Classification of hyperfinite factors up to completely bounded isomorphism of their preduals

Magdalena Musat
University of Copenhagen

Copenhagen, January 28, 2010

Can one distinguish the type of a von Neumann algebra factor by the cb-isomorphism class of its predual?

Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. A linear map $\phi\colon \mathcal{M}_* \to \mathcal{N}_*$ is cb if and only if its adjoint $\phi^*\colon \mathcal{N} \to \mathcal{M}$ is cb, and in this case,

$$\|\phi^*\|_{\text{cb}} = \|\phi\|_{\text{cb}}.$$

In particular, if $\mathcal{M}_* \overset{cb}{\simeq} \mathcal{N}_*$, then clearly $\mathcal{M} \overset{cb}{\simeq} \mathcal{N}$.

Problem: What about the converse statement, i.e., if $\mathcal{M} \overset{cb}{\simeq} \mathcal{N}$, does it follow that $\mathcal{M}_* \overset{cb}{\simeq} \mathcal{N}_*$?

In general, this is not true. For a simple counterexample, note that

$$L^\infty([0,1]) \overset{cb}{\simeq} l^\infty(\mathbb{N}),$$

however, their preduals $L^1([0,1])$ and $l^1(\mathbb{N})$, respectively, are <u>not</u> even Banach space isomorphic. Indeed, $l_1(\mathbb{N})$ has a (unique, up to equivalence) unconditional basis, namely, the unit vector basis, while it was proved by Pelczynski that $L^1([0,1])$ is not isomorphic to a subspace of a space with an unconditional basis.

Theorem (Christensen, Sinclair 1989):

Let $\mathcal{M}$ and $\mathcal{N}$ be infinite dimensional *injective* factors with separable preduals. Then

$$\mathcal{M} \overset{cb}{\simeq} \mathcal{N}.$$

i.e., there is $\phi\colon \mathcal{M} \to \mathcal{N}$ linear bijection, so that $\|\phi\|_{\text{cb}}\|\phi^{-1}\|_{\text{cb}} < \infty$.

Theorem (Kirchberg 1993):

Let $\mathcal{A}$ and $\mathcal{B}$ be simple, separable, nuclear, non-type I C*-algebras. Then $\mathcal{A} \overset{cb}{\simeq} \mathcal{B}$.

Remark: Injectivity is preserved under cb-isomorphisms, i.e., if $\mathcal{M}$ is injective and $\mathcal{M} \overset{cb}{\simeq} \mathcal{N}$, then $\mathcal{N}$ is injective, as well.

Theorem (Haagerup, Rosenthal, Sukochev 2003):

Let $\mathcal{M}$ be a II_1-factor and $\mathcal{N}$ a II_∞-factor with separable preduals $\mathcal{M}_*$ and $\mathcal{N}_*$, respectively. Then $\mathcal{M}_*$ and $\mathcal{N}_*$ are not isomorphic as Banach spaces. In particular,

$$\mathcal{M}_* \overset{cb}{\not\simeq} \mathcal{N}_* \,.$$

Proof: It is shown that the space of trace-class op. $S_1 \colon = (\mathcal{B}(H))_*$ <u>does not</u> Banach space embed into the predual of any <u>finite</u> von Neumann algebra. Therefore,

$$S_1 \not\hookrightarrow \mathcal{M}_* \,.$$

On the other hand, since $\mathcal{N}$ is a II_∞-factor, then

$$\mathcal{N} \simeq \mathcal{N} \bar{\otimes} \mathcal{B}(H).$$

This implies that S_1 <u>does</u> Banach space embed into $\mathcal{N}_*$. Indeed, by general theory of (von Neumann algebras) tensor products, there is a normal conditional expectation onto

$$E : \mathcal{N} \bar{\otimes} \mathcal{B}(H) \to 1_{\mathcal{N}} \bar{\otimes} \mathcal{B}(H) \simeq \mathcal{B}(H) \,.$$

Hence the predual maps yields an embedding

$$E_* : (\mathcal{B}(H))_* \hookrightarrow (\mathcal{N} \bar{\otimes} \mathcal{B}(H))_* \simeq \mathcal{N}_* \,,$$

that is, $S_1 \hookrightarrow \mathcal{N}_*$, and the conclusion follows. $\square$

Theorem (Pisier 2004 + Junge 2006):

Let $\mathcal{M}$ be a semifinite von Neumann algebra, and let R_∞ be the unique injective type III_1-factor (with separable predual). Then

$$(R_\infty)_* \overset{\text{cb}}{\not\hookrightarrow} \mathcal{M}_*.$$

In particular, $(R_\infty)_* \overset{\text{cb}}{\not\simeq} \mathcal{M}_*$.

