Preliminaries on von Neumann algebras and operator spaces

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Von Neumann algebras were introduced by John von Neumann in 1929-1930 as *rings of operators.* It was Dieudonné's suggestion to Dixmier to call them *von Neumann algebras.* The theory of von Neumann algebras on a separable Hilbert space was developed by Murray and von Neumann in 1936-1943.

$$C^* - \text{algebras} \sim \textit{noncommutative topological spaces.}$$

(Gelfand): A unital commutative C^*-algebra is isometrically isomorphic to $C(\Delta)$, for some compact Hausdorff topological space Δ.

A projection is the indicator function of a clopen set.

$$\text{von Neumann algebras} \sim \textit{noncommutative measure spaces.}$$

(von Neumann): A commutative von Neumann algebra is isometrically isomorphic to $L^\infty(\Omega, \mu)$, for some standard Borel measure space (Ω, μ).

A projection is the indicator function of a measurable set.

Although any von Neumann algebra is a C^*-algebra, the converse is seldom true. The behavior of a generic von Neumann algebra differs strikingly from that of an arbitrary C^*-algebra. For example:

- There are many C^*-algebras without non-trivial projections.
  Non-simple: $C(X)$, where X is connected, compact Hausdorff.
  Simple: $C_r^*(\mathbb{F}_n)$, $n \geq 2$ (Pimsner-Voiculescu).

  In contrast, any von Neumann algebra is generated by projections.

- Infinite-dimensional von Neumann algebras are non-separable, while the interesting examples of infinite-dimensional C^*-algebras are separable. E.g., $C[0, 1]$ is separable, while $L^\infty[0, 1]$ is not.

Topologies on $\mathcal{B}(H)$

Let H be a (separable) Hilbert space with inner product $\langle \cdot, \cdot \rangle$.

- Norm topology:

$$x_n \xrightarrow{\|\cdot\|} x \quad \text{iff} \quad \|x_n - x\| \longrightarrow 0 \text{ as } n \to \infty\,.$$

- Strong operator topology (s.o.t.):

$$x_\alpha \xrightarrow{\text{s.o.t.}} x \quad \text{iff} \quad \|(x_\alpha - x)\xi\| \underset{\alpha}{\longrightarrow} 0\,, \quad \forall \xi \in H\,.$$

- Weak operator topology (w.o.t.):

$$x_\alpha \xrightarrow{\text{w.o.t.}} x \quad \text{iff} \quad |\langle (x_\alpha - x)\xi\,, \eta\rangle| \underset{\alpha}{\longrightarrow} 0\,, \quad \forall \xi\,, \eta \in H\,.$$

- w^*-topology is the topology induced by $\mathcal{B}(H) = S_1^*$ (where S_1 are the trace-class operators on H):

$$x_\alpha \xrightarrow{w^*} x \quad \text{iff} \quad \operatorname{tr}(x_\alpha y) \longrightarrow \operatorname{tr}(xy)\,, \quad \forall y \in S_1\,.$$

The norm topology is stronger than s.o.t., which, in turn, is stronger than w.o.t. Norm topology is also stronger than w^*-topology, which, in turn, is stronger than w.o.t.

Example: For $n \geq 1$, let

$$p_n := \begin{pmatrix} 1 & \dots & 0 & 0 & \dots \\ \vdots & \ddots & \vdots & \vdots & \\ 0 & \dots & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix} \in \mathcal{B}(l_2(\mathbb{N}))\,.$$

Then $p_n \xrightarrow{\text{s.o.t.}} I_{l_2(\mathbb{N})}$, but not in norm.

Example: Let s be the unilateral shift on $l_2(\mathbb{N})$, i.e., $se_n = e_{n+1}$, for $n \geq 1$, where e_n are the canonical unit vectors in $l_2(\mathbb{N})$. Then $s^n \xrightarrow{\text{w.o.t.}} 0$ as $n \to \infty$, but $(s^n)_{n\geq 1}$ does not converge (at all) in s.o.t.

