Stability properties for a-T-menability, part II

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Context

In this talk:

- G, H, Γ, N, Q will denote countable groups;
- $\mathcal{H}$ will denote a real (or complex) Hilbert space;

Suppose $\Gamma \curvearrowright (X, d)$ by isometries. Let $x_0 \in X$.

Recall

The action is (metrically) proper if, for all $R > 0$, the set $\{\gamma \in \Gamma : d(x_0, \gamma x_0) \leq R\}$ is finite. We write $\lim_{\gamma \to \infty} d(x_0, \gamma x_0) = +\infty$.

This does not depend on the choice of x_0.

Extensions vs. Haagerup property(1)

Remark

a-T-menability is stable under taking:

1. subgroups;
2. direct products;
3. free products.

Proposition (Jolissaint)

*Let $1 \to N \to \Gamma \to Q \to 1$ be a short exact sequence. If N has the Haagerup property and if Q is **amenable**, then G has the Haagerup property.*

Extensions vs. Haagerup property(2)

Remark

The Haagerup property is not stable under semi-direct product:

- $\mathbb{Z}^2$ has the Haagerup property;
- $SL_2(\mathbb{Z})$ has the Haagerup property;
- $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$ does not have the Haagerup property.

Indeed, Margulis proved that the pair $(\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z}), \mathbb{Z}^2)$ has relative property (T).

Wreath products

Let X be a (countable) G-set.

Definition

The wreath product of G and H over X is the group $H \wr_X G := H^{(X)} \rtimes G$, where $H^{(X)}$ is the set of finitely supported functions $X \to H$ and G acts by shifting indices: $(g \cdot w)(x) = w(g^{-1}x)$ for $g \in G$ and $w \in H^{(X)}$.

- Recall that every element $\gamma \in H \wr_X G$ admits a unique decomposition $\gamma = wg$ with $g \in G$ and $w \in H^{(X)}$.
- If $X = G$, endowed with action by left translations, we write $H \wr G$ instead of $H \wr_G G$.

Results

Suppose Q is a quotient of G.

Theorem (Cornulier-S.-Valette)

If G, H, Q all have the Haagerup property, then the wreath product $H \wr_Q G$ has the Haagerup property.

Example

$(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{F}_2$ has the Haagerup property.

Theorem (Chifan-Ioana)

Suppose $H \neq \{1\}$ and Q does not have the Haagerup property. Then, $H \wr_Q G$ does not have the Haagerup property.

Measured walls structures

- Let X be a countable set. Recall that the power set of X identifies with $2^X = \{0,1\}^X$, which is a Cantor set.
- Let $A, B \subseteq X$. Say A cuts B, denoted $A \vdash B$, if $A \cap B \neq \emptyset$ and $A^c \cap B \neq \emptyset$.

Definition

A measured walls structure on X is a Borel measure μ on 2^X such that, for all $x, y \in X$

$$d_\mu(x,y) := \mu\{A \subseteq X : A \vdash \{x,y\}\} + \infty\,.$$

Then, the kernel $d_\mu : X \times X \to \mathbb{R}_+$ is a pseudo-distance on X.

a-T-menability vs. measured walls structures

Theorem

Let $\Gamma = BS(m,n) = \langle a, b | ab^m a^{-1} = b^n \rangle$ *for* $m, n \in \mathbb{N}^*$*.*

- *(Gal-Januszkiewicz)* Γ *is a-T-menable.*
- *(Haglund) Provided* $m \neq n$*, there is no proper* Γ*-action on a space with walls.*

Theorem (Robertson-Steger)

The following are equivalent:

- Γ *is a-T-menable;*
- *there exists a left-invariant measured walls structure on* Γ *such that* d_μ *is proper.*

See also: Cherix-Martin-Valette; Chatterji-Drutu-Haglund

Idea of proof: $\Gamma := (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{F}_2$ is Haagerup

Identify $\mathbb{F}_2$ to its Cayley tree. If A is a half-space and if $f : A^c \to \mathbb{Z}/2\mathbb{Z}$ is finitely supported, set

$$E(A, f) := \{\gamma = wg \in (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{F}_2 : g \in A \text{ and } w|_{A^c} = f\} .$$

One shows then that these half-spaces define a left-invariant structure of space with walls on Γ whose associated pseudodistance is proper.

Stability properties of a-T-menability II

By Y. Stalder ; end of the talk

Thm A (Cornulier – S. – Valette)

Suppose Q is a quotient group of G. If G, H, Q are a-T-menable, then so is $H \wr_Q G$.

Lemma 1 Let X be a G-set, $W = H^{(X)}$ and μ be a G-invariant MWS on X. Then, there exists a $W \rtimes G$-invariant MWS on $W \times X$ st

$$d_{\tilde{\mu}}(w_1 x_1, w_2 x_2) = \mu \{ A \subseteq X \;\; A \text{ cuts } \operatorname{Supp}(w_1^{-1} w_2) \cup \{x_1, x_2\} \}$$

Proof is skipped.