Proof: The key tool is Pisier's operator Hilbert space OH .
Pisier (2004) showed that

$$OH \overset{\text{cb}}{\not\hookrightarrow} \mathcal{M}_*\,,$$

while Junge (2006) proved that

$$OH \overset{\text{cb}}{\hookrightarrow} (R_\infty)_*\,. \qquad \square$$

Connes' classification of injective factors

Type	**Model**
I_n	$M_n(\mathbb{C})$
I_∞	$\mathcal{B}(l^2(\mathbb{N}))$
II_1	$R = \bigotimes_{n=1}^{\infty} (M_2(\mathbb{C}), \tau_2)$ **Murray-von Neumann, 1940**
II_∞	$R \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$ **Connes, 1976**
III_0	Krieger factors $L^\infty(\Omega, \mu) \rtimes \mathbb{Z}$, classified by non-transitive ergodic flows (uncountably many). **Connes and Krieger, 1976**
III_λ $0 < \lambda < 1$	Powers factors (unique one for each λ): $R_\lambda = \bigotimes_{n=1}^{\infty} (M_2(\mathbb{C}), \phi_\lambda)\,, \quad \phi_\lambda(x) = \mathrm{Tr}\left(\begin{pmatrix} \frac{\lambda}{1+\lambda} & 0 \\ 0 & \frac{1}{1+\lambda} \end{pmatrix} x\right).$ **Connes, 1976**
III_1	Unique one: the Araki-Woods factor $R_\infty \simeq R_{\lambda_1} \bar{\otimes} R_{\lambda_2}\,, \quad \frac{\log \lambda_1}{\log \lambda_2} \notin \mathbb{Q}\,.$ **Connes and Haagerup, 1986-87**

Main Theorem (Haagerup, M. 2007):

(**1**) Let $\mathcal{M}$ and $\mathcal{N}$ be hyperfinite factors with separable preduals $\mathcal{M}_*$ and $\mathcal{N}_*$, respectively. If $\mathcal{M}$ is semifinite and $\mathcal{N}$ is type III, then

$$\mathcal{M}_* \overset{\text{cb}}{\not\simeq} \mathcal{N}_* \,.$$

(**2**) The predual $\mathcal{N}_*$ of a hyperfinite type III-factor (on a separable Hilbert space) is cb-isomorphic to $(R_\infty)_*$, if and only if there exists an invariant normal state on the *flow of weights* for $\mathcal{N}$.

(**3**) There exists a one-parameter family $(\mathcal{M}_t)_{0\le t<2}$ of type III_0 hyperfinite factors (on separable Hilbert spaces) with mutually non-cb-isomorphic preduals.

Remark: As a corollary of (**2**) (cf. also Haagerup, Rosenthal, Sukochev 2003), it follows that for all $0 < \lambda < 1$,

$$(R_\lambda)_* \overset{\text{cb}}{\simeq} (R_\infty)_* \,.$$

Ingredients in the proof of Main Theorem

Lemma A (Pelczynski): Let X, Y be Banach spaces such that

1) $X \simeq Y \oplus F$, for some Banach space F
2) $Y \simeq X \oplus E$, for some Banach space E
3) $X \simeq X \oplus X$ and $Y \simeq Y \oplus Y$

Then $X \simeq Y$.

Proof: $X \simeq X \oplus X \simeq Y \oplus Y \oplus F \oplus F \simeq Y \oplus F \oplus F \simeq X \oplus F$.
Therefore, $X \simeq Y \oplus F \simeq X \oplus E \oplus F \simeq X \oplus F \oplus E \simeq X \oplus E \simeq Y$.