Theorem (von Neumann's bicommutant theorem):

Suppose $\mathcal{M}$ is a $*$-closed, unital subalgebra of $\mathcal{B}(H)$. Then the following are equivalent:

(i) $\mathcal{M} = \mathcal{M}''$.

(ii) $\mathcal{M}$ is w.o.t. closed.

(iii) $\mathcal{M}$ is s.o.t. closed.

Definition: A von Neumann algebra $\mathcal{M}$ on a separable Hilbert space H is a $*$-closed unital subalgebra of $\mathcal{B}(H)$ which is w.o.t.-closed.

The algebra $K(H)$ of compact operators on an infinite dimensional (separable) Hilbert space H is not a von Neumann algebra. The identity operator $I : H \to H$ belongs to $(K(H))''$, but I is not compact.

Examples of von Neumann algebras:

(i) $\mathcal{B}(H)$ and matrix algebras.

(ii) $L^\infty([0,1], dm)$ is a von Neumann algebra acting on the (separable) Hilbert space $H := L^2([0,1], dm)$ by multiplication, i.e.,

$$f \in L^\infty([0,1], dm) \rightsquigarrow M_f : H \to H \,,$$

where $M_f(g) = fg \in H \,, \forall g \in H$.

(iii) If G is a group and $g \in G \overset{U}{\longmapsto} u_g \in \mathcal{U}(H)$ is a unitary repres. of G on a (sep.) Hilbert space H , then the bicommutant $\{u_g : g \in G\}''$ is a von Neumann algebra.

In the special case when U is the *left regular representation* of G on $l^2(G) := \{\xi : G \to \mathbb{C} : \sum_{t\in G} |\xi(t)|^2 < \infty\}$, i.e.,

$$(u_g(\xi))(t) := \xi(g^{-1}t)\,, \quad \xi \in l^2(G), t \in G,$$

then the von Neumann algebra $L(G) := \{u_g : g \in G\}''$ is called the group von Neumann algebra of G.

(iv) If $\mathcal{M}$ and $\mathcal{N}$ are vN algebras on Hilbert spaces H and K, respectively, then there are obvious notions of direct sum $\mathcal{M} \oplus \mathcal{N}$ and tensor product $\mathcal{M}\bar{\otimes}\mathcal{N}$ acting on $H \oplus K$ and $H \otimes K$, respectively.

The Gelfand-Naimark-Segal (GNS) representation provides a useful procedure of constructing von Neumann algebras.

The necessary ingredients are a unital C^*-algebra $\mathcal{A}$ and a positive linear functional ϕ on $\mathcal{A}$. Then there exists a Hilbert space H_ϕ , a unital $*$-representation $\pi : \mathcal{A} \to \mathcal{B}(H_\phi)$ and a vector $\xi \in H_\phi$ with $\|\xi\|^2 = \|\phi\|$ such that

$$\phi(x) = \langle \pi(x)\xi, \xi \rangle\,, \quad x \in \mathcal{A}\,.$$

Moreover, $H_\phi = \overline{(\pi(\mathcal{A})\xi)}^{\|\cdot\|}$ and $\mathcal{A}$ acts on H_ϕ by left multiplication.

The completion $\overline{\pi(\mathcal{A})}^{\text{w.o.t}} \subseteq \mathcal{B}(H_\phi)$ is a von Neumann algebra.

An abstract characterization of von Neumann algebras:

Theorem (Sakai):

A von Neumann algebra $\mathcal{M}$ is a C^*-algebra which, as a Banach space is a dual space, i.e., there exists a Banach space $\mathcal{M}_*$ such that

$$\mathcal{M} = (\mathcal{M}_*)^* .$$

Moreover, the Banach space $\mathcal{M}_*$ with this property is unique (up to isomorphism), and it is called the *predual* of $\mathcal{M}$.

