New Action-Induced Nested Classes of Groups and Jump (Co)homology

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Copenhagen, January 27, 2010

Outline

1. $\mathcal{N}(\mathcal{P}, \mathcal{A}, \mathcal{X})$-groups
   - Motivation
   - $\mathcal{N}^{cell}(\mathcal{P})$-groups

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2. $\mathcal{N}_1^{cell}(\mathcal{P}_R)$-groups
   - Construction
   - Jump (co)homology

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Let $\mathcal{F}$ be the class of finite groups. Then $H\mathcal{F}$ is closed under taking subgroups, (HNN-)extensions, countable directed unions, and amalgamated products.

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- Every torsion-free FP_∞-group in $H\mathcal{F}$ has finite cohomological dimension.
- Thompson's group F is not in $H\mathcal{F}$.
- Groups constructed by Arzhantseva, Bridson, Januszkiewicz, Leary, Minasyan, and Świątkowski in "Infinite groups with fixed point properties" are not in $H\mathcal{F}$.

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The condition X satisfies $\mathcal{P}$ is equivalent to requiring $X \in \mathcal{P}$, a chosen set of topological spaces.

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- When $\mathcal{P} \subset \{X | X \text{ is a finite dimensional CW-complex}\}$, $\mathcal{A}$ defines the action to be cellular, and $\mathcal{X} = \{\langle 1 \rangle\}$, denote $\mathcal{N}(\mathcal{P}, \mathcal{A}, \mathcal{X})$ by $\mathcal{N}^{cell}(\mathcal{P})$.

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- $\mathcal{N}^{cell}(\mathcal{P})$ is extension closed.

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Some known classes of groups

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Well-known classes

i	$\mathcal{P}_i$	$\mathcal{N}^{cell}(\mathcal{P}_i)$
1	$\{S^1\}$	finite solvable groups
2	$\{\mathbb{T}^m \mid m \in \mathbb{N}\}$	finite groups
3	$\{S^m \mid m \in \mathbb{N}\}$	finite groups
4	$\{S^1, \mathbb{R}\}$	polycyclic groups
5	$\{S^m, \mathbb{R} \mid m \in \mathbb{N}\}$	virtually polycyclic groups

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Theorem (P. 2010)

Let $\mathcal{P}_6 = \{X | X = S^m, m \in \mathbb{N}, \text{ or } X \text{ is a locally finite tree}\}$.

$\mathcal{N}^{cell}(\mathcal{P}_6)$-groups

Theorem (P. 2010)

Let $\mathcal{P}_6 = \{X | X = S^m, m \in \mathbb{N},$ or X is a locally finite tree$\}$. Then we have:

- $\mathcal{N}^{cell}(\mathcal{P}_6)$ contains all poly-$\mathbb{Z}$ and all countable locally finite groups.

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- Every group in $\mathcal{N}^{cell}(\mathcal{P}_6)$ either contains a free subgroup on two generators or it is countable elementary amenable.

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- Every group in $\mathcal{N}^{cell}(\mathcal{P}_6)$ either contains a free subgroup on two generators or it is countable elementary amenable.
- In particular, every Noetherian group in $\mathcal{N}^{cell}(\mathcal{P}_6)$ is virtually polycyclic.

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$\{S^1\}$	finite solvable gps	$= \mathcal{N}_\omega^{cell}(\mathcal{P}_1)$
$\{\mathbb{T}^m \mid m \in \mathbb{N}\}$	finite gps	$= \mathcal{N}_1^{cell}(\mathcal{P}_2)$
$\{S^m \mid m \in \mathbb{N}\}$	finite gps	$= \mathcal{N}_\omega^{cell}(\mathcal{P}_3)$
$\{S^1, \mathbb{R}\}$	polycyclic gps	$= \mathcal{N}_\omega^{cell}(\mathcal{P}_4)$
$\{S^m, \mathbb{R} \mid m \in \mathbb{N}\}$	v. polycyclic gps	$= \mathcal{N}_\omega^{cell}(\mathcal{P}_5)$
$\{S^m, m \in \mathbb{N}$, all l. f. trees$\}$	“alternative” gps	?

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$\{S^m, m \in \mathbb{N}, \text{all l. f. trees}\}$	“alternative” gps	?

Also, $\mathcal{N}_k^{cell}(\mathcal{P}_i) \subsetneq \mathcal{N}_{k+1}^{cell}(\mathcal{P}_i)$ for $i = 1, 3, 4, 5, 6$ and each $k \in \mathbb{N}$.

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(b) $H_k(X) = \mathbb{Z}^m \oplus F$, where F is an R-torsion-free finite group.

When $R = \mathbb{Q}$, CW-complexes that have f. g. homology groups, such as finitely dominated ones, satisfy both conditions.

Identifying $\mathcal{N}_1^{cell}(\mathcal{P}_R)$

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Theorem (P. 2009)

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Let $\mathcal{J}_\mathcal{R}$ be the class of groups with jump cohomology over R and let $\mathcal{VCD}$ denote the class of groups with finite virtual cohomological dimension. Then,

$$\mathcal{VCD} \subseteq \mathcal{N}_1^{cell}(\mathcal{P}_R) \subseteq \mathcal{J}_\mathcal{R}.$$

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In fact, because $\mathbb{Z}^\infty = \cup_{i=1}^{\infty} \mathbb{Z}^i$, it acts cellularly on a 1-dim contractible CW-complex with all stabilizer subgroups in $\mathcal{N}_1^{cell}(\mathcal{P}_R)$.

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In fact, because $\mathbb{Z}^\infty = \cup_{i=1}^{\infty} \mathbb{Z}^i$, it acts cellularly on a 1-dim contractible CW-complex with all stabilizer subgroups in $\mathcal{N}_1^{cell}(\mathcal{P}_R)$. Therefore, $\mathbb{Z}^\infty \in \mathcal{N}_2^{cell}(\mathcal{P}_R)$. Since $\mathbb{Z}^\infty$ does not have jump cohomology,

$$\mathcal{N}_1^{cell}(\mathcal{P}_R) \subsetneq \mathcal{N}_2^{cell}(\mathcal{P}_R).$$

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- When $R = \mathbb{Z}$, we will simply say that G has jump cohomology.

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- A finitely generated solvable group G has finite Hirsch length if and only if it has jump homology.
- A linear group has jump homology if and only if there is an upper bound on the Hirsch lengths of its finitely generated unipotent subgroups.

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- This holds when G is in $H\mathcal{F}$.

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- (Adem-Smith, 2001) A countable group G has periodic cohomology if and only if G acts freely and properly discontinuously on some $S^n \times \mathbb{R}^k$.

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- For torsion-free groups, it has been conjectured by Olympia Talelli that the notions of periodic cohomology and finite cohomological dimension are equivalent.
- (Adem-Smith, 2001) A countable group G has periodic cohomology if and only if G acts freely and properly discontinuously on some $S^n \times \mathbb{R}^k$.
- Are torsion-free $\mathcal{N}_1^{cell}(\mathcal{P}_{\mathbb{Z}})$-groups the same as torsion-free $H_1\mathcal{F}$-groups?

Thank You!