Stability of unitary representations 2.2.20.

A. Tom

Ulam's program: are algebraic structures up to ε always close to actual algebraic situations?

Def: let G be a ctbl gp, H a Hilbert sp.
A map $\pi : G \to \mathcal{U}(H)$ is an ε-unitary repr
if $\sup_{g,h\in G} \| \pi(gh) - \pi(g)\pi(h)\| < \varepsilon$

We say that $\pi, \sigma : G \to \mathcal{U}(H)$ are ε-close
if $\sup_{g\in G} \| \pi(g) - \sigma(g)\| < \varepsilon$

Q: Is it true, that ε-unitary repr are always close to unitary repr. (stability)

More precisely: Is it true that $\forall \varepsilon > 0$ $\exists \delta > 0$: all δ-unitary repr are ε-close to a unitary repr. (uniform stability)

Answer: This depends on G!

The uniqueness question leads to:
Is it true that $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t.
$\forall \pi, \sigma : G \to \mathcal{U}(H)$ unitary repr
π, σ δ-close $\Rightarrow$ $\exists u \in \mathcal{U}(H)$, $\|1 - u\| < \varepsilon$
and $u \pi(g) u^* = \sigma(g)$
(if so, could call it deformation rigidity)

Thm (Kazhdan, '82)
let G be amenable. Then for δ sufficiently small, any δ-unitary repr is 3δ-close to a unitary repr.

Thm: If G is amenable, then uniformly close repr. are conjugate by some unitary close to 1

Def: G amenable if $\exists\, m: \ell^\infty G \to \mathbb{C}$ s.t.
1) $m(1) = 1$
2) $m(f) \geq 0$ if $f \geq 0$
3) $m(f^g) = m(f)$ for $g \in G$, $f \in \ell^\infty G$

Pf of 2nd thm:
If π, σ are ε-close, then $1 \in B(H)$ is almost fixed by the action $g \cdot T = \pi(g) T \sigma(g)^{-1}$
We set $T = \int_G g \cdot 1 \, dm(g)$
Then $\pi(g) T \sigma(g)^{-1} = T$ for all $g \in G$, so T intertwiner, $\|1 - T\| \leq \varepsilon$ $\rightsquigarrow$ T inv'le
$\rightsquigarrow$ π, σ conjugate by unitary close to 1

idea of pf of 1st thm: (due to Stern)
let π be a ε-unitary repr of G
1) change π a little bit so that $\pi(g)^{-1} = \pi(g^{-1})$
2) replace π by π'
$\pi'(g) = \int_G \pi(gh^{-1}) \pi(h) \, dm(h)$

computation shows that π' is a $5\varepsilon^2$-repr (not nec unitary)
Doing this inductively improves ε more and more, the limit of these repr exists, is unitary and close to π

Thm (Rydlik, Szwarc)
There ex a one-parameter family of unitary repr. π_t, $t \in [0,1]$ ~~s.t.~~ of $\mathbb{F}_2$ s.t.
1) π_t is not equiv to π_s for $t \neq s$
2) $\sup_{g \in \mathbb{F}_2} \| \pi_t(g) - \pi_s(g) \| \leq |t-s|$

So $\mathbb{F}_2$ does not sat. deformation rigidity.

Thm:
Let $G = \mathbb{F}_2$. There ex 1-dim ε-unitary repr for all $\varepsilon > 0$ which are not close to unitary repr.

So $\mathbb{F}_2$ does not sat. (uniform) stability.

Def: $\varphi: G \to \mathbb{R}$ is called a quasimorphism if $\exists C > 0$: $|\varphi(gh) - \varphi(g) - \varphi(h)| < C$

Q: Is there a morph $\psi: G \to \mathbb{R}$ and $C' > 0$ s.t. $|\varphi(g) - \psi(g)| < C'$

observe: $g \mapsto \exp(i \cdot t \cdot \varphi(g))$ gives a $t \cdot C$ - unitary repr

We now have to construct a non-trivial quasi-morphism $\varphi: \mathbb{F}_2 \to \mathbb{R}$. Let $\mathbb{F}_2 = \langle a, b \rangle$, consider $\varphi : \mathbb{F}_2 \to \mathbb{Z}$, ~~st~~

$$\varphi(\omega) = \#(ab \subset \omega) - \#(ab \subset \omega^{-1})$$

If $\omega'' = \omega' \cdot \omega$, $\omega' = v' \cdot s$, $\omega = s^{-1} v$

$\omega'' = v' \sqcup v$, $|\varphi(\omega'') - \varphi(\omega') - \varphi(\omega)| \leq 3$ (or 4)

Can show that φ is not close to actual morphism

Thm: (joint work with Burger, Ozawa)

If $\mathbb{F}_2 \subset G$, then there ex ε-unitary repr, for all $\varepsilon > 0$ sufficiently small, which are not close to unitary repr.

Are the groups which share the stability property precisely the amenable groups?

Thm: If $\mathbb{F}_2 \subset G$, then G is not deformation rigid.

Thm: If G is unitarizable, then G is also deformation rigid.

Dixmier conjecture: G amenable $\Leftrightarrow$ G unitarizable