Note that $\mathcal{M}_* \subseteq \mathcal{M}^*$. A positive, linear functional $\phi : \mathcal{M} \to \mathbb{C}$ belongs to $\mathcal{M}_*$ if and only if ϕ is *normal*, i.e., whenever $(x_\alpha)_{\alpha \in I} \subseteq \mathcal{M}$ is an increasing net of positive elements converging s.o.t. to some $x \in \mathcal{M}$, then $\lim_\alpha \phi(x_\alpha) = \phi(x)$. It was proved by Dixmier that ϕ is normal if and only if it is continuous with respect to the w*-topology on $\mathcal{M}$.

Example: Let $\xi \in H$ and set

$$\phi_\xi(x) := \langle x\xi, \xi \rangle , \quad x \in \mathcal{M}.$$

Then $\phi_\xi : \mathcal{M} \to \mathbb{C}$ is normal.

In particular, if $\mathcal{M} = L^\infty([0,1], dm)$, since the function $f \equiv 1 \in L^2$, then functional ϕ: $L^\infty([0,1], dm) \to \mathbb{C}$ defined by

$$\phi(f) = \int f dm, \quad f \in L^\infty([0,1], dm)$$

is normal.

Definition: A von Neumann algebra $\mathcal{M}$ is called a *factor* if its center

$$Z(\mathcal{M}) \colon = \{x \in \mathcal{M} : \ xy = yx \,, \forall y \in \mathcal{M}\}$$

consists of scalar multiples of the identity of $\mathcal{M}$.

Examples of factors:

(i) $\mathcal{B}(H)$ and matrix algebras.

(ii) The group von Neumann algebra $L(G)$ of an ICC group G (e.g., $\mathbb{F}_n$, $n \geq 2$ or the group S_∞ of finite permutations of $\mathbb{N}$).

(iii) $\mathcal{M}\bar{\otimes}\mathcal{N}$, where $\mathcal{M}$ and $\mathcal{N}$ are factors.

(iv) Let $\mathcal{A}$ be the UHF algebra of type 2^∞ (=the CAR algebra), equipped with its natural trace. The von Neumann algebra obtained as the completion of $\mathcal{A}$ via the GNS construction is a factor.

Murray and von Neumann's classification of factors

Let $\mathcal{M}$ be a factor on a separable Hilbert space H. Let $\mathcal{P}(\mathcal{M})$ denote the set of (orthogonal) projections in $\mathcal{M}$, i.e.,

$$\mathcal{P}(\mathcal{M}) \colon = \{p \in \mathcal{M} : \ p^2 = p = p^*\} \,.$$

It can be shown that $\mathcal{P}(\mathcal{M})$ is a complete lattice.

Two projections p, q in $\mathcal{M}$ are called *equivalent* (written $p \sim q$) if there exists $u \in \mathcal{M}$ such that $p = u^*u$ and $q = uu^*$.

Note: If $\mathcal{M} = \mathcal{B}(H)$, then two projections p, q in $\mathcal{M}$ are equivalent if and only if their ranges have the same dimension.

This suggested the idea that equivalence classes of projections represent an abstract notion of dimension for an arbitrary factor.

Theorem (Murray-von Neumann 1936):

For every factor $\mathcal{M}$ on a separable Hilbert space H there exists a *dimension* function $d\colon \mathcal{P}(\mathcal{M}) \to [0,\infty]$ such that

1) If $p \perp q$, then $d(p+q) = d(p) + d(q)$.

2) $p \sim q$ iff $d(p) = d(q)$.

The dimension function is unique, up to multiplication by positive scalars.

By studying the possible range sets $d(\mathcal{P}(\mathcal{M}))$, Murray and von Neumann found that, after a suitable normalization of the dimension function, factors can be classified into the following types:

Type	$d(\mathcal{P}(\mathcal{M}))$
I_n	$\{0,1,2,\ldots,n\}, \quad n \in \mathbb{N}$
I_∞	$\{0,1,2,\ldots,\infty\}$
II_1	$[0,1]$
II_∞	$[0,\infty]$
III	$\{0,\infty\}$

The only factors of type I_n and type I_∞ are the matrix algebras $M_n(\mathbb{C})$ and $\mathcal{B}(H)$ (on an infinite dimensional Hilbert space H), respectively.

