac{1}{u} = \frac{1-\theta}{v} + \frac{\theta}{s}\,. \tag{3.80}$$

By Corollary 1.4 of [70] it follows that

$$S_q[S_u] = [\, S_v[S_v]\,, S_{\tilde{v}}[S_s]\,]_\theta\,. \tag{3.81}$$

By Proposition 3.3.12, $S_v[S_v]$ is UMD_p. Moreover, since $s' < 2 < p < \tilde{v}$, it follows by the result proved above that $S_{\tilde{v}}[S_s]$ is UMD_p. Therefore, by (65), interpolation with exponent θ shows that $S_q[S_u]$ is UMD_p. Since $\max\{u', q\} \leq \max\{u', q, 2\}$ it follows by interpolation that $S_q[S_v]$ is UMD_p, for $\max\{u', q\} \le p < \infty$. Thus, so far we showed that $S_q[S_u]$ is UMD$_p$, for $\min\{u', q\} \le p < \infty$. Let $1 < p < \min\{u', q\}$. This implies that $\min\{u, q'\} < p' < \infty$, where q' is the conjugate exponent of q. By what we proved above it follows that $S_{q'}[S_{u'}]$ is UMD$_{p'}$. By (5) of Proposition 3.2.10 and Corollary 1.8 of [70], it follows that $S_q[S_u]$ is UMD$_p$. The proof is now complete. □

Corollary 3.3.14 *Let $1 < p, q, u < \infty$. Then the spaces $C_u \otimes^h R_q$, $C_u \otimes^h C_q$, $R_u \otimes^h C_q$ and $R_u \otimes^h R_q$ are all UMD$_p$.*

Proof. By the injectivity and associativity properties of the Haagerup tensor product, we obtain

$$\begin{aligned} C_u \otimes^h R_q &\subseteq C_u \otimes^h C_q \otimes^h R_q \otimes^h R_u \\ &= S_u[S_q]\,. \end{aligned}$$

By (2) of Proposition 3.2.10 and Proposition 3.3.13, it follows that $C_u \otimes^h R_q$ is UMD$_p$. A similar argument applies for the spaces $C_u \otimes^h C_q$, $R_u \otimes^h C_q$ and $R_u \otimes^h R_q$. □

Recall that a C^*-algebra A has the *weak expectation property* (WEP) of Lance [55], if for the universal representation $A \subset A^{**} \subset \mathcal{B}(H)$ there exists a contraction $P : \mathcal{B}(H) \to A^{**}$ such that $P \restriction A = \mathrm{Id}_A$. A C^*-algebra B is said to be $QWEP$ if it is a *quotient* of a WEP C^*-algebra; more precisely, there exista a C^*-algebra A with the WEP and a closed two-sided ideal I such that $B = A/I$. It is a long standing problem whether every C^*-algebra is $QWEP$ (see Kirchberg [51] for many equivalent formulations). Note that an injective von Neumann algebra has the WEP, and, therefore, it is $QWEP$. Also, it was proved by Wassermann [86] that for $n \ge 2$, $VN(\mathbb{F}_n)$ is $QWEP$, where $\mathbb{F}_n$ is the free goup on n generators.

Proposition 3.3.15 *Let $(\mathcal{M}, \tau)$ be a QWEP von Neumann algebra equipped with an n.f. tracial state τ. Then $L_q(\mathcal{M}, \tau)$ is UMD$_p$, for $1 < p, q < \infty$.*

Proof. By results of Junge [46], it follows that $L_q(\mathcal{M}, \tau)$ is completely contractively complemented in an ultrapower $(S_q)_{\mathcal{U}}$. Therefore, combining Theorem 3.3.12 with (7) and (2) of Proposition 3.2.10, we obtain the conclusion. □

Remark 3.3.16 As a corollary, it follows that if G is an amenable group (in which case $VN(G)$ is an injective von Neumann algebra), or $G = \mathbb{F}_n \, (n \geq 2)$, then $L_q(VN(G), \tau)$ is UMD_p, for $1 < p, q < \infty$.

Corollary 3.3.17 *Let $1 < p, q, u < \infty$. Then, for every hyperfinite von Neumann algebra $(\mathcal{M}, \tau)$ and every QWEP von Neumann algebra $(\mathcal{N}, \phi)$ equipped with n.f. tracial states τ, respectively, ϕ, $L_q(\mathcal{M}; L_u(\mathcal{N}, \phi))$ is UMD$_p$.*

Proof. Since $\mathcal{M}$ is a hyperfinite von Neumann algebra, the spaces $S_q^m[L_u(\mathcal{N}, \phi)]$ are dense in $L_q(\mathcal{M}; L_u(\mathcal{N}, \phi))$. Thus, since UMD_p is a local property, it suffices to show that for all positive integers m, $S_q^m[L_u(\mathcal{N}, \phi)]$ is UMD_p, where the constant is independent on m. Fix such m. By Junge's result [46] and the injectivity and associativity properties of the Haagerup tensor product it folows that, completely contractively,

$$S_q^m[L_u(\mathcal{N}, \phi)] \hookrightarrow S_q^m[(S_u)_{\mathcal{U}}]\,. \tag{3.82}$$

By Lemma A.1.6 of [70], we have the complete isometry

$$S_q^m[(S_u)_{\mathcal{U}}] \quad = \quad (S_q^m[S_u])_{\mathcal{U}}\,. \tag{3.83}$$

Therefore, by Proposition 3.3.13, together with (7) and (1) of Proposition 3.2.10 it follows that $S_q^m[L_u(\mathcal{N}, \phi)]$ is UMD_p and, moreover,

$$c_p(S_q^m[L_u(\mathcal{N}, \phi)]) \quad \leq \quad c_p(S_q[S_u])\,.$$

This completes the proof. □

Proposition 3.3.18 *Let $(\mathcal{M}, \tau)$ be a hyperfinite von Neumann algebra, equipped with an n.f. tracial state τ. Then the associated Lorentz spaces $L_{q,s}(\mathcal{M}, \tau)$ are UMD$_p$, for $1 < q, s, p < \infty$.*

Proof. There exists $0 < \theta < 1$ and $1 < q_1, q_2 < \infty$, such that $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_2}$. Therefore, we have the complete isometry

$$L_{q,s}(\mathcal{M}, \tau) \;=\; [\, L_{q_1}(\mathcal{M}, \tau), L_{q_2}(\mathcal{M}, \tau)\,]_{\theta,s}\,. \tag{3.84}$$

We refer to Xu's paper [89] for details on the operator space structure of the real interpolation space $[\, L_{q_1}(\mathcal{M}, \tau), L_{q_2}(\mathcal{M}, \tau)\,]_{\theta,s}$. By Xu's results [89], we have a completely isometric embedding

$$[\, L_{q_1}(\mathcal{M}, \tau), L_{q_2}(\mathcal{M}, \tau)\,]_{\theta,s} \;\hookrightarrow\; l_s(\, \{L_{q_1}(\mathcal{M}, \tau) +_s 2^{-k} L_{q_2}(\mathcal{M}, \tau)\}_{k\in\mathbb{Z}}\,;\, 2^{-k\theta}\,)\,. \tag{3.85}$$

