Factorization and dilation problems for completely positive maps on von Neumann algebras

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Definition (Anantharaman-Delaroche, 2004):

Let (M, ϕ) and (N, ψ) be two von Neumann algebras equipped with normal, faithful states ϕ, ψ. A linear map $T\colon M \to N$ is called a (ϕ, ψ)-*Markov map* if

(a) T is completely positive

(b) $T(1_M) = 1_N$

(c) $\psi \circ T = \phi$

(d) $T \circ \sigma_t^{\phi} = \sigma_t^{\psi} \circ T\,, \quad t \in \mathbb{R}\,.$

If $(M, \phi) = (N, \psi)$, then T is called a ϕ-*Markov map* on M.

Note: A (ϕ, ψ)-*Markov map* $T\colon M \to N$ has an *adjoint* (ψ, ϕ)-Markov map $T^*\colon N \to M$ uniquely determined by

$$\psi(yT(x)) = \phi(T^*(y)x)\,, \quad x \in M\,, y \in N\,.$$

Definition (Anantharaman-Delaroche, 2004):

A (ϕ, ψ)-Markov map $T\colon M \to N$ is called *factorizable* if there exists a finite von Neumann algebra P with a normal, faithful state χ and two $*$-monomorphisms

$$\alpha\colon M \to P\,, \quad \beta\colon N \to P$$

such that α is (ϕ, χ)-Markov, β is (ψ, χ)-Markov and

$$T = \beta^* \circ \alpha\,.$$

Remarks:

(a) $\beta^* = \beta^{-1} \circ E_{\beta(N)}$, where $E_{\beta(N)}$ is the unique χ-preserving conditional expectation of P onto $\beta(N)$.

(b) If ϕ and ψ are traces and T is factorizable, then (P, χ) in the definition above can be chosen such that χ is also a trace. This can be achieved by replacing (P, χ) by $(P_\chi, \chi_{|P_\chi})$, where P_χ denotes the centralizer of χ , since both $\phi(M)$ and $\psi(N)$ are contained in P_χ .

Problem (Anantharaman-Delaroche, 2004):
Is every Markov map factorizable?

Markov maps on $(M_n(\mathbb{C})\,, \tau_n))$

Here $\tau_n = \frac{1}{n}\mathrm{Tr}$ is the normalized trace on $M_n(\mathbb{C})$.

A linear map $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is $(M_n(\mathbb{C})\,, \tau_n)$-Markov if

(a) T is completely positive

(b) $T(1) = 1$

(c) $\tau_n \circ T = \tau_n$.

By a result of Choi (1973) , condition (a) is equivalent to the fact that T has the form

$$Tx = \sum_{i=1}^{d} a_i^* x a_i \,, \quad x \in M_n(\mathbb{C})$$

where $a_1, \ldots, a_d \in M_n(\mathbb{C})$ can be chosen to be linearly independent. Note that in this case,

$$T(1) = 1 \iff \sum_{i=1}^{d} a_i^* a_i = 1\,,$$

$$\tau_n \circ T = \tau_n \iff \sum_{i=1}^{d} a_i a_i^* = 1\,.$$

Theorem 1 (Haagerup-M.):
Let $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a $(M_n(\mathbb{C})\,, \tau_n)$-Markov map, written in the form

$$Tx = \sum_{i=1}^{d} a_i^* x a_i\,, \quad x \in M_n(\mathbb{C})\,,$$

where $a_1, \ldots, a_d \in M_n(\mathbb{C})$ are linearly independent. Then the following conditions are equivalent:

(1) T is factorizable

(2) There exists a finite von Neumann algebra N with a normal faithful tracial state τ_N and a unitary $u \in M_n(N)$ such that

$$Tx = (\mathrm{id}_{M_n(\mathbb{C})} \otimes \tau_N)(u^*(x \otimes 1)u)\,, \quad x \in M_n(\mathbb{C})\,.$$

(3) There exists a finite von Neumann algebra N with a normal faithful tracial state τ_N and $v_1, \ldots, v_d \in N$ such that $u\colon = \sum_{i=1}^{d} a_i \otimes v_i$ is a unitary operator in $M_n(\mathbb{C}) \otimes N$ and

$$\tau_N(v_i^* v_j) = \delta_{ij}\,, \quad 1 \leq i, j \leq d\,.$$

Corollary 1:

Let $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a $(M_n(\mathbb{C})\,, \tau_n)$-Markov map of the form

$$Tx = \sum_{i=1}^{d} a_i^* x a_i\,, \quad x \in M_n(\mathbb{C})\,,$$

where $a_1\,, \ldots\,, a_d \in M_n(\mathbb{C})$. If $d \geq 2$ and the set $\{a_i^* a_j : 1 \leq i, j \leq d\}$ is linearly independent, then T is not factorizable.