Note: Pelczynski's lemma holds for operator spaces and cb-isomorphisms.

Lemma: If $\mathcal{M}$ is a properly infinite von Neumann algebra, then

$$\mathcal{M} \oplus \mathcal{M} \overset{\text{cb}}{\simeq} \mathcal{M}, \quad \mathcal{M}_* \oplus \mathcal{M}_* \overset{\text{cb}}{\simeq} \mathcal{M}_*.$$

Proof: There exist isometries $u_1, u_2 \in \mathcal{M}$ such that $u_1 u_1^*$ and $u_2 u_2^*$ are orthogonal projections with sum equal to 1.
Define $\phi\colon \mathcal{M} \to \mathcal{M} \oplus \mathcal{M}$ by

$$\phi(x)\colon = (u_1^* x, u_2^* x), \quad x \in \mathcal{M}$$

and $\psi\colon \mathcal{M} \oplus \mathcal{M} \to \mathcal{M}$ by

$$\psi(x, y)\colon = u_1 x + u_2 y, \quad x, y \in \mathcal{M}.$$

Then $\phi \circ \psi = \mathrm{Id}_{\mathcal{M} \oplus \mathcal{M}}$ and $\psi \circ \phi = \mathrm{Id}_{\mathcal{M}}$. Both ϕ and ψ are completely bounded, hence $\mathcal{M} \oplus \mathcal{M} \overset{\text{cb}}{\simeq} \mathcal{M}$. Also, since both ϕ and ψ are normal, we get $\mathcal{M}_* \oplus M_* \overset{\text{cb}}{\simeq} \mathcal{M}_*$. $\square$

Proposition:

Let $\mathcal{M}$ and $\mathcal{N}$ be *properly infinite injective* von Neumann algebras with separable preduals $\mathcal{M}_*$ and $\mathcal{N}_*$. TFAE:

(1) $\mathcal{M}_* \overset{\text{cb}}{\simeq} \mathcal{N}_*$.

(2) $\mathrm{Id}_{\mathcal{M}_*}$ cb-factors through $\mathcal{N}_*$ and $\mathrm{Id}_{\mathcal{N}_*}$ cb-factors through $\mathcal{M}_*$.

(3) $\mathrm{Id}_{\mathcal{M}}$ cb-factors through $\mathcal{N}$ and $\mathrm{Id}_{\mathcal{N}}$ cb-factors through $\mathcal{M}$, such that all four cb-maps involved are normal.

(4) There exist von Neumann algebra embeddings $i\colon \mathcal{M} \hookrightarrow \mathcal{N}$ and $j\colon \mathcal{N} \hookrightarrow \mathcal{M}$, and normal conditional expectations

$$E\colon \mathcal{N} \to i(\mathcal{M}), \quad F\colon \mathcal{M} \to j(\mathcal{N}).$$

Remark: The requirement (hence restriction) that $\mathcal{M}$ and $\mathcal{N}$ be injective in the statement of Proposition is due to our method of proof. The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow from Pelczynski's trick and duality, while (3) $\Leftrightarrow$ (4) uses Stinespring-type decompositions.

Lemma B (Stinespring-Kasparov-type decomposition):

Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras with separable preduals. Assume that $\mathcal{N}$ is properly infinite. If

$$\alpha\colon \mathcal{M} \to \mathcal{N}$$

is a normal completely positive map, then there exists $\pi\colon \mathcal{M} \to \mathcal{N}$ normal unital $*$-representation and $V \in \mathcal{N}$ such that

$$\alpha(x) = V^*\pi(x)V\,, \quad x \in \mathcal{M}\,.$$

Lemma C (Stinespring-Paulsen-type decomposition):

Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras with separable preduals. Assume that $\mathcal{N}$ is properly infinite and injective. If

$$\beta\colon \mathcal{M} \to \mathcal{N}$$

is a normal completely bounded map, then there exists $\pi\colon \mathcal{M} \to \mathcal{N}$ normal unital $*$-representation and $R, S \in \mathcal{N}$ such that

$$\beta(x) = R\pi(x)S\,, \quad x \in \mathcal{M}$$

and $\|R\|\|S\| = \|\beta\|_{\mathrm{cb}}\,.$

Proof of (1) **in Main Theorem**:

Let $\mathcal{M}$, $\mathcal{N}$ be hyperfinite factors with separable preduals $\mathcal{M}_*$ and $\mathcal{N}_*$, where $\mathcal{M}$ is semifinite and $\mathcal{N}$ is type III.