Let k be a positive integer. The space $L_{q_1}(\mathcal{M}, \tau) +_s 2^{-k} L_{q_2}(\mathcal{M}, \tau)$ is defined in [89] as a quotient of $L_{q_1}(\mathcal{M}, \tau) \oplus_s 2^{-k} L_{q_2}(\mathcal{M}, \tau)$. Then, by (2.6)' of [70], $l_s(\, \{L_{q_1}(\mathcal{M}, \tau) +_s 2^{-k} L_{q_2}(\mathcal{M}, \tau)\}_{k\in\mathbb{Z}}\,;\, 2^{-k\theta}\,)$ is a quotient of the operator space $l_s(\, \{L_{q_1}(\mathcal{M}, \tau) \oplus_s 2^{-k} L_{q_2}(\mathcal{M}, \tau)\}_{k\in\mathbb{Z}}\,;\, 2^{-k\theta}\,)$. We have the complete isometry

$$l_s(\, \{L_{q_1}(\mathcal{M}) \oplus_s 2^{-k} L_{q_2}(\mathcal{M})\}_{k\in\mathbb{Z}}\,;\, 2^{-k\theta}\,) \;=\; l_s(L_{q_1}(\mathcal{M})) \oplus_s l_s(2^{-k} L_{q_2}(\mathcal{M})\}_{k\in\mathbb{Z}}\,;\, 2^{-k\theta})\,.$$

Therefore, by (8) of Proposition 3.2.10 it suffices to prove that both spaces $l_s(L_{q_1}(\mathcal{M}, \tau))$ and $l_s(2^{-k} L_{q_2}(\mathcal{M}, \tau)\}_{k\in\mathbb{Z}}\,;\, 2^{-k\theta})$ are UMD_p. Indeed, as explained in [70], the space $l_s(\, L_{q_1}(\mathcal{M}, \tau)\,)$ is completely isometric to the subspace of $S_s[\, L_{q_1}(\mathcal{M}, \tau)\,]$ formed by all the diagonal matrices, and, moreover, the usual projection onto this subspace is a complete contraction. Since $\mathcal{M}$ is an injective von Neumann algebra, it follows by Corollary 3.3.17 that $S_s[\, L_{q_1}(\mathcal{M}, \tau)\,]$ is UMD_p. Thus, by (1) of Proposition 3.2.10, it follows that $l_s(\, L_{q_1}(\mathcal{M}, \tau)\,)$ is UMD_p. Similar arguments show that the space $l_s(\, 2^{-k} L_{q_2}(\mathcal{M}, \tau)\}_{k\in\mathbb{Z}}\,;\, 2^{-k\theta})$ is UMD_p, and the proof is complete. □

Question: Is C (the column Hilbert space of l_2) UMD_p for some (all) $1 < p < \infty$? By Lemma 3.2.12, if C is UMD_p for some $1 < p < \infty$, then $S_p[C]$ is UMD as a Banach space. We can prove that $S_p[C]$ is superreflexive. We are indebted to Timur Oikhberg for showing us a short proof of this result. Our original proof uses Proposition A.1.3. and will appear elsewhere.

Appendix A

A.1 Ultraproducts

The ultraproduct construction has been extensively applied in functional analysis; it is a powerful tool in studying the local theory of Banach spaces and it has profound applications in the theory of operator spaces.

We first recall some definitions. Let I be an infinite index set. A nonempty collection $\mathcal{F}$ of subsets of I is called a *filter* of I if the following conditions are satisfied:

(i) $\emptyset \notin \mathcal{F}$

(ii) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

(iii) If $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.

The collection of filters on I is partially ordered by set inclusion. A maximal filter (with respect to this partial order) is called an *ultrafilter*. If $\mathcal{U}$ is an ultrafilter on I, then for every subset A of I, either $A \in \mathcal{U}$, or $I \backslash A \in \mathcal{U}$. An ultrafilter is called *trivial* (or *non-free*) if it is generated by a single element $i_0 \in I$, i.e., $A \in \mathcal{U}$ if and only if $i_0 \in A$. There exists always a *free* ultrafilter on I.
The conjunction between ultrafilters and compact sets is shown by the following result (see [38]).

Proposition A.1.1 *Let K be a compact Hausdorff space and $\mathcal{U}$ an ultrafilter on I. Then for each family $(x_i)_{i\in I}$ with $x_i \in K$, the limit*

$$\lim_{\mathcal{U}} x_i = x$$

exists in K. This means, there is a unique point $x \in K$ such that, for each neighborhood U of x, the set $\{i \in I : x_i \in U\}$ belongs to $\mathcal{U}$.

The notion of a *Banach space ultraproduct* was introduced by Bretagnolle, Dacunha-Castelle and Krivine [9], [53], [18]. Let $(V_i)_{i\in I}$ be a family of Banach spaces. Denote by $l_\infty(I, V_i)$ the space of families $(x_i)_{i\in I}$ with $x_i \in V_i$ ($i \in I$) and

$$\|(x_i)\| \quad = \quad \sup_{i\in I} \|x_i\| \ < \ \infty\,. \tag{A.1}$$

It follows that $l_\infty(I, V_i)$ is a Banach space under the norm given by (A.1). Let

$$N_{\mathcal{U}} \quad = \quad \{(x_i)_{i\in I} \in l_\infty(I, V_i) : \lim_{\mathcal{U}} \|x_i\| \ = \ 0\}\,.$$

Then $N_{\mathcal{U}}$ is a closed subspace of $l_\infty(I, V_i)$. The quotient space

$$(V_i)_{\mathcal{U}} \quad = \quad l_\infty(I, V_i)/N_{\mathcal{U}}\,,$$

equipped with the canonical quotient norm is a Banach space, called the *ultraproduct* of the family $(V_i)_{i\in I}$. Often the ultraproduct of $(V_i)_{i\in I}$ is denoted by $\prod_{i\in I} V_i/\mathcal{U}$. If $V_i = V$, for all $i \in I$, then we speak of an *ultrapower*, denoted by $(V)_{\mathcal{U}}$ (or $V^I/\mathcal{U}$).

For an element $(x_i)_{i\in I} \in l_\infty(I, V_i)$ we denote by $(x_i)_{\mathcal{U}}$ the corresponding equivalence class in the ultraproduct $(V_i)_{\mathcal{U}}$. The norm of $(x_i)_{\mathcal{U}}$ is given by

$$\|(x_i)_{\mathcal{U}}\| \quad = \quad \lim_{\mathcal{U}} \|x_i\|\,.$$

Note that if $\mathcal{U}$ is a trivial ultrafilter generated by the element $i_0 \in I$, then the ultraproduct $(V_i)_{\mathcal{U}}$ is isometrically isomorphic to V_{i_0}.

Let $(V_i)_{i\in I}$ and $(W_i)_{i\in I}$ be families of Banach spaces indexed by the same set I. For $i \in I$, let $T_i \in \mathcal{B}(V_i, W_i)$ be such that

$$\sup_{i\in I} \|T_i\| \ < \ \infty\,.$$

Then the map

$$(V_i)_{\mathcal{U}} \ \ni \ (x_i)_{\mathcal{U}} \ \mapsto \ (\, T_i(x_i)\,)_{\mathcal{U}} \ \in \ (W_i)_{\mathcal{U}}$$

defines a bounded linear operator $(T_i)_{\mathcal{U}}$ from $(V_i)_{\mathcal{U}}$ to $(W_i)_{\mathcal{U}}$, called the *ultraproduct* of the family of operators $(T_i)_{i\in I}$. Moreover, we have

$$\|(T_i)_{\mathcal{U}}\| \;=\; \lim_{\mathcal{U}} \|T_i\| \,.$$

Let $(V_i)_{i\in I}$ be a family of Banach spaces. Then there is a canonical isometric embedding J of the ultraproduct of the dual spaces $(V_i^*)_{\mathcal{U}}$ into the dual of the ultraproduct $(V_i)_{\mathcal{U}}^*$, defined by

$$\langle J(\,(f_i)_{\mathcal{U}}\,), (x_i)_{\mathcal{U}}\rangle \;=\; \lim_{\mathcal{U}} \langle f_i, x_i\rangle \,,$$

for all $(f_i)_{\mathcal{U}} \in (V_i^*)_{\mathcal{U}}$ and $(x_i)_{\mathcal{U}} \in (V_i)_{\mathcal{U}}$ (see [38]). Henson and Moore [40] gave conditions under which $(V_i^*)_{\mathcal{U}}$ is all of $(V_i)_{\mathcal{U}}^*$. Results obtained independently by Kürsten [54] and Stern [78] on the local structure of the dual of an ultraproduct, referred to as *local duality of ultraproducts* show that, in the general case, the duality between $(V_i)_{\mathcal{U}}$ and $(V_i^*)_{\mathcal{U}}$ still holds 'locally'.