Proof: Assume that T is factorizable. By Theorem 1, there exists a finite von Neumann algebra N with a normal faithful tracial state τ_N and $v_1\,, \ldots\,, v_d \in N$ such that

$$u \colon = \sum_{i=1}^{d} a_i \otimes v_i$$

is unitary. Since $\sum_{i=1}^{d} a_i^* a_i = 1$, it follows that

$$\sum_{i,j=1}^{d} a_i^* a_j \otimes (v_i^* v_j - \delta_{ij} 1_N) = u^* u - \left(\sum_{i=1}^{d} a_i^* a_i \right) \otimes 1_N = 0\,.$$

By the linear independence of the set $\{a_i^* a_j : 1 \leq i, j \leq d\}$,

$$v_i^* v_j - \delta_{ij} 1_N = 0\,, \quad 1 \leq i, j \leq d\,.$$

Since $d \geq 2$, it follows in particular that $v_1^* v_1 = v_2^* v_2 = 1$ and $v_1^* v_2 = 0$. Since N is finite, v_1 and v_2 are unitary operators, which gives rise to a contradiction. This proves that T is <u>not</u> factorizable.

Example 1 (Haagerup-M.): Set

$$a_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad a_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$a_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then $\sum_{i=1}^3 a_i^* a_i = \sum_{i=1}^3 a_i a_i^* = 1$. Hence the operator T defined by

$$Tx : = \sum_{i=1}^{3} a_i^* x a_i \,, \quad x \in M_3(\mathbb{C})$$

is a $(M_3(\mathbb{C}), \tau_3)$-Markov map. The set

$$\{a_i^* a_j : 1 \leq i, j \leq 3\}$$

is linearly independent. Hence, by Corollary 1, T is <u>not</u> factorizable.

Remark: Let $\mathcal{FM}(M_n(\mathbb{C}), \tau_n)$ be the set of factorizable $(M_n(\mathbb{C}), \tau_n)$-Markov maps. It can be checked that

$$\operatorname{conv}(\operatorname{Aut}(M_n(\mathbb{C}), \tau_n))) \subset \mathcal{FM}(M_n(\mathbb{C}), \tau_n)\,. \tag{1}$$

All automorphisms of $M_n(\mathbb{C})$ are inner. The map T from above is an example of a completely positive, unital, trace-preserving map on $M_3(\mathbb{C})$ which is not a convex combination of inner automorphisms.

Question: Is the inclusion (1) strict?

Proposition 1 (Haagerup-M.):

Let $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a $(M_n(\mathbb{C})\,, \tau_n)$-Markov map written in the form

$$Tx = \sum_{i=1}^{d} a_i^* x a_i\,, \quad x \in M_n(\mathbb{C})\,,$$

where $a_1\,, \ldots\,, a_d \in M_n(\mathbb{C})$ are linearly independent. Then the following conditions are equivalent:

(a) $T \in \mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C}))\,.$

(b) T satisfies condition (2) of Theorem 1 with N abelian.

(c) T satisfies condition (3) of Theorem 1 with N abelian.

Corollary 2:

Let $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a $(M_n(\mathbb{C})\,, \tau_n)$-Markov map of the form

$$Tx = \sum_{i=1}^{d} a_i^* x a_i\,, \quad x \in M_n(\mathbb{C})\,,$$

where $a_1\,, \ldots\,, a_d \in M_n(\mathbb{C})$ are self-adjoint, $\sum_{i=1}^{d} a_i^2 = 1$ and satisfy $a_i a_j = a_j a_i\,, 1 \le i, j \le d\,.$ Then the following hold:

(a) T is factorizable.

