Examples of quantum homogeneous spaces.

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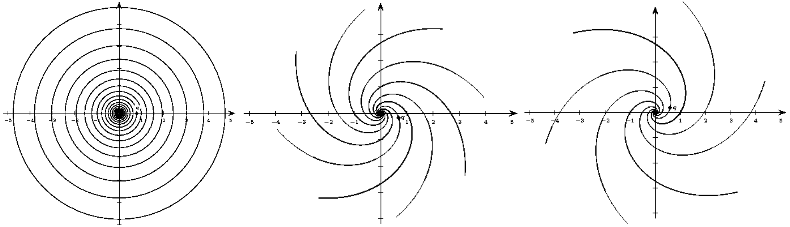
THE QUANTUM "$az + b$" GROUP

- A — C*-algebra generated by two normal elements a and b affiliated with it such that a is invertible,

$$ab = q^2 ba, \qquad ab^* = b^* a,$$

and a and b have spectrum contained in one of these:



- Comultiplication $\Delta \in \mathrm{Mor}(A, A \otimes A)$ is defined by

$$\Delta(a) = a \otimes a, \qquad \Delta(b) = a \otimes b \dotplus b \otimes \mathbf{1}.$$

- $\mathbb{G} = (A, \Delta)$ — locally compact quantum group.

More details

- The spectra of a and b are equal to $\Gamma \cup \{0\}$, with Γ a multiplicative subgroup of $\mathbb{C} \setminus \{0\}$.
- A is isomorphic to $\mathrm{C}_0(\Gamma \cup \{0\}) \rtimes \Gamma$.
- We have a surjective $\pi \in \mathrm{Mor}\big(A, \mathrm{C}_0(\Gamma)\big)$:

  $$\pi(a) = \boldsymbol{u}, \qquad \pi(b) = 0,$$

  where $\boldsymbol{u}$ is the coordinate on Γ.
- Γ is a closed quantum subgroup of $\mathbb{G} = (A, \Delta)$, i.e.

  $$\begin{array}{ccc} A & \xrightarrow{\quad\Delta\quad} & A \otimes A \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi\otimes\pi} \\ \mathrm{C}_0(\Gamma) & \xrightarrow{\quad\Delta_\Gamma\quad} & \mathrm{C}_0(\Gamma) \otimes \mathrm{C}_0(\Gamma) \end{array}$$

  where $\Delta_\Gamma(f)(s,t) = f(st)$.

THE SPACE $\mathbb{G}/\Gamma$

- We define $\mathbb{G}/\Gamma$ as the "quantum space" corresponding to the following C*-algebra:

$$B = \left\{ x \in \mathrm{M}(A) \,\middle|\, \begin{array}{l} \bullet\ (\mathrm{id} \otimes \pi)\Delta(x) = x \otimes \mathbf{1}, \\ \bullet\ xy \in A \ \forall\ y \in \mathrm{C}^*(\Gamma) \subset \mathrm{M}(A) \\ \bullet\ \gamma \mapsto U_\gamma x U_\gamma^* \text{ is continuous} \end{array} \right\} \subset \mathrm{M}(A).$$

- $(U_\gamma)_{\gamma\in\Gamma}$ are the unitaries in $\mathrm{M}(A) = \mathrm{C}_0(\Gamma \cup \{0\}) \rtimes \Gamma$ implementing the action of Γ, each U_γ is a certain function of the generator a.

THE SPACE $\mathbb{G}/\Gamma$

THEOREM

1. $B = \left\{ f(b) \middle| f \in \mathrm{C}_0(\Gamma \cup \{0\}) \right\} \subset \mathrm{M}(A)$, *(commutative!)*
2. $\Delta(B) \subset \mathrm{M}(A \otimes B)$, $\alpha = \Delta\big|_B \in \mathrm{Mor}(B, A \otimes B)$,
3. $(\mathrm{id} \otimes \alpha)\circ\alpha = (\Delta \otimes \mathrm{id})\circ\alpha$, *(action of $\mathbb{G}$)*
4. $(\epsilon \otimes \mathrm{id})\circ\alpha = \mathrm{id}$, *where ϵ is the counit of $\mathbb{G}$,*
5. *let $\boldsymbol{z}$ be the generator of $B \cong \mathrm{C}_0(\Gamma \cup \{0\})$, then*
$$\alpha(\boldsymbol{z}) = a \otimes \boldsymbol{z} \dotplus b \otimes \mathbf{1},$$
6. *for any $c \in A$ and $x \in B$ we have* *(continuity)*
$$(c \otimes \mathbf{1})\alpha(x) \in A \otimes B.$$

- The proof of point 6. is based on analysis of commutation relations between a and b and certain special function

The quantum E(2) group

- $\mathscr{H}$ — Hilbert space with o.n. basis $(e_{i,j})_{i,j\in\mathbb{Z}}$.
- Two operators: $(q\in]0,1[)$

$$ve_{i,j}=e_{i-1,j},\quad ne_{i,j}=q^{i}e_{i,j+1}.$$

- A — the closure in $\mathrm{B}(\mathscr{H})$ of the set of finite sums

$$\sum f_k(n)v^k$$

with $f_k\in\mathrm{C}_0(\text{spectrum of }n)$.