We prove that $\mathcal{M}_*$ and $\mathcal{N}_*$ are <u>not</u> cb-isomorphic. Assume by contradiction that

$$\mathcal{M}_* \overset{\text{cb}}{\simeq} \mathcal{N}_* \,.$$

Note that $\mathcal{N}$ is properly infinite.

If $\mathcal{M}$ is properly infinite, then by (1) $\Leftrightarrow$ (4) in the Proposition, there exists a von Neumann algebra embedding

$$i \colon \mathcal{N} \hookrightarrow \mathcal{M}$$

and a normal conditional expectation

$$E \colon \mathcal{M} \to i(\mathcal{N}) \,.$$

It was proved by Tomiyama (1959) (see also Sakai (1957)), that if $\mathcal{M}$ is semifinite and $\mathcal{N}$ is of type III, no such normal conditional expectation exists. Hence the conclusion follows.

If $\mathcal{M}$ is not properly infinite, one can use the same argument on $\mathcal{M}\bar{\otimes}\mathcal{B}(l^2(\mathbb{N}))$, instead. $\square$

The flow of weights of a type III-factor

(Connes-Takesaki, 1977)

Let $\mathcal{M}$ be a type III-factor with separable predual $\mathcal{M}_*$. Let ϕ_0 be a normal, faithful state on $\mathcal{M}$. Set

$$\mathcal{N} := \mathcal{M} \rtimes_{\sigma^{\phi_0}} \mathbb{R}\,.$$

Generators of $\mathcal{N}$: $x \in \mathcal{M}$, $(\lambda(t))_{t\in\mathbb{R}}$.

Dual action $(\widetilde{\theta}_s)_{s\in\mathbb{R}}$ on $\mathcal{N}$:

$$\begin{aligned} \widetilde{\theta}_s(x) &= x\,, \quad x \in \mathcal{M} \\ \widetilde{\theta}_s(\lambda(t)) &= e^{ist}\lambda(t)\,, \quad t \in \mathbb{R}\,. \end{aligned}$$

Let $Z(\mathcal{N})$ denote the center of $\mathcal{N}$ and set

$$\theta_s := \widetilde{\theta}_s|_{Z(\mathcal{N})}\,, \quad s \in \mathbb{R}\,.$$

Then $(\theta_s)_{s\in\mathbb{R}}$ is an ergodic action of $\mathbb{R}$ on $Z(\mathcal{N})$.

$(Z(\mathcal{N}), (\theta_s)_{s\in\mathbb{R}})$ is called *the flow of weights* for $\mathcal{M}$. It is independent of the choice of the state ϕ_0.

Remark: Since $Z(\mathcal{N}) \simeq L^\infty(\Omega, \mu)$, for some standard Borel measure space (Ω, μ), the flow $\theta := (\theta_s)_{s\in\mathbb{R}}$ can be realized as

$$(\theta_s f)(x) = f(\sigma_s^{-1} x)\,, \quad f \in L^\infty(\Omega, \mu)\,, x \in \Omega\,,$$

for a one-parameter family (flow) $(\sigma_s)_{s\in\mathbb{R}}$ of Borel transformations of Ω, which preserve the measure class $[\mu]$ of μ.