The connection between finite representability and ultrapowers was independently discovered by Henson and Moore [41] and Stern [77]. Recall that a Banach space W is called *finitely representable* in a Banach space V , if for each finite dimensional subspace $W_0 \subseteq W$ and each $\varepsilon > 0$, there exists a finite dimensional subspace $V_0 \subseteq V$, which is $(1+\varepsilon)$-isomorphic to W_0 .

Theorem A.1.2 (Henson-Moore [41], Stern [77]) *A Banach space W is finitely representable in a Banach space V if and only if there is an ultrafilter $\mathcal{U}$ such that W is isometric to a subspace of $(V)_{\mathcal{U}}$.*

The notion of finite representability was introduced by R. C. James [42] in connection with the study of superreflexivity. Recal that a Banach space V is called *superreflexive* if each Banach space W which is finitely representable in V is reflexive. The following result from [38] is a consequence of Theorem A.1.2.

Proposition A.1.3 *A Banach space V is superreflexive if and only if each ultrapower $(V)_{\mathcal{U}}$ is reflexive.*

We refer to Heinrich's survey [38] for more results on the structure of ultraproducts, their applications in Banach space theory and further references.

The ultraproduct theory for operator spaces is due to Pisier (see [67], [68] and [71]). Let $(E_i)_{i\in I}$ be a family of operator spaces and $\mathcal{U}$ an ultrafilter on I. Denote by $\hat{E} = (E_i)_{\mathcal{U}}$ the associated ultraproduct in the category of Banach spaces. For all positive integers m, define

$$M_m(\hat{E}) \quad = \quad (M_m(E_i))_{\mathcal{U}}\,. \tag{A.2}$$

as a vector space. We equip $M_m(\hat{E})$ with the norm $\|\cdot\|_m$ induced via this identification by the norm of the Banach space $(M_m(E_i))_{\mathcal{U}}$. The matrix norms $\{\|\cdot\|_m\,, m\geq 1\}$ verify the Ruan axioms (1.12), hence determine an operator space structure on $\hat{E}$. Note that, with the operator space structure of $\hat{E}$ defined as above, we have the following isometric identity

$$F\check{\otimes}(E_i)_{\mathcal{U}} \quad = \quad (F\check{\otimes}E_i)_{\mathcal{U}}\,,$$

valid if $F = M_m$. It was proved in [67] that this identity fails to be isometric in general, for instance when $F = OH_m$, or $F = S_2^m$. This is closely related to the absence of 'local reflexivity' for an arbitrary operator space (see [21]).

Let $(E_i)_{i\in I}$ and $(F_i)_{i\in I}$ be two families of operator spaces indexed by the same set I. By Proposition 10.3.2 of [29], each family $(\phi_i)_{i\in I} \in l_\infty(I, \mathcal{CB}(E_i, F_i))$ determines a mapping $(\phi_i)_{\mathcal{U}}$ defined by

$$(E_i)_{\mathcal{U}} \ni (v_i)_{\mathcal{U}} \;\mapsto\; (\,\phi_i(v_i)\,)_{\mathcal{U}} \in (F_i)_{\mathcal{U}}\,,$$

which satisfies

$$\|\,(\phi_i)_{\mathcal{U}}\,\|_{cb} \quad\leq\quad \lim_{\mathcal{U}}\|\phi_i\|_{cb}\,. \tag{A.3}$$

Therefore, we obtain a well-defined complete contraction

$$(\mathcal{CB}(E_i,F_i))_{\mathcal{U}} \quad\to\quad \mathcal{CB}((E_i)_{\mathcal{U}},(F_i)_{\mathcal{U}})\,. \tag{A.4}$$

However, in contrast to the Banach space theory, this map need not be a completely isometric injection. This problem is related to the notion of *exactness* for operator spaces. We refer to Pisier

[67] for details and related results. In particular, it is proved in [67] that if $F_i = M_m$, for all $i \in I$, where m is a positive integer, then the map (A.4) is a completely isometric injection. As a consequence, the following result holds (see [29]).

Proposition A.1.4 *If $(E_i)_{i\in I}$ is a family of operator spaces and $\mathcal{U}$ is an ultrafilter on I, then we have a completely isometric inclusion*

$$(E_i^*)_{\mathcal{U}} \quad \hookrightarrow \quad ((E_i)_{\mathcal{U}})^* .$$

Groh [36] proved that given a von Neumann algebra $\mathcal{M}$, every ultrapower of its predual $\mathcal{M}_*$ is isometric to the predual of a von Neumann algebra $\mathcal{N}$. Raynaud [75] studied ultrapowers of the Haagerup non-commutative L_p-spaces, for $1 < p < \infty$.

Theorem A.1.5 (Raynaud [75]) *Let $\mathcal{M}$ be a von Neumann algebra, $1 \le p < \infty$ and $\mathcal{U}$ an ultrafilter on the index set I. The ultrapower $L_p(\mathcal{M})^I/\mathcal{U}$ is isometric to the $L_p(\mathcal{N})$ space associated with the von Neumann algebra $\mathcal{N} = (\, L_1(\mathcal{M})^I/\mathcal{U}\,)^*$.*

Pisier [70] considered ultraproducts of non-commutative vector-valued L_p-spaces and proved the following.

Lemma A.1.6 (Pisier [70]) *Let $(E_i)_{\alpha\in I}$ be a family of operator spaces and let $\mathcal{U}$ be a nontrivial ultrafilter on I. Let $\hat{E}$ be the corresponding ultraproduct. Let $m \ge 1$ be a fixed integer. Consider a matrix $[\hat{x}_{j\,k}]$ in $M_m(\hat{E})$. Let $(x^i_{j\,k})_{i\in I}$ be a representative of $\hat{x}_{j\,k}$ with $x^i_{j\,k} \in E_i,$. Then we have*

$$\|[\hat{x}_{j\,k}]\|_{S_p^m[\hat{E}]} \quad = \quad \lim_{\mathcal{U}} \|(x^i_{j\,k})\|_{S_p^m[E_i]} .$$

Therefore, we have a completely isometric identity

$$S_p^m[\hat{E}] \quad = \quad (S_p^m[E_i])_{\mathcal{U}} . \tag{A.5}$$

We refer to Effros and Ruan [29] for more results and applications of ultraproduct techniques to the study of operator spaces.

A.2 Kouba's interpolation theorem

In this section we will give a detailed proof to the following interpolation result from Pisier [68] on the Haagerup tensor product.

Theorem A.2.1 (Kouba) *Let* (E_0, E_1) *and* (F_0, F_1) *be two compatible couples of operator spaces. Then* $(E_0 \otimes^h F_0, E_1 \otimes^h F_1)$ *is a compatible couple of operator spaces, and for all* $0 < \theta < 1$ *we have a complete isometry*

$$[\, E_0 \otimes^h F_0, E_1 \otimes^h F_1 \,]_\theta \quad = \quad E_\theta \otimes^h F_\theta \,, \tag{A.6}$$

where $E_\theta = [E_0, E_1]_\theta$, *respectively,* $F_\theta = [F_0, F_1]_\theta$.