Proof of thm A

Take λ, μ, ν left-invar MWS on G, Q, H s.t. d_λ, d_μ, d_ν are proper (they exist by the Robertson-Steger theorem).

Consider the maps

$$p : W \rtimes G \longrightarrow G$$

$$\rho : W \rtimes G \longrightarrow W \times Q$$

$$\pi_q : W \rtimes G \longrightarrow H \quad \text{for } q \in Q$$

$$wg \longmapsto w(q)$$

and set

$$\omega := p^* \lambda + \rho^* \tilde{\mu} + \sum_{q \in Q} \pi_q^* \nu$$

[Def : pull-back of a MWS

$f : E \longrightarrow F$; ξ MWS on F

$f^\# : 2^F \longrightarrow 2^E$; $A \longmapsto f^{-1}(A)$

$f^* \xi := (f^\#)_* \xi$ (push-forward of measure)

Properties :

a) $d_{f^* \xi}(x, y) = d_\xi(f(x), f(y))$

b) If f is Γ-equivariant and ξ is Γ-invariant, then $f^* \xi$ is Γ-invariant.]

Hence ω is $W \rtimes G$ - invariant since $p^*\lambda$, $\rho^*\tilde{\mu}$ and $\sum_{q \in Q} \pi_q^* \nu$ are. It remains to prove that d_ω is proper.

Fix $R > 0$; suppose $d_\omega(1, wg) \leq R$. Then:

* $d_{\tilde{\mu}}((1_W, 1_Q), (w, g_Q)) \leq R$, hence $Supp(w) \subseteq \overline{B}_\mu(1_Q, R)$ by Lemma 1.
* $d_\lambda(1_G, g) \leq R$, hence $g \in \overline{B}_\lambda(1_G, R)$
* $\forall q \in Q$, $d_\nu(1_H, w(q)) \leq R$, hence $w(q) \subseteq \overline{B}_\nu(1_H, R)$

These three balls are finite since d_λ, d_μ, d_ν are proper.

Thus $\{wg \in W \rtimes G : d_\omega(1, wg) \leq R\}$ is finite; this proves that d_ω is proper. $\square$

A result about compression

Let $\alpha : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a non-decreasing subadditive function.

Let H be a finitely generated group

$\mathbb{F}$ be a f.g. free group

Prop (Parry) For the natural finite generating set of $H \wr \mathbb{F}$, word lengths on $\mathbb{F}$, H and $H \wr \mathbb{F}$ satisfy

$$|wg|_{H \wr \mathbb{F}} = m(w,g) + \sum_{x \in \mathbb{F}} |w(x)|_H$$

where $m(w,g)$ is the length of a shortest path from 1 to g in (the Cayley graph of) $\mathbb{F}$ covering $\mathrm{Supp}(w)$.

Thm B (Cornulier - S. - Valette)

If H admits a left-invariant MWS σ s.t. $d_\sigma(1,g) \geq \alpha(|g|)$ $\forall g \in H$, then so does $H \wr \mathbb{F}$.

Proof Technical extra assumption: $\alpha(r) \leq \frac{r}{2}$ $\forall r \in \mathbb{R}_+$. Take the MWS μ on $\mathbb{F}$ induced by its tree structure; it is left-invariant and d_μ is the tree distance on $\mathbb{F}$.

Set $\omega = \tilde{\mu} + \sum_{x \in \mathbb{F}} \pi_x^* \sigma$, where:

- $\pi_x^* : H^{(\mathbb{F})} \rtimes \mathbb{F} \longrightarrow H$
  $w\, g \longmapsto w(x)$
- $\tilde{\mu}$ is MWS on $H^{(\mathbb{F})} \rtimes \mathbb{F}$ given by Lemma 1.

This is a left-invar MWS on $H \wr \mathbb{F}$ and Lemma 1 gives

$$d_\omega(1, wg) = n + \sum_{x \in \mathbb{F}} d_\sigma(1, w(x))$$

where $n = \#\{\text{edges cutting } \operatorname{Supp}(w) \cup \{1, g\}\}$

Observations:

a) these edges form a subtree

b) it is covered by a loop of length $2n$

Hence, $m(w, g) \leq 2n$

Consequently, one has $\forall wg \in H \wr \mathbb{F}$

$$d_\omega(1, wg) \geq \frac{1}{2} m(w,g) + \sum_{x \in \mathbb{F}} \alpha(|w(x)|)$$

$$\geq \alpha(m(w,g)) + \sum_{x \in \mathbb{F}} \alpha(|w(x)|)$$

$$\geq \alpha\left(m(w,g) + \sum_{x \in \mathbb{F}} |w(x)| \right)$$

$$= \alpha(|wg|) \text{ by Parry's prop.}$$

QED