$\mathcal{M}$ is type $\mathrm{III}_0 \;\Leftrightarrow\; (Z(\mathcal{N}), (\theta_s)_{s\in\mathbb{R}})$ is a non-transitive ergodic flow

(μ is not concentrated on a single orbit)

$$\mathcal{M} \text{ is type } \mathrm{III}_\lambda \;\Leftrightarrow\; \begin{cases} Z(\mathcal{N}) \simeq L^\infty(\mathbb{R}/(-\log\lambda)\mathbb{Z}), \\ \theta_s = \text{ translation by } s \end{cases}$$

$0 < \lambda < 1$

$$\mathcal{M} \text{ is type } \mathrm{III}_1 \;\Leftrightarrow\; \begin{cases} Z(\mathcal{N}) \simeq \mathbb{C}1 \\ \theta_s = \text{ trivial action.} \end{cases}$$

Theorem (Connes-Krieger 1976): The map

$$\mathcal{M} \mapsto (Z(\mathcal{N}), (\theta_s)_{s\in\mathbb{R}})$$

is a bijection of the set of (isomorphism classes of) injective type III_0 factors onto the set of (isomorphism classes of) non-transitive ergodic flows on standard measure spaces.

Proof of (2) **in Main Theorem**:
Let $\mathcal{M}$ be a hyperfinite type III-factor. We prove that

$$\mathcal{M}_* \overset{\text{cb}}{\simeq} (R_\infty)_*$$

if and only if there exists a normal *invariant* state $\widetilde{\phi}$ on the flow of weights for $\mathcal{M}$ (i.e., $\widetilde{\phi}\circ\theta_s = \widetilde{\phi}$, for all $s \in \mathbb{R}$).

Proof of ”$\Rightarrow$” :

If $\mathcal{M}_* \overset{\text{cb}}{\simeq} (R_\infty)_*$, then by (1) $\Rightarrow$ (4) in the Proposition,

$$R_\infty \simeq \mathcal{M}_0 \subseteq \mathcal{M}\,,$$

where $\mathcal{M}_0$ is the range of a normal conditional expectation E on $\mathcal{M}$. Moreover, since $\mathcal{M}$ is of type III, E can be chosen to be faithful.

Choose a normal faithful state ϕ on $\mathcal{M}_0$ and set

$$\psi := \phi \circ E\,.$$

By Takesaki's duality for crossed products we obtain an embedding

$$\mathcal{N}_0 \colon = \mathcal{M}_0 \rtimes_{\sigma^\phi} \mathbb{R} \ \hookrightarrow \ \mathcal{M} \rtimes_{\sigma^\psi} \mathbb{R} =: \mathcal{N}\,,$$

and E extends naturally to a normal faithful conditional expectation

$$\widetilde{E} \colon \mathcal{N} \to \mathcal{N}_0\,,$$

which intertwines the two dual actions, i.e.,

$$\widetilde{\theta}^0_s \circ \widetilde{E} = \widetilde{E} \circ \widetilde{\theta}_s\,, \quad s \in \mathbb{R}\,.$$

Moreover, $\widetilde{E}(Z(\mathcal{N})) \subseteq Z(\mathcal{N}_0)$.

Since $\mathcal{M}_0 \simeq R_\infty$ is type III_1, it follows that $Z(\mathcal{N}_0) = \mathbb{C}1$ and the action θ^0_s, $s \in \mathbb{R}$ is trivial. Hence

$$\widetilde{E}(x) = \widetilde{\phi}(x)1\,, \quad x \in Z(\mathcal{N})$$

for a θ_s-invariant normal state $\widetilde{\phi}$ on $Z(\mathcal{N})$.

Proof of "$\Leftarrow$" :

For $\phi, \psi \in S_{\text{nor}}(\mathcal{M})$, one writes $\phi \sim \psi$ if and only if

$$\inf_{u \in \mathcal{U}(\mathcal{M})} \|u\phi u^* - \psi\| = 0 .$$

More generally, for $k, l \in \mathbb{N}$ and $\phi \in S_{\text{nor}}(M_k(\mathcal{M}))$, $\psi \in S_{\text{nor}}(M_l(\mathcal{M}))$, we write $\phi \sim \psi$ if and only if

$$\inf_{u \in \mathcal{U}(M_{lk}(\mathcal{M}))} \|u\phi u^* - \psi\| = 0 ,$$

where $\mathcal{U}(M_{lk}(\mathcal{M}))$ is the set of $l \times k$ matrices u over $\mathcal{M}$, for which $u^*u = 1_{M_k(\mathcal{M})}$ and $uu^* = 1_{M_l(\mathcal{M})}$.

