The Howe-Moore property and isometric actions on HS 3.2.201
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1. Howe - Moore property And S

Workshop

G loc-cpct, 2nd ctbl v.N. alg

Def: G is Howe-Moore, if for every unitary repr π of G either π has (non-zero) fixed vectors or π has C_0-coefficients (i.e. $\forall\ \xi, \eta \in H_\pi : \lim_{g \to \infty} \langle \pi(g)\xi, \eta \rangle = 0$)

Ex 1 (Howe-Moore thm '79):
Ctd, non-cpct, simple Lie gps are HM
More generally, if F is a local field, G a simple algebraic gp / F, $G = \mathbb{G}(F)^+$ (subgp gen by unipotent ?) is HM

Ex 2 (Lubotzky-Mozes '91)
let T be a regular or bi-regular tree, bi-colored, $G = \mathrm{Aut}(T)$ - gp of bicolor preserving autom is HM

obs: If G is HM, $N \triangleleft G$, N cld, $N \neq G$, then N is cpct (so „small"). For this look at repr of G/N

obs: G HM. Every proper opn subgp of G is cpct.

Open question: Does there exist an inf discrete gp with HM?
(Such a gp must be quasi-finite, i.e. every proper subgp is finite)
quasi-finite gps exist, look at Tarski monsters

2. Ergodic theory

Consider p.m.p. action of a gp G on (X, μ)

Def 1: Ergodic if every G-inv measurable subset A has either $\mu(A) = 0$ or $\mu(A) = 1$
2: Mixing if $\forall A, B \in \mathcal{B}$ $\lim_{g \to \infty} \mu(A \cap gB) =$
$= \mu(A)\mu(B)$

Obs: • Mixing $\Rightarrow$ ergodic
• In restriction to a cld, non-cpct subgp a mixing action remains mixing

π repr on $L^2(X, \mu) = \mathbb{C}1 \oplus L^2_0(X, \mu)$
π_0 repr on $L^2_0(X, \mu)$

Thm (Koopman): Ergodicity $\Leftrightarrow$ π_0 has no non-zero inv vector.
Mixing $\Leftrightarrow$ π_0 is a C_0-repr

Obs: If G is HM, every ergodic action is mixing.
So restriction to cld, non-cpct subgp keeps ergodicity

Prop: This property characterizes HM property
pf:
Let G be s.t. every ergodic action is mixing.
a) G is minimally almost periodic, i.e. every f.d. unitary repr is trivial
Let $\rho: G \to \mathcal{U}(n)$, set $K = \overline{\rho(G)}$ cpct Lie gp.
G acts ergodically on K by left multiplication
$L^2_0(K)$ decomposes into f.d. repr (Peter-Weyl), so coeff not C_0 unless $K = \{1\}$.
b) Let π be a unitary repr of G without fixed vectors. Must proove that π has C_0 coeff. Use Gaussian construction:
$\exists$ (X, μ) s.t. assoc. rep on $L^2_{\mathbb{R}}(X, \mu)$ decomp as $\bigoplus_{k=0}^{\infty} S^k \pi$, where $S^k \pi$ = symmetric tensor product of π.
(need real repr, so look at underlying orthogonal repr from beginning)
Enough to proove that G-action on X is ergodic, i.e. (by Koopman) $\forall k \geq 1$ S^k_π has no (non-zero) invariant vector.
If $S^k \pi$ has non-zero inv vector, then π contains a f.d. repr, i.e. the

trivial repr by a)
(contradiction to assumptions on π

3. Property BP

Def: G has property (BP) if every isometric action of G on HS either has a fixed point or is proper.
(equivalently, every 1-cocycle on G is either bounded or proper)

Obs: • (T) $\Rightarrow$ (BP)
• If G has (BP), but not (T), then G has prop (H) (the Haagerup property)

Thm (Shalom 2001):
$SO(n,1)$, $SU(n,1)$ have (BP)

Prop: Howe-Moore $\Rightarrow$ (BP)

Def: Let H be cld subgp of G, say (G,H) has the (relative) ~~Haagerup~~ HM property if for every unitary repr π of G, $\pi|_H$ either has non-zero fixed vectors or it has C_0-coeff.

So G has prop (H) iff (G,G) has rel ~~H~~ HM

Thm: If $N \triangleleft G$, N cld, cocpct (i.e. G/N cpct), and (G,N) has rel. prop (HM), then G has (BP).

Ex: $\mathrm{Isom}(\mathbb{E}^n) = \mathbb{R}^n \rtimes O(n)$ has (BP) bc/ $(G, \mathbb{R}^n)$ has rel HM.

pf (of thm)
Let b be 1-cocycle on G, b unbounded. Must proove that b is proper. Since N is cocpct, enough to show that $b_{|N}$ is proper. Let $\varphi_t(g) := e^{-t\|b(g)\|^2}$ (!)
By Schonberg, φ_t is positive definit.
By GNS-construction $\varphi_t(g) = \langle \pi_t(g)\xi_t, \xi_t \rangle$ (ξ_t a cyclic vector).

$\pi_{t|N} = \underset{\text{N-fixed vectors}}{H^N} \oplus H^\perp \qquad \xi_t = \xi_0 \oplus \xi^\perp$

$\langle \pi_t(n)\xi_t, \xi_t \rangle = \|\xi_0\|^2 + \langle \pi^\perp(n)\xi^\perp, \xi^\perp \rangle$

$\pi^\perp$ is a G-repr without N-fixed vectors
by rel HM, the coeff go to zero
Let $(n_k)_{k\geq 1}$ be a seqn in N s.t. $\|b(n_k)\| \to \infty$ $(k \to \infty)$.
$\to \langle \pi_t(n_k)\xi_t, \xi_t \rangle \to 0$ $(k \to \infty)$
$\to \xi_0 = 0$
$\to H^N = 0$ because ξ_t cyclic
so φ_t is C_0-fct

4. Classification result

Thm: let F be either $\mathbb{R}$ or $\mathbb{Q}_p$, let G be a cld subgp of $Gl_n(F)$. Let $N \trianglelefteq G$ cld, non-cpct. Then (G, N) has rel HM iff either

1) $N \cong F^n$ and the G-repr on F^n is non-trivial, irred

or 2) $N \cong \mathbb{G}(F)^+$ for some simple, algebraic gp $\mathbb{G}$ def over F