The proof given in [68] of the fact that (A.6) is an *isometry* in the finite dimensional case is based on the observation that the Banach space underlying the Haagerup tensor product of two operator spaces is a 'γ-tensor product' of the kind studied in [66]. Consequently, the results of Kouba from [52] apply in this case. We will give a different proof, based on Pisier's approach from [66], using factorization of vector-valued analytic functions. In order to pass to the infinite dimensional case, ideas from Kouba's paper [52] are used. The arguments showing that (A.6) is a *complete* isometry are given in [68].

We begin with the following more general results on the Haagerup tensor product that will be useful in the proof of the theorem, more precisely, in showing that $(E_0 \otimes^h E_1, F_0 \otimes^h F_1)$ forms a compatible couple for interpolation.

Lemma A.2.2 *Let* E, F *be operator spaces. If* $u \in E \otimes^h F$ *and* $(f \otimes Id_F)(u) = 0$ *for all* $f \in E_1^*$, *then* $u = 0$.

Proof. Note that we have $(f \otimes g)(u) = 0$, for $f \in E_1^*$ and $g \in F_1^*$. Indeed,

$$(f \otimes g)(u) \,=\, (f \otimes \mathrm{Id}_F)(\mathrm{Id}_E \otimes g)(u) \,=\, g(f \otimes \mathrm{Id}_F)(u) \,=\, 0\,.$$

By Proposition 9.4.9 in [29], we have

$$\|u\|_h \quad = \quad \sup\{\|(f \odot g)(u)\| : f \in M_{1,p}(E^*)_1, g \in M_{p,1}(F^*)_1, p \in \mathbb{N}\}\,. \tag{A.7}$$

Recall that, by (1.14) we have

$$M_{1,p}(E^*)_1 \;=\; CB(E, M_{1,p})_1 \quad \text{and} \quad M_{p,1}(F^*)_1 \;=\; CB(F, M_{p,1})_1 \,.$$

Thus, if $f = [f_1, \ldots, f_p] \in M_{1,p}(E^*)_1$ and $g = \begin{bmatrix} g_1 \\ \vdots \\ g_p \end{bmatrix} \in M_{p,1}(F^*)_1$, then $(f \odot g)(u) = \sum_{i=1}^{p} (f_i \otimes g_i)(u) = 0$. By (A.7), this implies that $\|u\|_h = 0$, i.e., $u = 0$. □

Proposition A.2.3 *Let $E_i, F_i\,, i = 0,1$ be operator spaces. Let $\phi_i : E_i \to F_i\,, i = 0,1$ be complete contractions and $1-1$. Then $\phi_0 \otimes \phi_1 : E_0 \otimes^h E_1 \to F_0 \otimes^h F_1$ is a complete contraction and $1-1$.*

Proof. The fact that $\phi_0 \otimes \phi_1$ is a complete contraction follows from the injectivity of the Haagerup tensor product. To show that $\phi_0 \otimes \phi_1$ is $1-1$, note that $\phi_0 \otimes \phi_1 = (\phi_0 \otimes \mathrm{Id}_{F_1}) \circ (\mathrm{Id}_{E_0} \otimes \phi_1)$, so it suffices to prove that both $\phi_0 \otimes \mathrm{Id}_{F_1}$ and $\mathrm{Id}_{E_0} \otimes \phi_1$ are $1-1$.
Let $u \in E_0 \otimes^h E_1$ such that $(\mathrm{Id}_{E_0} \otimes \phi_1)(u) = 0$. Then, for all $f \in (E_0)^*_1$ we have

$$0 \;=\; (f \otimes \mathrm{Id}_{E_1})(\mathrm{Id}_{E_0} \otimes \phi_1) \;=\; (f \otimes \phi_1)(u) \;=\; \phi_1(f \otimes \mathrm{Id}_{E_1})(u)\,.$$

Since ϕ_1 is $1-1$, it follows that $(f \otimes \mathrm{Id}_{E_1})(u) \;=\; 0$. By Lemma A.2.2, this implies that $u = 0$. Thus $\mathrm{Id}_{E_0} \otimes \phi_1$ is $1-1$. The proof for $\phi_0 \otimes \mathrm{Id}_{E_1}$ is similar. □

Proof of Theorem A.2.1. We first show that $(E_0 \otimes^h F_0, E_1 \otimes^h F_1)$ is a compatible couple of operator spaces. Indeed, we claim that both spaces $E_0 \otimes^h F_0$ and $E_1 \otimes^h F_1$ embed continuously into $(E_0 + E_1) \otimes^h (F_0 + F_1)$. Let

$$\begin{aligned} N_0 &= \{x = (x_0, x_1) \in E_0 \times E_1 : x_0 + x_1 = 0\}\,, \\ N_1 &= \{y = (y_0, y_1) \in F_0 \times F_1 : y_0 + y_1 = 0\}\,. \end{aligned}$$

Denote by ϕ_0, ψ_0 the inclusion maps $\phi_0 : V_0 \to E_0 + E_1$ given by $\phi_0(x) = (x, 0) + N_0\,, x \in E_0\,,$ respectively $\psi_0 : E_1 \to E_0 + E_1$, given by $\psi_0(x) = (0, x) + N_0\,, x \in E_1$. Similarly, denote by ϕ_1, ψ_1 the inclusion maps $\phi_1 : F_0 \to F_0 + F_1$ given by $\phi_1(x) = (x, 0) + N_1\,, x \in F_0\,,$ respectively,

$\psi_1 : F_1 \to F_0 + F_1$, given by $\psi_1(x) = (0, x) + N_1\,, x \in F_1$. It is immediate to see that $\phi_0, \psi_0, \phi_1, \psi_1$ are all complete contractions and $1-1$. By Proposition A.2.3, it follows that the maps

$$\phi_0 \otimes \phi_1 : E_0 \otimes^h F_0 \to (E_0 + E_1) \otimes^h (F_0 + F_1) \quad \text{and} \quad \psi_0 \otimes \psi_1 : E_1 \otimes^h F_1 \to (E_0 + E_1) \otimes^h (F_0 + F_1)$$

are complete contractions and $1-1$, which proves the claim. Hence, for $0 < \theta < 1$, we can consider the interpolation space $[\, E_0 \otimes^h F_0, E_1 \otimes^h F_1\,]_\theta$.

We now show that (A.6) holds isometrically at the Banach space level.

1. The finite dimensional case:

Note that we can assume, without loss of generality, that

$$\begin{cases} \dim E_0 & = & \dim E_1 \; = \; n\,, \\ \dim F_0 & = & \dim \mathcal{F}_1 \; = \; m\,. \end{cases}$$

Indeed, by general results on complex interpolation, if we denote the closure of $E_0 \cap E_1$ in E_0 by E_0^0 and, respectively, the closure of $E_0 \cap E_1$ in E_1 by E_1^0, we then have

$$E_\theta \;=\; [E_0^0, E_1]_\theta \;=\; [E_0, E_1^0]_\theta \;=\; [E_0^0, E_1^0]_\theta\,.$$