As an application of the main result of [Haagerup, Størmer 1990], we show that there exists a normal invariant state on the flow of weights for $\mathcal{M}$ if and only if $\exists\, \phi \in S_{\text{nor}}(\mathcal{M})$ such that for all $n \in \mathbb{N}$,

$$\phi \;\sim\; \frac{1}{n}\begin{pmatrix} \phi & & & 0 \\ & \phi & & \\ & & \ddots & \\ 0 & & & \phi \end{pmatrix} \in S_{\text{nor}}(M_n(\mathcal{M})) .$$

This can be used to construct an embedding $i\colon R_\infty \hookrightarrow \mathcal{M}$ with a normal faithful conditional expectation $E\colon \mathcal{M} \to i(R_\infty)$.

Since $\mathcal{M}\bar{\otimes}R_\infty \simeq R_\infty$, there exists also an embedding $j\colon \mathcal{M} \hookrightarrow R_\infty$ with a normal faithful conditional expectation $F\colon R_\infty \to j(\mathcal{M})$.

By (4) $\Rightarrow$ (1) in Proposition, we then conclude that

$$\mathcal{M}_* \overset{\text{cb}}{\simeq} (R_\infty)_* . \qquad \square$$

Remark: The relation $\sim$ is an equivalence relation on $S_{\mathrm{nor}}(\mathcal{M})$, and thus $S_{\mathrm{nor}}(\mathcal{M})/\!\sim$ becomes a metric space with

$$d([\phi]\,,[\psi])\colon = \inf_{u\in\mathcal{U}(\mathcal{M})} \|u\phi u^* - \psi\|\,.$$

Theorem (Connes, Haagerup, Størmer 1983):

If $\mathcal{M}$ is a factor of type III_λ, where $0 \leq \lambda \leq 1$, then

$$\text{diameter}\,(S_{\mathrm{nor}}(\mathcal{M})/\!\sim) \;=\; 2\,\frac{1-\sqrt{\lambda}}{1+\sqrt{\lambda}}\,.$$

Therefore the diameter of $S_{\mathrm{nor}}(\mathcal{M})/\!\sim$ determines uniquely the parameter λ for a type III factor.

Proof of (3) **in Main Theorem**:

Lemma D:

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be hyperfinite type III-factors with separable preduals. Let $(Z(\mathcal{N}_1)\,,\theta^{(1)})\,,\,(Z(\mathcal{N}_2)\,,\theta^{(2)})$ be their flow of weights. If

$$(\mathcal{M}_1)_* \overset{\text{cb}}{\simeq} (\mathcal{M}_2)_*\,,$$

then there exist normal unital positive maps

$$R\colon Z(\mathcal{N}_1) \to Z(\mathcal{N}_2), \qquad S\colon Z(\mathcal{N}_2) \to Z(\mathcal{N}_1)$$

intertwining the actions $\theta^{(1)}$ and $\theta^{(2)}$, i.e., for all $s \in \mathbb{R}$,

$$R \circ \theta_s^{(1)} = \theta_s^{(2)} \circ R\,, \quad S \circ \theta_s^{(2)} = \theta_s^{(1)} \circ S\,.$$

Lemma E:

For each $0 \le t < 2$, there exists a non-transitive ergodic flow

$$(A_t, (\theta_s^{(t)})_{s\in\mathbb{R}})$$

with separable predual $(A_t)_*$ such that for all $\omega \in S_{\text{nor}}(A_t)$,

$$\lim_{n\to\infty} \|\omega \circ \theta_{2^n}^{(t)} - \omega\| = t\,.$$

We now prove (**3**) in Main Theorem, based on Lemmas D and E:

Let $(\mathcal{M}_t)_{0\le t<2}$ be the hyperfinite III_0 factors with flow of weights equal to $(A_t, (\theta_s^{(t)})_{s\in\mathbb{R}})$, respectively, as given by Lemma E.