(b) If $d \ge 3$ and the set $\{a_i a_j : 1 \le i, j \le d\}$ is linearly independent, then $T \notin \mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))\,.$

Schur multipliers

If $B = (b_{ij})_{i,j=1}^n$ is a positive semi-definite matrix, then the map $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ given by

$$Tx\colon = (b_{ij}x_{ij})_{1\le i,j\le n}\,, \quad x = (x_{ij})_{i,j=1}^n \in M_n(\mathbb{C})$$

is called the Schur multiplier associated to the matrix B. Note that T is completely positive. If, moreover,

$$b_{11} = b_{22} = \ldots = b_{nn} = 1\,,$$

then $T(1) = 1$ and $\tau_n \circ T = \tau_n$. Hence T is an $(M_n(\mathbb{C})\,, \tau_n)$-Markov map.

Example 2 (Haagerup-M.): Let $\beta = 1/\sqrt{5}$ and set

$$B\colon = \begin{pmatrix} 1 & \beta & \beta & \beta & \beta & \beta \\ \beta & 1 & \beta & -\beta & -\beta & -\beta \\ \beta & \beta & 1 & \beta & -\beta & -\beta \\ \beta & -\beta & \beta & 1 & \beta & -\beta \\ \beta & -\beta & -\beta & \beta & 1 & \beta \\ \beta & \beta & -\beta & -\beta & \beta & 1 \end{pmatrix}.$$

We can show that the associated Schur multiplier T_B satisfies the hypotheses of Corollary 2, hence T_B is a factorizable Markov map on $M_6(\mathbb{C})$, but

$$T_B \notin \mathrm{conv}(\mathrm{Aut}(M_6(\mathbb{C})))\,.$$

Example 3 (Haagerup-M.): Let $0 < s < 1$ and set

$$B(s) \colon = \begin{pmatrix} 1 & \sqrt{s} & \sqrt{s} & \sqrt{s} \\ \sqrt{s} & s & s & s \\ \sqrt{s} & s & s & s \\ \sqrt{s} & s & s & s \end{pmatrix} + (1-s) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \omega & \overline{\omega} \\ 0 & \overline{\omega} & 1 & \omega \\ 0 & \omega & \overline{\omega} & 1 \end{pmatrix},$$

where $\omega = e^{i2\pi/3} = -1/2 + i\sqrt{3}/2$ and $\overline{\omega}$ is its complex conjugate. Then $B(s)$ is positive semi-definite matrix of rank 2. Moreover,

$$T_{B(s)}(x) = \sum_{i=1}^{2} a_i(s)^* x a_i(s) \,, \quad x \in M_4(\mathbb{C}) \,,$$

where $a_1(s) = \mathrm{diag}(1\,, \sqrt{s}\,, \sqrt{s}\,, \sqrt{s})\,,\, a_2(s) = \sqrt{1-s}\,\mathrm{diag}(0\,, 1\,, \omega\,, \overline{\omega})\,.$ The set $\{a_i^* a_j : i, j = 1, 2\}$ is linearly independent, hence $T_{B(s)}$ is <u>not</u> factorizable, by Corollary 1.

Furthermore, set

$$L = \frac{dB(s)}{ds}_{|s=1} = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 3 - i\sqrt{3} & 3 + i\sqrt{3} \\ 1 & 3 + i\sqrt{3} & 0 & 3 - i\sqrt{3} \\ 1 & 3 - i\sqrt{3} & 3 + i\sqrt{3} & 0 \end{pmatrix}.$$

Then

$$N(t) := \left(e^{-L_{ij} t}\right)_{1 \le i,j \le 4} \,, \quad t \ge 0$$

is a semigroup of positive definite matrices having 1 on the diagonal. Hence

$$T(t) \colon = T_{N(t)} \,, \quad t \ge 0$$

is a semigroup of Schur multipliers which are $(M_4(\mathbb{C}), \tau_4)$-Markov maps.

For $t > 0$, $N(t)$ has rank 4, and therefore Corollary 1 cannot be applied. Using a different method we can obtain from Theorem 1 that there exists $t_0 > 0$ such that $T(t)$ is not factorizable, for any $0 < t < t_0$.

Remarks:

(1) Eric Ricard proved in 2007 that if a $(M_n(\mathbb{C}), \tau_n)$-Markov map T is a Schur multiplier $T = T_B$ associated to a matrix B having real entries, then T is always factorizable.