Therefore, we may also assume that the spaces E_0 and E_1, respectively, F_0 and F_1 have dense intersection. Next, observe that $\dim (E_0 \otimes F_0) \;=\; \dim (E_1 \otimes F_1) \;=\; nm$. To prove this, let $\{e_i^0\}_{i=1}^n$ be a basis for E_0, and $\{f_j^0\}_{j=1}^m$ be a basis for F_0. Clearly, the vectors $\{e_i^0 \otimes f_j^0\,, 1 \le i \le n, 1 \le j \le m\}$ are linearly independent over $E_0 \otimes F_0$. Moreover, they span $E_0 \otimes F_0$. Indeed, for any $v \;=\; x \otimes y \;\in\; E_0 \otimes F_0$, it follows that $x \;=\; \sum_{i=1}^n \alpha_i e_i^0$, respectively, $y \;=\; \sum_{j=1}^m \beta_j f_j^0$, for some $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m \in \mathbb{C}$. Hence

$$v \;=\; \left(\sum_{i=1}^n \alpha_i e_i^0 \right) \otimes \left(\sum_{j=1}^m \beta_j f_j^0 \right) \;=\; \sum_{i,j=1}^{nm} \alpha_i \beta_j e_i^0 \otimes f_j^0\,.$$

Furthermore, if $u \in E_0 \otimes^h F_0$, it follows by Lemma 1.2.6 that there exists a representation

$$u \;=\; \sum_{i=1}^{l} x_i \otimes y_i \,, \tag{A.8}$$

where $\{x_1, \ldots, x_l\}$ are linearly independent in E_0 , and $\{y_1, \ldots, y_l\}$ are linearly independent in F_0 . In particular, this implies that $l \leq \min\{n, m\}$. So, by (possibly) adding zeroes, we may assume that every element $u \in E_0 \otimes^h F_0$ has a representation of the form (A.8), where $l = \min\{n, m\}$. A similar argument applies for the space $E_1 \otimes^h F_1$ .In particular, it follows that both spaces $E_0 \otimes^h F_0$ and $E_1 \otimes^h F_1$ embed continuously into $\mathbb{C}^{nm}$.

<u>Claim 1</u>: Let $u \in E_\theta \otimes^h F_\theta$ with $\|u\|_h < 1$. We will show that $u \in [\, E_0 \otimes^h F_0, E_1 \otimes^h F_1 \,]_\theta$ and $\|u\|_\theta < 1$.

By Lemma 1.2.6, there exist $x = [x_1, \ldots, x_l] \in M_{1,l}(E_\theta)$ and $y = \begin{bmatrix} y_1 \\ \vdots \\ y_l \end{bmatrix} \in M_{l,1}(F_\theta)$ with $\|x\| , \|y\| < 1$, such that $u = x \odot y$, where $l = \min\{n, m\}$, as proved above. By (1.18), we have

$$M_{1,l}(E_\theta) \;=\; [M_{1,l}(E_0), M_{1,l}(E_1)]_\theta \,.$$

Therefore, there exists a map $f = [f_1, \ldots, f_l] \in \mathcal{F}_{M_{1,l}(E_0)+M_{1,l}(E_1)}$, such that $f(\theta) = x$ and $\|f\|_{\mathcal{F}} < 1$. Respectively, since

$$M_{l,1}(F_\theta) \;=\; [M_{l,1}(F_0), M_{l,1}(F_1)]_\theta \,,$$

it follows that there exists a map $g = \begin{bmatrix} g_1 \\ \vdots \\ g_l \end{bmatrix} \in \mathcal{F}_{M_{l,1}(F_0)+M_{l,1}(F_1)}$, such that $g(\theta) = y$ and $\|g\|_{\mathcal{F}} < 1$. Note that for $1 \leq i, j \leq l$, the map

$$f_i \otimes g_j : S \;\to\; E_0 \otimes^{\mathrm{h}} F_0 + E_1 \otimes^{\mathrm{h}} F_1$$

is analytic in S_0 and satisfies $f_i \otimes g_j(\theta) = x_i \otimes y_j$.

Define $F : S \to E_0 \otimes^h F_0 + E_1 \otimes^h F_1$ by

$$F \;=\; f \odot g \;=\; [f_1, \ldots, f_l] \odot \begin{bmatrix} g_1 \\ \vdots \\ g_l \end{bmatrix} \;=\; \sum_{i=1}^{l} f_i \otimes g_i \,.$$

Then $F \in \mathcal{F}_{E_0 \otimes^h F_0 + E_1 \otimes^h F_1}$ and $F(\theta) = u$. This implies that $u \in [\, E_0 \otimes^h F_0, E_1 \otimes^h F_1 \,]_\theta$. Moreover,

$$\begin{aligned} \|u\|_\theta &\le \|F\|_{\mathcal{F}_{E_0 \otimes^h F_0 + E_1 \otimes^h F_1}} \\ &= \max\{ \sup_{t \in \mathbb{R}} \|F(it)\|_{E_0 \otimes^h F_0} \,, \sup_{t \in \mathbb{R}} \|F(1+it)\|_{E_1 \otimes^h F_1} \} \,. \end{aligned}$$

By the definition of the Haagerup tensor product norm and (1.4), we have for all $t \in \mathbb{R}$,

$$\begin{aligned} \|F(it)\|_{E_0 \otimes^h F_0} &\le \|f(it)\|_{M_{1,l}(E_0)} \|g(it)\|_{M_{l,1}(F_0)} \\ &\le \|f\|_{\mathcal{F}_{M_{1,l}(E_0) + M_{1,l}(E_1)}} \|g\|_{\mathcal{F}_{M_{l,1}(F_0) + M_{l,1}(F_1)}} \;<\; 1 \,, \end{aligned}$$

and, respectively,

$$\begin{aligned} \|F(1+it)\|_{E_1 \otimes^h F_1} &\le \|f(1+it)\|_{M_{1,l}(E_1)} \|g(it)\|_{M_{l,1}(F_1)} \\ &\le \|f\|_{\mathcal{F}_{M_{1,l}(E_0) + M_{1,l}(E_1)}} \|g\|_{\mathcal{F}_{M_{l,1}(F_0) + M_{l,1}(F_1)}} \;<\; 1 \,. \end{aligned}$$

Thus $\|u\|_\theta < 1$ and the claim is proved.

Claim 2: Let $u \in [E_0 \otimes^h F_0, E_1 \otimes^h F_1]_\theta$ with $\|u\|_\theta < 1$. We will show that $u \in E_\theta \otimes^h F_\theta$ and $\|u\|_h < 1$.

By the assumption, it follows that there exists a map

$$f : S \to E_0 \otimes^h F_0 + E_1 \otimes^h F_1 \hookrightarrow (E_0 + E_1) \otimes^h (F_0 + F_1) \,,$$

such that $f \in \mathcal{F}_{E_0 \otimes^h F_0 + E_1 \otimes^h F_1}$, satisfying $f(\theta) = x$ and $\|f\|_{\mathcal{F}_{E_0 \otimes^h F_0 + E_1 \otimes^h F_1}} < 1$.
We wish to obtain a representation

$$f \;=\; g \odot h\,,$$

where $g : S \to E_0 + E_1$, $h : S \to F_0 + F_1$, and both are analytic in S_0. For this, recall that the strip S is conformally equivalent to the closed unit disc $D = \{z \in \mathbb{C} : |z| \leq 1\}$. Denote by D_0 the interior of D. If X is an operator space, we denote by $A(X)$ the set

$$A(X) \;=\; \{f : D \to X : f \text{ is analytic in } D_0 \text{ and continuous on } D\}\,.$$

Then $A(X)$ is a Banach space in the sup norm

$$\|f\|_{A(X)} \;=\; \sup_{z \in D} \|f(z)\|_X \;=\; \sup_{t \in \partial D} \|f(t)\|_X\,, \quad f \in A(X)\,.$$

In fact, $A(X)$ is an operator space, with matrix norms defined for all positive integers m by

$$M_m(A(X)) \;=\; A(M_m(X))\,.$$

Our goal is to prove that given finite dimensional operator spaces X_1 and X_2, if $f \in A(X_1 \otimes^h X_2)$ with $\|f\|_{A(X_1 \otimes^h X_2)} < 1$, then

$$f \,\in\, A(X_1) \otimes^h A(X_2) \quad \text{and} \quad \|f\|_{A(X_1) \otimes^h A(X_2)} \,<\, 1\,.$$

For this, we will follow the argument in [66], Appendix B. As explained there, we may assume without loss of generality that $X_2 = \overline{X_2}$.
In the following we will show that for a finite dimensional operator space X, if $f \in A(X \otimes^h \overline{X})$ with $\|f\|_{A(X \otimes^h \overline{X})} < 1$, then

$$f \,\in\, A(X) \otimes^h A(\overline{X}) \quad \text{and} \quad \|f\|_{A(X) \otimes^h A(\overline{X})} \,<\, 1\,.$$

The previous statement follows then by taking $X = X_1 \oplus X_2$.

