1.2.2010
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And 4
Workshop
v.N. alg

1. Isometric actions on Hilbert spaces

G - σ-cpct, loc-cpct gp

X - real Hilbert space, viewed as a metric space, $Isom(X)$ = gp of isometries

Thm (Mazur, Ulam)

If E is a real Banach sp, then every isom. of E is affine, i.e. composition of lin map and translation

pf:

for strictly cvx Banach sp:

in such a space get a metric charact. of segments

$[x,y] = \{z \in E : \|x-z\| + \|z-y\| = \|x-y\|\}$

$\to$ isometries preserve segments

$\to$ (Darboux) isometries are affine

$Isom(X) = X \rtimes O(X)$ for $\mathbb{R}$-Hilbert sp X

H real Hilbert sp $\leftrightarrow$ gp of translations

$O(H)$ gp of lin, orthog. transfo

Q: How can G act isometrically on H?

$\alpha : G \to Isom(H)$

$\to \pi : G \to Isom(H) \twoheadrightarrow O(H)$

π orthog. repr

$\alpha(g)v = \pi(g)v + b(g) \qquad v \in H, g \in G$

$b(g)$ - the translation part

$\alpha(gh) = \alpha(g)\alpha(h)$

$\Leftrightarrow$ 1-cocycle condition for b:

$b(gh) = \pi(g)b(h) + b(g)$

$Z^1(G,\pi) := \{ b : G \to H$ cts, satisfying the 1-cocycle relation $\}$

$B^1(G,\pi) := \{ b \in Z^1(G,\pi) : \exists v \in H$ s.t. $b(g) = \pi(g)v - v \}$

- the 1 coboundaries
- uninteresting since the corresponding affine action is conjugate to π via translation

$H^1(G,\pi) := Z^1(G,\pi)/B^1(G,\pi)$

- parametrizes affine actions with lin. part π up to conjugation by translations

2) a-T-menability / property T

Def: G is a-T-menable, or has the Haagerup property if G admits a proper isom. action of Hilbert sp, i.e.

$\lim_{g \to \infty} \|\alpha(g)v\| = \infty \quad \forall v \in H$

Def: G has property T, or is a Kazhdan gp, if every isom. action of G on Hilbert sp has a fixed pt.

a-T-menable
- compact (these are precisely the gps that also have prop T)
- amenable
- free gps (Haagerup)
- cld subgps of $SO(n,1)$ & $SU(n,1)$
- Coxeter gps
- gps acting properly on CAT(0) cubical cpx
- countable subgps of Gl_2(field)

property T
- simple algebraic gps of rank ≥ 2 over $\mathbb{R}$ or $\mathbb{Q}_p$ (?) e.g. $Sl_n(F)$, $Sp_{2n}(F)$
- lattices in these simple gps
  eg. $Sl(n, \mathbb{Z})$ $n \geq 3$, $Sp(2n, \mathbb{Z})$ $n \geq 2$
- Gromov's random gps (expanders)

3) Who cares about a-T-menability?

Thm (Higson-Kasparov '96)
A-T-menable gps satisfy the strongest form of the Baum-Connes conjecture (i.e. with coefficients)

$\Rightarrow$ Novikov conj, idempotent conj on gp rings of torsion-free gps

For G discrete, a-T-menability can be characterized through the v.N. alg $L(G)$

4) How to establish a-T-menability?

Def: (Haglund, Paulin)
spaces with walls:
Let X be a set, endowed with a family $\mathcal{W}$ of partitions of X into 2 classes, s.t.

$\forall x, y \in X$, $x \neq y$
$w(x,y) = \# \{\text{walls in } \mathcal{W} \text{ separating } x \text{ from } y\}$
is finite

- Note that w is a pseudo-metric:
$w(x,y) \leq w(x,z) + w(z,y)$
- The elements of $\mathcal{W}$ are called walls

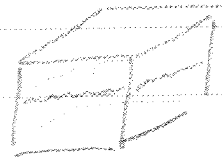
G acts on $(X, \mathcal{W})$, if G acts on X and $G(\mathcal{W}) = \mathcal{W}$.
G acts properly if $\forall x_0 \in X$
$\lim_{g \to \infty} w(g x_0, x_0) = \infty$

Ex:
1) trees $X = (V, E)$ $\mathcal{W} = E$
(every edge separates V into two parts)
$w(x,y) = d(x,y)$ - distance in the graph
2) CAT(0) cubical complexes, obtained by gluing euclidean cubes along sub-cubes, s.t. the resulting length metric is CAT(0)

CAT(0) means: non-positively curved in combinatorical sense

extend walls from local to global (first to neighboring cubes, ...)

separating hyperplane



Thm (Sageev)
Walls separate into two connected comp.

Prop: If G acts properly on a space with walls, then G is a-T-menable.

pf:
half-spaces = one class of the partition defined by a wall
Σ = set of half spaces
π = permutation repr of G on $\ell^2(\Sigma)$
$\forall x \in X$, Σ_x = set of half-spaces through x
χ_x = characteristic function of Σ_x
(since Σ_x is infinite, usually $\chi_x \notin \ell^2(\Sigma)$)
but $\chi_x - \chi_y$ has finite support (by wall condition)
$x_0 \in X$ base point
$b(g) := \chi_{gx_0} - \chi_{x_0} \in Z^1(G, \pi)$
$\|b(g)\|_2 = 2\omega(gx_0, x_0)$
proper 1-cocycle