(2) By a result of Kümmerer and Maassen (1987), it follows that if

$$T(t) \colon = e^{-Lt}, \quad t \geq 0$$

is a one-parameter semigroup of $(M_n(\mathbb{C}), \tau_n)$-Markov maps satisfying

$$T(t)^* = T(t), \quad t \geq 0,$$

then

$$T(t) \in \operatorname{conv}(\operatorname{Aut}(M_n(\mathbb{C}))), \quad t \geq 0.$$

In particular, $T(t)$ is factorizable, for all $t \geq 0$.

On the connection between Anantharaman-Delaroche's work and Kümmerer's work (Communicated by Claus Koestler, May 2008)

Definition (Kümmerer, JFA 1985):

Let (M, ϕ) be a von Neumann algebra with a normal, faithful state ϕ. A ϕ-Markov map $T\colon M \to M$ has a *dilation* if there exists

- (N, ψ) von Neumann algebra with a normal faithful state ψ
- $i\colon M \to N$ (ϕ, ψ)-Markov $*$-monomorphism
- $\alpha \in \mathrm{Aut}(N, \psi)$

such that

$$T^n = i^* \circ \alpha^n \circ i\,, \quad n \geq 1\,.$$

Combining results from Anantharaman-Delaroche (2004) with results from Kümmerer's unpublished Habilitationsschrift (1986), one gets the following

Theorem (Anantharaman-Delaroche, 2004 + Kümmerer, 1986):

Let $T\colon M \to M$ be a ϕ-Markov map. The following are equivalent:

(1) T is factorizable.

(2) T has a dilation.

In his Habilitationsschrift (1986), Kümmerer constructs examples of τ_n-Markov maps on $M_n(\mathbb{C})$ having <u>no</u> dilation. His examples are similar to our examples 1 and 3, but he does not consider the one-parameter semigroup case.

Proposition (Kümmerer, 1986, cf. Cor. 3.4.5 and Prop. 3.5.5):
(1) Let $T\colon M_3(\mathbb{C}) \to M_3(\mathbb{C})$ be the τ_3-Markov map

$$Tx\colon = \sum_{i=1}^{3} a_i^* x a_i \,, \quad x \in M_3(\mathbb{C})$$

where

$$a_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad a_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$a_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Then T has <u>no</u> dilation.

(2) Let $n \geq 4$ and $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be the τ_n-Markov map

$$Tx\colon = \sum_{i=1}^{2} a_i^* x a_i \,, \quad x \in M_n(\mathbb{C})$$

where

$$a_1 = \operatorname{diag}\left(1\,, \frac{1}{\sqrt{2}}\,, \frac{1}{\sqrt{2}}\,, 0\,, \ldots\,, 0\right), a_2 = \operatorname{diag}\left(0\,, \frac{1}{\sqrt{2}}\,, \frac{i}{\sqrt{2}}\,, 1\,, \ldots\,, 1\right).$$

Then T is a Schur multiplier which has <u>no</u> dilation.

The noncommutative Rota dilation property

Definition (Junge, Le Merdy, Xu, 2006):

Let (M, τ) be a (finite) von Neumann algebra with a normal, faithful tracial state τ. A τ-Markov map $T\colon M \to M$ has the *Rota dilation property* if there exists

- N von Neumann algebra with a normal faithful tracial state τ_N
- $(N_n)_{n\geq 1}$ decreasing sequence of von Neumann subalgebras of N
- $i\colon M \hookrightarrow N$ trace-preserving embedding

such that

$$T^n = i^* \circ E_{N_n} \circ i\,, \quad n \geq 1\,,$$

where E_{N_n} is the trace-preserving conditional expect. of N onto N_n.

Remark: If $T\colon M \to M$ has the Rota dilation property, then T is factorizable. The converse is not necessarily true, as shown by following example:

Let $T\colon M_2(\mathbb{C}) \to M_2(\mathbb{C})$ given by

$$T\left(x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\right) = \begin{pmatrix} x_{11} & -x_{12} \\ -x_{21} & x_{22} \end{pmatrix}, \quad x \in M_2(\mathbb{C})\,.$$

Then $T \in \mathrm{Aut}(M_2(\mathbb{C}))$, and hence it is factorizable, but T does not have the Rota dilation property, since it is not positive (as an operator on $L_2(M_2(\mathbb{C})\,, \tau_2))$.