By the assumption, the map $f : D \to X \otimes^h \overline{X}$ is analytic in D_0 and continuous on D. Denote by f, as well, the restriction of f to the boundary ∂D. Moreover, for all $t \in \partial D$, we have

$$\|f(t)\|_{X \otimes^h \overline{X}} \;\leq\; \|f\|_{A(X \otimes^h \overline{X})} \;<\; 1\,. \tag{A.9}$$

Let $\dim X = n$. By the definition of the Haagerup tensor product norm, it follows from (A.9) that there exist $v(t) \in M_{1,n}(X)$ and $w(t) \in M_{1,n}(X)$, both with norm < 1 such that

$$f(t) \;=\; [v_1(t), \ldots, v_n(t)] \odot \begin{bmatrix} \overline{w_1(t)} \\ \vdots \\ \overline{w_n(t)} \end{bmatrix} \;=\; v(t) \odot \overline{w(t)}^{\,*}\,.$$

Note that a priori the maps $t \mapsto v(t)$ and, respectively, $t \mapsto w(t)$ are not necessarely continuous. However, by Michael's selection theorem (see [60]), there exist continuous selections $\partial D \ni t \mapsto v(t) \in M_{1,n}(X)$ and, respectively, $\partial D \ni t \mapsto w(t) \in M_{1,n}(X)$ such that for all $t \in \partial D$,

$$\|v(t)\|_{M_{1,n}(X)} < 1 \quad \text{and} \quad \|w(t)\|_{M_{1,n}(X)} < 1\,. \tag{A.10}$$

Let $\{e_i\,, 1 \leq i \leq n\}$ be an orthonormal basis for X (over $\mathbb{C}$). We identify $X \otimes^h \overline{X}$ with the space $\mathcal{F}(X^*, \overline{X})$ of finite rank operators which are $\sigma(X^*, X)-$continuous. The identification is given by the following correspondence

$$X \otimes^h \overline{X} \;\ni\; u \;=\; \sum_{i=1}^{n} x_i \otimes y_i \quad \mapsto \quad T_u : X^* \to \overline{X}\,,$$

where $T_u(f) \;=\; \sum\limits_{i=1}^{n} f(x_i) y_i$, for all $f \in X^*$. Let

$$u \;=\; \left[\frac{e_1}{\sqrt{n}}, \ldots, \frac{e_1}{\sqrt{n}}\right] \odot \begin{bmatrix} \frac{e_1}{\sqrt{n}} \\ \vdots \\ \frac{e_n}{\sqrt{n}} \end{bmatrix} \;\in\; X \otimes^h \overline{X}\,.$$

It follows that

$$\|u\|_{X\otimes^h \overline{X}} \;\le\; \left\| \left[\frac{e_1}{\sqrt{n}}, \dots, \frac{e_1}{\sqrt{n}} \right] \right\| \cdot \left\| \begin{bmatrix} \frac{e_1}{\sqrt{n}} \\ \vdots \\ \frac{e_n}{\sqrt{n}} \end{bmatrix} \right\| \;=\; 1\,.$$

By (A.9) and (A.10), there exists $\varepsilon > 0$ such that for all $t \in \partial D$,

$$\|f(t)\|_{X\otimes^h \overline{X}} \;<\; 1-\epsilon \quad \text{and} \quad \|v(t)\odot \overline{v(t)}^{\,*}\|_{X\otimes^h \overline{X}} \;<\; 1-\epsilon\,.$$

Let $t \in \partial D$. For $1 \le i \le n$, we have

$$v_i(t) \;=\; \sum_{k=1}^{n} v_i^k(t) e_k\,,$$

for some $v_i^k(t) \in \mathbb{C}\,, 1 \le k \le n$. It follows that

$$\begin{aligned} v(t)\odot \overline{v(t)}^{\,*} \;&=\; \sum_{i=1}^{n} v_i(t) \otimes \overline{v_i(t)} \\ &=\; \sum_{i=1}^{n} \left(\sum_{k=1}^{n} v_i^k(t)\, e_k \right) \otimes \left(\sum_{j=1}^{n} \overline{v_i^j(t)}\, e_j \right) \\ &=\; \sum_{i=1}^{n} \sum_{k,j=1}^{n} v_i^k(t) \overline{v_i^j(t)}\, e_k \otimes e_j\,. \end{aligned}$$

Therefore, the matrix associated to $v(t)\odot \overline{v(t)}^{\,*}$ (in the basis $\{e_i, 1 \le i \le n\}$) is

$$\begin{pmatrix} \sum_{i=1}^{n} v_i^1(t)\overline{v_i^1(t)} & \sum_{i=1}^{n} v_i^1(t)\overline{v_i^2(t)} & \dots & \sum_{i=1}^{n} v_i^1(t)\overline{v_i^1(t)} \\ \vdots & \vdots & & \vdots \\ \sum_{i=1}^{n} v_i^n(t)\overline{v_i^1(t)} & \sum_{i=1}^{n} v_i^n(t)\overline{v_i^2(t)} & \dots & \sum_{i=1}^{n} v_i^n(t)\overline{v_i^1(t)} \end{pmatrix}.$$

Note that this is a positive definite scalar matrix. Define a map Q by

$$\partial D \ni t \;\mapsto\; Q(t) \;=\; v(t)\odot \overline{v(t)}^{\,*} + \varepsilon u \;\in\; X \otimes^h \overline{X}\,.$$

The following properties are satisfied for all $t \in \partial D$:

1. The scalar matrix associated to $Q(t)$ is positive definite.

2. $\|Q(t)\|_{X \otimes^h \overline{X}} < 1$.

3. $Q(t)$ is bounded away from zero.