The factor constructed in example (iv) above is of type II_1. Moreover, it is hyperfinite. (A factor $\mathcal{M}$ is called *hyperfinite* if there is an increasing sequence of finite-dimensional von Neumann subalgebras of $\mathcal{M}$ whose union is w.o.t. dense in $\mathcal{M}$.) It is a deep result of Murray and von Neumann (1940) that there is a unique (up to algebraic isomorphism) hyperfinite II_1 factor, denoted by $\mathcal{R}$. The factor $L(S_\infty)$ is also hyperfinite of type II_1, and therefore $L(S_\infty) \cong \mathcal{R}$. However, it can be proved that $L(\mathbb{F}_n) \ncong \mathcal{R}$, $n \geq 2$.

Furthermore, Murray and von Neumann proved that every factor $\mathcal{M}$ of type II_∞ is of the form $\mathcal{M} = \mathcal{N} \bar{\otimes} \mathcal{B}(H)$, for some factor $\mathcal{N}$ of type II_1.

Murray and von Neumann constructed examples of factors of type III, as well. However, factors of type III remained a mystery until 1970's, when using the developments of the Tomita-Takesaki theory, Connes classified them into types III_λ, with $0 \leq \lambda \leq 1$.

In 1975, Connes proved his celebrated result that a factor $\mathcal{M} \subseteq \mathcal{B}(H)$ is hyperfinite if and only if it is injective, i.e., there exists a projection of norm 1 from $\mathcal{B}(H)$ onto $\mathcal{M}$. As a corollary, all hyperfinite factors of type II_∞ are isomorphic to $\mathcal{R} \bar{\otimes} \mathcal{B}(l^2(\mathbb{N}))$.
In the same fundamental paper, Connes completed the classification of hyperfinite factors of type III_λ, for all $0 \leq \lambda < 1$. Haagerup (1986) settled in the affirmative the long-standing problem of uniqueness of hyperfinite type III_1 factors.

Operator spaces and completely bounded maps

Let H be a Hilbert space and $E \subseteq \mathcal{B}(H)$ a closed subspace. Then E becomes an *operator space* with the sequence of norms on $M_n(E)$ inherited from $\mathcal{B}(H^n)$, $n \in \mathbb{N}$, via the isometric embeddings

$$M_n(E) \subseteq M_n(\mathcal{B}(H)) = \mathcal{B}(H^n)\,.$$

An *abstract operator space* is a vector space E equipped with matrix norms $\|\cdot\|_n$ on $M_n(E)$, for each $n \geq 1$, satisfying the axioms

$$\begin{aligned} \|x \oplus y\|_{n+m} &= \max\{\|x\|_n, \|y\|_m\}\,, \\ \|\alpha x \beta\|_n &\leq \|\alpha\|\|x\|_n\|\beta\|\,, \end{aligned}$$

for all $x \in M_n(E)$, $y \in M_m(E)$ and $\alpha\,, \beta \in M_n(\mathbb{C})$.

Let E, F be operator spaces, $\phi\colon E \to F$ linear, bounded. Consider

$$\phi \otimes \mathrm{Id}_n\colon M_n(E) \to M_n(F), \quad n \in \mathbb{N}\,.$$

The map ϕ is called *completely bounded* (for short, c.b.) if

$$\|\phi\|_{\mathrm{cb}}\colon = \sup_{n\in\mathbb{N}} \|\phi \otimes \mathrm{Id}_n\| < \infty.$$

The map ϕ is a *complete isometry* if all ϕ_m are isometries, and a *complete isomorphism* if it is an isomorphism with $\|\phi\|_{cb}\,, \|\phi^{-1}\|_{cb} < \infty$.

Theorem (Ruan 1985):

If E is an abstract operator space, then there is a Hilbert space H, a concrete operator space $F \subseteq \mathcal{B}(H)$, and a complete isometry ϕ from E onto F. If E is separable as a normed space, then we can let $H = l^2$.