We prove that if $t_1 \neq t_2$, then

$$(\mathcal{M}_{t_1})_* \overset{\text{cb}}{\not\simeq} (\mathcal{M}_{t_2})_* .$$

Assume by contradiction that $(\mathcal{M}_{t_1})_* \overset{\text{cb}}{\simeq} (\mathcal{M}_{t_2})_*$. By Lemma D, there exist normal, positive, unital maps $R\colon A_{t_1} \to A_{t_2}$ and $S\colon A_{t_2} \to A_{t_1}$ which are $\theta^{(t_1)}$, respectively, $\theta^{(t_2)}$-intertwiners, i.e.,

$$\begin{aligned} R\circ\theta_s^{(t_1)} &= \theta_s^{(t_2)}\circ R\,, \quad s\in\mathbb{R} \\ S\circ\theta_s^{(t_2)} &= \theta_s^{(t_1)}\circ S\,, \quad s\in\mathbb{R}\,. \end{aligned}$$

Since $\|S\|\le 1$, it follows from Lemma E that for all $\omega\in S_{\mathrm{nor}}(A_{t_1})$,

$$\begin{aligned} t_2 &= \lim_{n\to\infty} \|(\omega\circ S)\circ\theta_{2^n}^{(t_2)} - (\omega\circ S)\| \\ &= \lim_{n\to\infty} \|(\omega\circ\theta_{2^n}^{(t_1)} - \omega)\circ S\| \\ &\le \lim_{n\to\infty} \|\omega\circ\theta_{2^n}^{(t_1)} - \omega\| \\ &= t_1\,. \end{aligned}$$

Similarly, $t_1\le t_2$. Hence $t_1 = t_2$, which gives a contradiction. $\square$

Proof of Lemma E:
$(A_t, (\theta_s^{(t)})_{s\in\mathbb{R}})$ is obtained by building a flow under the constant ceiling function $\phi(x) = 1$:

For $t \in [0, 2)$ write

$$t = 2 - 4a\,, \quad a \in (0, 1/2]\,.$$

Set

$$(\Omega, \mu_a):\ = \left(\{0,1\}^\infty, \otimes_{n=1}^\infty \nu_a\right)\,,$$

where

$$\nu_a:\ = a\delta_0 + (1-a)\delta_1\,.$$

Let g be the dyadic odometer transformation (= dyadic adding machine) on Ω, namely,

For $x = (x_1, x_2, x_3, \dots) \in \Omega = \{0,1\}^\infty$ put

$$\begin{aligned} g(0, x_2, x_3, \dots) &= (1, x_2, x_3, \dots) \\ g(1, 0, x_3, \dots) &= (0, 1, x_3, \dots) \\ &\vdots \\ g(1, 1, \dots, 1, 0, x_{n+1}, \dots) &= (0, 0, \dots, 0, 1, x_{n+1}, \dots) \\ g(1, 1, 1, \dots) &= (0, 0, 0, \dots) \end{aligned}$$

Then g is an ergodic transformation on (Ω, μ_a).

Define

$$\widetilde{\Omega} := \Omega \times [0, 1)\,, \quad \widetilde{\mu}_a := \mu_a \times dm\,,$$

where dm is the Lebesgue measure on $[0, 1)$.

The flow $(\widetilde{g}_s)_{s\in\mathbb{R}}$ on $(\widetilde{\Omega}\,,\widetilde{\mu}_a)$ built from (g,Ω,μ) under the constant ceiling function $\phi(x)=1$ is given by

$$\widetilde{g}_s(x,y)\colon = (g^n x, r)\,, \quad (x,y)\in\widetilde{\Omega}\,,$$

where $n\colon = [s-t]\in\mathbb{Z}$ and $r\colon = s-t-[s-t]\in[0,1)\,.$

Now let

$$A_t\colon = L^\infty(\widetilde{\Omega}\,,\widetilde{\mu}_a)$$

and

$$(\theta_s^{(t)} f)(z)\colon = (f\circ\widetilde{g}_{-s})(z)\,, \quad f\in A_t\,, z\in\Omega\times[0,1)\,.$$

Then $(A_t,(\theta_s^{(t)})_{s\in\mathbb{R}})$ satisfies the condition in Lemma E, namely,

$$\lim_{n\to\infty}\|\omega\circ\theta_{2^n}^{(t)}-\omega\| = t\,, \quad \forall\,\omega\in S_{\mathrm{nor}}(A_t) \qquad \square$$