Thus, Q can be viewed as a map from ∂D into the set of all positive definite, invertible matrices. Hence, Devinatz's theorem (see [39], Theorem 17) applies. It follows that Q is factorable, i.e., there exists $F : D \to \mathcal{B}(l_2^n, X)$ an analytic operator-valued function, such that for all $t \in \partial D$ we have

$$Q(t) \;=\; F(t) \odot \overline{F(t)}^{\,*}.$$

Moreover, as explained in [39], the factorization can be chosen in such a way that F is outer (as a matrix-valued function) and invertible. So that the inverse function $z \mapsto F^{-1}(z)$ is also analytic and bounded away from zero.
Furthermore, recall that for all $t \in \partial D$, we have $v(t) \odot \overline{v(t)}^{\,*} \leq Q(t)$. Therefore we can write

$$v(t) \;=\; F(t)H(t)$$

for some operator-valued function $H(t) : l_2^n \to l_2^n$ with $\|S(t)\| < 1$. This implies that

$$f(t) \;=\; v(t) \odot \overline{w(t)}^{\,*} \;=\; F(t)H(t) \odot \overline{w(t)}^{\,*}.$$

For all $t \in \partial D$, let $G(t) \;=\; H(t) \odot \overline{w(t)}^{\,*}$. Since the map $t \mapsto F^{-1}(t)f(t)$ is the boundary value of the analytic function $z \mapsto F^{-1}(z)f(z)$, it follows that G must extend analytically in D, i.e., $G \in A(\overline{X})$. Moreover, since for all $t \in \partial D$ we have

$$\|F(t)\| < 1 \quad \text{and} \quad \|G(t)\| < 1,$$

it follows by the maximum modulus principle that

$$\|F\|_{A(X)} \;\leq\; 1 \quad \text{and} \quad \|G\|_{A(\overline{X})} \;\leq\; 1.$$

Therefore we have obtained a factorization $f = F \odot G \in A(X) \otimes^h A(\overline{X})$ such that

$$\begin{aligned} \|f\|_{A(X \otimes^h \overline{X})} &= \|F \odot G\|_{A(X) \otimes^h A(\overline{X})} \\ &\leq \|F\|_{A(X)} \cdot \|G\|_{A(\overline{X})} \leq 1 . \end{aligned}$$

This ends the proof of the final dimensional case.

2. The infinite dimensional case:

The key result in the proof of the infinite dimensional case is Lemma 4.1 in [52]. Let us first recall some notation from the paper [52], that will be used only throughout this proof. Let (X_0, X_1) be an interpolation couple, G a (finite dimensional) subspace of $X_0 \cap X_1$ and $0 < \theta < 1$. Then, for $k = 0, 1$, denote by G_k the space G considered as a subspace of X_k, and by G_θ the space $[G_0, G_1]_\theta$, and not considered as a subspace of X_θ. We now recall the aforementioned result.

Lemma A.2.4 *Let (X_0, X_1) be an interpolation couple, X a subspace of $X_0 \cap X_1$ which is dense in both X_0 and X_1, and $0 < \theta < 1$. Then, for every $0 < \varepsilon < 1$ and every finite dimensional subspace G of X, there exists a finite dimensional subspace $\tilde{G}$ of X containing G and such that for all $x \in G$*

$$(1 - \varepsilon)\|x\|_{\tilde{G}_\theta} \leq \|x\|_{X_\theta} \leq \|x\|_{\tilde{G}_\theta} .$$

Denote by E the space $E_0 \cap E_1$, and, respectively, by F the space $F_0 \cap F_1$. By the injectivity of the Haagerup tensor product it follows that $E \otimes F$ is a norm dense subspace of $E_\theta \otimes^h F_\theta$. Moreover, note that $E \otimes F$ is also norm dense in the interpolation space $[E_0 \otimes^h F_0, E_1 \otimes^h F_1]_\theta$. Indeed, by Theorem 4.2.2 of [3], the intersection $(E_0 \otimes^h F_0) \cap (E_1 \otimes^h F_1)$ is a dense subspace of $[E_0 \otimes^h F_0, E_1 \otimes^h F_1]_\theta$. Thus, by applying Lemma A.2.4 to the couple $(E_0 \otimes^h F_0, E_1 \otimes^h F_1)$ and the subspace $E \otimes F$ of $(E_0 \otimes^h F_0) \cap (E_1 \otimes^h F_1)$, we obtain the above statement. Hence, it suffices to show that for all $u \in E \otimes F$ we have

$$\|u\|_{[E_0 \otimes^h F_0, E_1 \otimes^h F_1]_\theta} \leq \|u\|_{E_\theta \otimes^h F_\theta} \leq \|u\|_{[E_0 \otimes^h F_0, E_1 \otimes^h F_1]_\theta} . \tag{A.11}$$

Let $u \in E \otimes F$. Then

$$u = \sum_{i=1}^{n} v_i \otimes w_i ,$$

for some $n \in \mathbb{N}, v_i \in E, w_i \in F, 1 \leq i \leq n$. Define

$$X = \operatorname{Span}\{v_i, 1 \leq i \leq n\} \subseteq E \quad \text{and} \quad Y = \operatorname{Span}\{w_i, 1 \leq i \leq n\} \subseteq F. \tag{A.12}$$

We now prove the right hand-side inequality in (A.11). For this we apply Lemma A.2.4 for the interpolation couple $(E_0 \otimes^h F_0, E_1 \otimes^h F_1)$ and the subspace $E \otimes F$ of $(E_0 \otimes^h F_0) \cap (E_1 \otimes^h F_1)$, which is dense in both $E_0 \otimes^h F_0$ and $E_1 \otimes^h F_1$. Let $\varepsilon > 0$. Since $X \otimes^h Y$ is a finite dimensional subspace of $E \otimes F$, it follows that there exists a finite dimensional subspace of $E \otimes F$, containing $X \otimes^h Y$, which we may assume to be of the form $G \otimes H$ (where G is a subspace of E and H is a subspace of F), such that

$$\|u\|_{[(G\otimes H)_0, (G\otimes H)_1]_\theta} \leq (1+\varepsilon)\|u\|_{[E_0 \otimes^h F_0, E_1 \otimes^h F_1]_\theta}. \tag{A.13}$$

Note that, for $k = 0, 1$, the identity map $\mathrm{Id} : (G \otimes H)_k \to G_k \otimes^h H_k$ is an isometry. By interpolation we obtain

$$\|u\|_{[G_0 \otimes^h H_0, G_1 \otimes^h H_1]_\theta} \leq \|u\|_{[(G\otimes H)_0, (G\otimes H)_1]_\theta}. \tag{A.14}$$

By the finite dimensional case it follows that, isometrically,

$$[G_0 \otimes^{\mathrm{h}} H_0, G_1 \otimes^{\mathrm{h}} H_1]_\theta = G_\theta \otimes^{\mathrm{h}} H_\theta,$$

which implies that

$$\|u\|_{[G_0\otimes^h H_0, G_1\otimes^h H_1]_\theta} = \|u\|_{G_\theta \otimes^h H_\theta}. \tag{A.15}$$

Therefore, by (A.15), (A.14) and (A.13) it follows that

$$\begin{aligned} \|u\|_{E_\theta \otimes^h F_\theta} &\leq \|u\|_{G_\theta \otimes^h H_\theta} \\ &\leq (1+\varepsilon)\|u\|_{[E_0 \otimes^h F_0, E_1 \otimes^h F_1]_\theta}. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrarily chosen, the right hand-side inequality in (A.11) is proved.

We now show the left hand-side inequality in (A.11). For this, we need to choose a particular representation of u as element of the algebraic tensor product $E \otimes F$. Namely, by Proposition 9.4.9 in [29], there exist a positive integer n and vectors $v = [v_1, \dots, v_n] \in M_{1,n}(E) \subseteq M_{1,n}(E_\theta)$, respectively, $w = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \in M_{n,1}(F) \subseteq M_{n,1}(F_\theta)$ such that

$$\|u\|_{E_\theta \otimes^h F_\theta} = \| [v_1, \dots, v_n] \|_{M_{1,n}(E_\theta)} \cdot \left\| \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \right\|_{M_{n,1}(F_\theta)} .$$

As before, let X and Y be defined by (A.12), and let $\varepsilon > 0$. We apply Lemma A.2.4 to the interpolation couple $(M_{1,n}(E_0), M_{1,n}(E_1))$, the subspace $M_{1,n}(E) = M_{1,n}(E_0) \cap M_{1,n}(E_1)$ of $[M_{1,n}(E_0), M_{1,n}(E_1)]_\theta$, and the finite dimensional subspace $M_{1,n}(X)$ of $M_{1,n}(E)$. Hence, there exists a finite dimensional subspace $\tilde{X}$ with $M_{1,n}(X) \subseteq \tilde{X} \subseteq M_{1,n}(E)$ such that, for all $x \in M_{1,n}(X)$, we have

$$\|x\|_{\tilde{X}_\theta} \leq (1 + \varepsilon) \|x\|_{M_{1,n}(E_\theta)} . \tag{A.16}$$

We now construct a finite dimensional space $\hat{X}$ such that

$$X \subseteq \hat{X} \subseteq E \quad \text{and} \quad \tilde{X} \subseteq M_{1,n}(\hat{X}) .$$

Let $m = \dim \tilde{X}$. Then $\tilde{X} = \operatorname{Span}\{x_1, \dots, x_m\}$, where for $1 < i < m$, $x_i = [x_1^i, \dots, x_n^i]$, for some $x_j^i \in E$, $1 \leq j \leq n$. Define $\hat{X}$ by

$$\hat{X} = \operatorname{Span}\{x_j^i , 1 \leq i \leq m , 1 \leq j \leq n\} .$$

It follows that $\dim \hat{X} \leq nm$, and $\hat{X}$ clearly satisfies the desired properties.