- There exists a unique $\Delta\in\mathrm{Mor}(A,A\otimes A)$ such that

$$\Delta(v)=v\otimes v,\qquad\Delta(n)=v\otimes n\dotplus n\otimes v^{*}$$

($v\in\mathrm{M}(A)$ and n is affiliated with A).

- $\mathbb{G}=(A,\Delta)$ — locally compact quantum group.

SUBGROUP

- We have a surjective $\pi \in \mathrm{Mor}\big(A, \mathrm{C}(\mathbb{T})\big)$ given by

$$\pi(v) = \boldsymbol{u}, \qquad \pi(n) = 0,$$

  where $\boldsymbol{u}$ is the coordinate on $\mathbb{T}$.
- π identifies $\mathbb{T}$ as a closed subgroup of $\mathbb{G}$:

$$\begin{array}{ccc} A & \xrightarrow{\quad\Delta\quad} & A \otimes A \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi\otimes\pi} \\ \mathrm{C}(\mathbb{T}) & \xrightarrow{\quad\Delta_{\mathbb{T}}\quad} & \mathrm{C}(\mathbb{T}) \otimes \mathrm{C}(\mathbb{T}) \end{array}$$

  where $\Delta_{\mathbb{T}}$ is the standard comultiplication on $\mathrm{C}(\mathbb{T})$.

THE SPACE $\mathbb{G}/\mathbb{T}$

- We define

$$B = \left\{ x \in \mathrm{M}(A) \,\middle|\, \begin{array}{l} \bullet\ (\mathrm{id} \otimes \pi)\Delta(x) = x \otimes \mathbf{1}, \\ \bullet\ xy \in A \ \ \forall\, y \in \mathrm{C}^*(\mathbb{T}) \subset \mathrm{M}(A) \end{array} \right\} \subset \mathrm{M}(A).$$

THEOREM

1. *The* C**-algebra B is the closure of the set of finite sums of the form*

$$\sum f_k(n)v^k$$

where for each $k \in \mathbb{Z}$ the function $f_k \in \mathrm{C}_0(\text{spectrum of } n)$ is such that

$$f_k(\mu z) = \mu^k f_k(z)$$

for all $z \in$ spectrum of n and $\mu \in \mathbb{T}$,

2. *the operator vn is affiliated with B,*
3. *B is generated by vn.*

THE SPACE $\mathbb{G}/\mathbb{T}$

- B is isomorphic to the C*-algebra

$$\left\{ \begin{bmatrix} x & y \\ z & u \end{bmatrix} \middle| \, x \in \mathscr{T}, \; y, z, u \in \mathscr{K} \right\}$$

where

- $\mathscr{T}$ = the Toeplitz algebra,
- $\mathscr{K}$ = the compact operators $\subset \mathscr{T}$.

THEOREM

1. $\Delta(B) \subset \mathrm{M}(A \otimes B)$ *and* $\alpha = \Delta\big|_B \in \mathrm{Mor}(B, A \otimes B)$,
2. $(\mathrm{id} \otimes \alpha) \circ \alpha = (\Delta \otimes \mathrm{id}) \circ \alpha$,
3. $(\epsilon \otimes \mathrm{id}) \circ \alpha = \mathrm{id}$,
4. *for any* $b \in B$ *and* $a \in A$ *we have* $(a \otimes \mathbf{1})\alpha(b) \in A \otimes B$.

A GENERAL CONSTRUCTION

- The second example generalizes to the following setup:
  - (A, Δ) — bisimplifiable Hopf C*-algebra:
$$\overline{\operatorname{span}\left\{\Delta(a)(\mathbf{1}\otimes a')\middle| a, a' \in A\right\}} = A \otimes A,$$
$$\overline{\operatorname{span}\left\{(a\otimes \mathbf{1})\Delta(a')\middle| a, a' \in A\right\}} = A \otimes A,$$
  - $\mathbb{K} = (C, \Delta_C)$ — compact quantum group,
  - $\pi \in \operatorname{Mor}(A, C)$ — surjective, $\Delta_C \circ \pi = (\pi \otimes \pi) \circ \Delta$.
- We then define
  - $B = \left\{x \in A \middle| (\mathrm{id} \otimes \pi)\Delta(x) = x \otimes \mathbf{1}\right\}$,
  - $\alpha = \Delta\big|_B$.
- It follows that
  - $\alpha \in \operatorname{Mor}(B, A \otimes B)$,
  - $(\mathrm{id} \otimes \alpha) \circ \alpha = (\Delta \otimes \mathrm{id}) \circ \alpha$,
  - for any $b \in B$ and $a \in A$ we have $(a \otimes \mathbf{1})\alpha(b) \in A \otimes B$.