Theorem (Anantharaman-Delaroche, 2004, cf. E. Ricard, 2007):
If $T\colon M \to M$ is a factorizable Markov map and $T^* = T$, then T^2 has the Rota dilation property.

Remark: If M is abelian, then any Markov map $T\colon M \to M$ is automatically factorizable. If, moreover, $T = T^*$, then the Rota dilation for T^2 in the above theorem can be chosen such that N is abelian. This is the classical Rota dilation theorem.

Theorem 2 (Haagerup-M.):
For some large $n \in \mathbb{N}$, there exists a Markov map T on $(M_n(\mathbb{C}), \tau_n)$ such that $T^* = T$, but T^2 is not factorizable. In particular, T^2 does <u>not</u> have the Rota dilation property.

Key Lemma: Let $n, d \in \mathbb{N}$ with $d \geq 5$ and set

$$Tx\colon = \sum_{i=1}^{n} a_i^* x a_i\,, \quad x \in M_n(\mathbb{C})\,,$$

where $a_1, \ldots, a_d \in M_n(\mathbb{C})$ satisfy:

(1) $a_i = a_i^*$, $1 \leq i \leq d$

(2) $\sum_{i=1}^{d} a_i^2 = 1$

(3) $a_i^2 a_j = a_j a_i^2$, $1 \leq i, j \leq d$

(4) $A\colon = \{a_i a_j : 1 \leq i, j \leq d\}$ is linearly independent

(5) $B\colon = \cup_{i=1}^{6} B_i$ is linearly independent, where

$B_1 \colon = \{a_i a_j a_k a_l : i \neq j \neq k \neq l\}$, $B_2 \colon = \{a_i a_j a_k^2 : i \neq j \neq k \neq k\}$, $B_3 \colon = \{a_i^3 a_j : i \neq j\}$, $B_4 \colon = \{a_i a_j^3 : i \neq j\}$, $B_5 \colon = \{a_i^2 a_j^2 : i < j\}$, $B_6 \colon = \{a_i^4 : 1 \leq i \leq d\}$.
Then T is a $(M_n(\mathbb{C}), \tau_n)$-Markov map, but T^2 is not factorizable. In particular, T^2 does <u>not</u> have the Rota dilation property.

Remark: Operators $a_1, \dots, a_d$ satisfying conditions $(1) - (5)$ can be realized in

$$L_\infty(S^{d-1}) \bar{\otimes} L(\mathbb{Z}_2 * \dots * \mathbb{Z}_2),$$

namely as

$$a_i = b_i \otimes u_i, \quad 1 \leq i \leq d$$

where $b_1, \dots, b_d$ are the coordinate functions on S^{d-1} (the unit sphere in $\mathbb{R}^d$) and $u_1, \dots, u_d \in L(\mathbb{Z}_2 * \dots * \mathbb{Z}_2)$ are the self-adjoint unitaries corresponding to the generators $g_1, \dots, g_d$ of $\mathbb{Z}_2 * \dots * \mathbb{Z}_2$. Using the fact that this group is residually finite, it is possible to get examples of $n \times n$ matrices $a_1, \dots, a_d$ satisfying $(1) - (5)$ for large values of n.

Theorem 2 (Haagerup-M.):

Let M be a finite von Neumann algebra with normal faithful tracial state τ, and let $S \colon M \to M$ be a τ-Markov map on M. Then the following are equivalent:

(1) S has the Rota dilation property

(2) S has a Rota dilation of order 1

(3) $S = T^* T$, where $T \colon M \to N$ is a factorizable (τ, τ_N)-Markov map, for some von Neumann algebra N with a normal faithful tracial state τ_N.

Further results

Recall the noncommutative little Grothendieck inequality (cb-version):

Theorem (Pisier–Shlyakhtenko, 2002, Haagerup-M, 2008):
Let A be a C*-algebra. If $T : A \to OH(I)$ is a completely bounded linear map, then there exist states f_1, f_2 on A such that

$$\|T(x)\| \leq \sqrt{2}\|T\|_{\mathrm{cb}} f_1(xx^*)^{1/4} f_2(x^*x)^{1/4}, \quad x \in A\,.$$

Problem: What is the best constant C_0 in the inequality

$$\|T(x)\| \leq C\|T\|_{\mathrm{cb}} f_1(xx^*)^{1/4} f_2(x^*x)^{1/4}, \quad x \in A\,. \tag{2}$$

for all choices of A and T.