Similarly, by applying Lemma A.2.4 to the interpolation couple $(M_{n,1}(F_0), M_{n,1}(F_1))$, the subspace

$M_{n,1}(F) = M_{n,1}(F_0) \cap M_{n,1}(F_1)$ of $[M_{n,1}(F_0)\,, M_{n,1}(F_1)]_\theta$, and the finite dimensional subspace $M_{n,1}(Y)$ of $M_{n,1}(F)$, there exists a finite dimensional subspace $\tilde{Y}$ with $M_{n,1}(Y) \subseteq \tilde{Y} \subseteq M_{n,1}(F)$ such that, for all $y \in M_{n,1}(Y)$, we have

$$\|y\|_{\tilde{Y}_\theta} \leq (1+\varepsilon)\,\|y\|_{M_{n,1}(F_\theta)}\,. \tag{A.17}$$

In the same way as above, we construct a finite dimensional space $\hat{Y}$ such that

$$Y \subseteq \hat{Y} \subseteq F \quad \text{and} \quad \tilde{Y} \subseteq M_{n,1}(\hat{X})\,.$$

By (A.16) and (A.17) it follows that

$$\begin{aligned}
\|u\|_{\hat{X}\otimes^h \hat{Y}} &\leq \|\,[v_1,\ldots,v_n]\,\|_{M_{1,n}(\hat{X})} \cdot \left\| \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \right\|_{M_{n,1}(\hat{Y}_\theta)} \\
&\leq \|\,[v_1,\ldots,v_n]\,\|_{\tilde{X}_\theta} \cdot \left\| \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \right\|_{\tilde{Y}_\theta} \\
&\leq (1+\varepsilon)^2\,\|\,[v_1,\ldots,v_n]\,\|_{M_{1,n}(E_\theta)} \cdot \left\| \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \right\|_{M_{n,1}(F_\theta)} \\
&= (1+\varepsilon)^2\,\|u\|_{E_\theta\otimes^h F_\theta}\,.
\end{aligned} \tag{A.18}$$

On the other hand, for $k = 0, 1$, we have completely isometric embeddings

$$\hat{X}_k \otimes \hat{Y}_k \hookrightarrow E_k \otimes^h F_k\,.$$

Therefore, by interpolation, we obtain a contraction

$$[\,\hat{X}_0 \otimes^h \hat{Y}_0\,, \hat{X}_1 \otimes^h \hat{Y}_1\,]_\theta \;\to\; [\,E_0 \otimes^h F_0\,, E_1 \otimes^h F_1\,]_\theta\,. \tag{A.19}$$

By the finite dimensional case, it follows that, isometrically,

$$[\,\hat{X}_0 \otimes^h \hat{Y}_0\,, \hat{X}_1 \otimes^h \hat{Y}_1\,]_\theta \;\;=\;\; \hat{X}_\theta \otimes^h \hat{Y}_\theta\,. \tag{A.20}$$

This implies that $u \in [\,E_0 \otimes^h F_0\,, E_1 \otimes^h F_1\,]_\theta$ and, moreover, by (A.19), (A.20) and (A.19) we have

$$\begin{aligned}\|u\|_{[\,E_0 \otimes^h F_0\,, E_1 \otimes^h F_1\,]_\theta} &\leq \|u\|_{[\,\hat{X}_0 \otimes^h \hat{Y}_0\,, \hat{X}_1 \otimes^h \hat{Y}_1\,]_\theta} \\ &= \|u\|_{\hat{X}_\theta \otimes^h \hat{Y}_\theta} \\ &\leq (1+\varepsilon)^2\, \|u\|_{E_\theta \otimes^h F_\theta}\,.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrarily chosen, the left hand-side inequality in (A.11) follows. The proof of the infinite dimensional case is complete.

We now show that (A.6) holds *completely* isometrically. Recall that if V and W are operator spaces, then, for every positive integer m, we have a natural isometry (see [6])

$$M_m(V \otimes^h W) \;\;=\;\; M_{m,1}(V) \otimes^h M_{1,m}(W)\,. \tag{A.21}$$

In particular, we obtain the isometry

$$M_m(E_\theta \otimes^h F_\theta) \;\;=\;\; M_{m,1}(E_\theta) \otimes^h M_{1,m}(F_\theta)\,. \tag{A.22}$$

Furthermore, by (1.18), it follows that, isometrically,

$$M_{m,1}(E_\theta) \otimes^h M_{1,m}(F_\theta) \;\;=\;\; [\,M_{m,1}(E_0)\,, M_{m,1}(E_1)\,]_\theta \otimes^h [\,M_{1,m}(F_0)\,, M_{1,m}(F_1)\,]_\theta\,. \tag{A.23}$$

By (A.6) applied to the compatible couples $(M_{m,1}(E_0)\,, M_{m,1}(E_1)\,)$ and $(M_{1,m}(F_0)\,, M_{1,m}(F_1)\,)$, we obtain the isometry

$$[M_{m,1}(E_0), M_{m,1}(E_1)]_\theta \otimes^h [M_{1,m}(F_0)\,, M_{1,m}(F_1)]_\theta = [M_{m,1}(E_0) \otimes^h M_{1,m}(F_0)\,, M_{m,1}(E_1) \otimes^h M_{1,m}(F_1)]_\theta\,.$$

Furthermore, by (A.22) we have the isometries

$$M_{m,1}(E_0) \otimes^h M_{1,m}(F_0) \;=\; M_m(E_0 \otimes^h F_0)\,, \tag{A.24}$$

$$M_{m,1}(E_1) \otimes^h M_{1,m}(F_1) \;=\; M_m(E_1 \otimes^h F_1)\,. \tag{A.25}$$

Combining (A.25) and (A.25) with (1.17), it follows that, isometrically,

$$\begin{aligned}[\,M_{m,1}(E_0) \otimes^h M_{m,1}(E_1), M_{1,m}(F_0) \otimes^h M_{1,m}(F_1)\,]_\theta &= [M_m(E_0 \otimes^h F_0)\,, M_m(E_1 \otimes^h F_1)\,]_\theta \\ &= M_m(\,[\,E_0 \otimes^h F_0\,, E_1 \otimes^h F_1\,]_\theta\,)\,.\end{aligned}$$

Hence, we have proved that, isometrically,

$$M_m(E_\theta \otimes^h F_\theta) \;=\; M_m(\,[\,E_0 \otimes^h F_0\,, E_1 \otimes^h F_1\,]_\theta\,)\,. \tag{A.26}$$

This shows that (A.6) holds completely isometrically, and the theorem is proved. $\square$

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Vita

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