Note: $1 \leq C_0 \leq \sqrt{2}$.

Theorem 4 (Haagerup-M): $C_0 > 1$.

More precisely,

(1) There exists $T\colon M_3(\mathbb{C}) \to OH(\{1,2,3\})$ such that (2) does not hold with $C = 1$, for any choice of states f_1, f_2.

(2) There exists $T\colon l_\infty\{1,2,3,4\} \to OH(\{1,2\})$ such that (2) does not hold with $C = 1$, for any choice of states f_1, f_2.

On the asymptotic quantum Birkhoff conjecture

Classical Birkhoff theorem (Birkhoff, 1946):

Every doubly stochastic matrix is a convex combination of permutation matrices.

Consider the abelian von Neumann algebra $D := l_\infty(\{1, 2, \ldots, n\})$ with trace given by $\tau(\{i\}) = 1/n$, $1 \le i \le n$. The positive unital trace-preserving maps on D are the linear operators on D which are given by doubly stochastic $n \times n$ matrices. Note that every automorphism of D is given by a permutation of $\{1, 2, \ldots, n\}$.

The quantum Birkhoff conjecture:

Does every *completely positive* unital trace-preserving map

$$T\colon (M_n(\mathbb{C}), \tau_n) \to (M_n(\mathbb{C}), \tau_n), \quad n \ge 1$$

lie in $\mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C}))$?

This turns out to be false for $n \ge 3$ (see, e.g, Example 1), and it was first shown by Landau-Streater (1993).

The asymptotic quantum Birkhoff conjecture:

Let $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a τ_n-Markov map, $n \ge 1$. Then

$$\lim_{k\to\infty} d_{\mathrm{cb}}\left(\bigotimes_{i=1}^{k} T\,, \mathrm{conv}(\mathrm{Aut}(\bigotimes_{i=1}^{k} M_n(\mathbb{C})))\right) = 0\,. \tag{3}$$

Theorem 5 (Haagerup-M):

Let $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a τ_n-Markov map, $n \geq 1$. Then

$$d_{\text{cb}}\left(\bigotimes_{i=1}^{k} T\,, \text{conv}(\text{Aut}(\bigotimes_{i=1}^{k} M_n(\mathbb{C})))\right) \geq d_{\text{cb}}(T\,, \mathcal{FM}(M_n(\mathbb{C})))\,.$$

Hence, if T is not factorizable, then (3) does not hold. Therefore the asymptotic quantum Birkhoff conjecture is not true.

Proof: It suffices to prove that given $m, n \geq 1$, then for any τ_n-Markov map T on $M_n(\mathbb{C})$ and any τ_m-Markov map S on $M_m(\mathbb{C})$,

$$d_{\text{cb}}(T \otimes S\,, \text{conv}(\text{Aut}(M_n(\mathbb{C})) \otimes M_m(\mathbb{C}))) \geq d_{\text{cb}}(T\,, \mathcal{FM}(M_n(\mathbb{C})))\,.$$

Let $i\colon M_n(\mathbb{C}) \to M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ be given by

$$i(x)\colon = x \otimes 1\,, \quad x \in M_n(\mathbb{C})\,.$$

It is easily checked that

$$i^*(T \otimes S)i = T\,,$$

where i^* is the adjoint of i. Since $\|i\|_{\text{cb}} = \|i^*\|_{\text{cb}} = 1$, we get

$$d_{\text{cb}}(T \otimes S\,, \text{conv}(\text{Aut}(M_n(\mathbb{C})) \otimes M_m(\mathbb{C}))) \geq \qquad (4)$$

$$d_{\text{cb}}(T\,, i^*\text{conv}(\text{Aut}(M_n(\mathbb{C})) \otimes M_m(\mathbb{C}))i)\,.$$

Since for every $u \in \mathcal{U}(M_n(\mathbb{C}) \otimes M_m(\mathbb{C}))$, the map $i^* \circ \text{ad}(u) \circ i$ is factorizable, and $\mathcal{FM}(M_n(\mathbb{C}))$ is a convex set, we deduce that

$$i^*\text{conv}(\text{Aut}(M_n(\mathbb{C})) \otimes M_m(\mathbb{C}))i \subset \mathcal{FM}(M_n(\mathbb{C}))\,,$$

which together with (4) completes the proof.