Let $\mathrm{CB}(E,F) := \{\phi : E \to F : \|\phi\|_{\mathrm{cb}} < \infty\}$.

If E is an operator space, then the dual $E^* = \mathcal{B}(E,\mathbb{C}) = \mathrm{CB}(E,\mathbb{C})$ endowed with matrix norms given by

$$M_n(E^*) := \mathrm{CB}(E, M_n(\mathbb{C}))\,, \quad n \geq 1$$

is again an operator space, called the *operator space dual* of E.

Examples of operator spaces:

1. C*-algebras are operator spaces.
2. The predual $\mathcal{M}_*$ of a von Neumann algebra $\mathcal{M}$ is an operator space, with norms inherited from the isometric embedding
$$M_n(\mathcal{M}_*) \subseteq M_n(\mathcal{M}^*) :\; = \mathrm{CB}(\mathcal{M}, M_n(\mathbb{C}))\,, \quad n \in \mathbb{N}.$$
3. Let $e_1, e_2, \dots$ be the standard unit vector basis in $l^2(\mathbb{N})$.

   (i) The *row Hilbert space* R is $l^2(\mathbb{N})$ as a Banach space, and for $n \in \mathbb{N}$,
$$\left\| \sum_{i=1}^{r} x_i \otimes e_i \right\|_{M_n(R)} :\; = \left\| \sum_{i=1}^{r} x_i x_i^* \right\|^{1/2},$$
   for all $r \in \mathbb{N}$ and $x_1, \dots, x_r \in M_n(\mathbb{C})$.

   (ii) The *column Hilbert space* C is $l^2(\mathbb{N})$ as a Banach space, and for $n \in \mathbb{N}$,
$$\left\| \sum_{i=1}^{r} x_i \otimes e_i \right\|_{M_n(C)} :\; = \left\| \sum_{i=1}^{r} x_i^* x_i \right\|^{1/2},$$
   for all $r \in \mathbb{N}$ and $x_1, \dots, x_r \in M_n(\mathbb{C})$.

The following simple computation shows that R and C are different operator spaces.

Let $x_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $x_2 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{C})$. Then

$$\|x_1 \otimes e_1 + x_2 \otimes e_2\|_{M_2(R)} = \|x_1 x_1^* + x_2 x_2^*\|^{1/2} = \left\| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\|^{1/2} = 1\,,$$

while

$$\|x_1 \otimes e_1 + x_2 \otimes e_2\|_{M_2(C)} = \|x_1^* x_1 + x_2^* x_2\|^{1/2} = \left\| \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \right\|^{1/2} = \sqrt{2}\,.$$

Fact: $R^* \cong C$ and $C^* \cong R$ (complete isometries).

Theorem (Pisier):

There exists an operator space, called OH, such that

(1) OH is isometric to $l^2(\mathbb{N})$ (as a Banach space)

(2) The canonical identification between OH and $\overline{OH^*}$ (corresponding to the canonical identification between $l^2(\mathbb{N})$ and $\overline{l^2(\mathbb{N})^*}$) is a complete isometry.

Moreover, OH is the unique operator space (up to complete isometry) satisfying (1) and (2). For $n \in \mathbb{N}$, $r \in \mathbb{N}$ and $x_1, \ldots, x_r \in M_n(\mathbb{C})$,

$$\left\| \sum_{i=1}^{r} x_i \otimes e_i \right\|_{M_n(OH)} := \left\| \sum_{i=1}^{r} x_i \otimes \overline{x_i} \right\|_{M_n(\mathbb{C}) \otimes \overline{M_n(\mathbb{C})}}^{1/2}.$$

By the classical Khintchine inequalities, the Rademacher functions $r_n(t) := \mathrm{sgn}(\sin(2^n t\pi))$, $t \in [0,1]$, $n \geq 1$ span an isomorphic copy of $l^2(\mathbb{N})$ inside L^1. Moreover, (by using Gaussian random variables), the embedding of $L^2([0,1])$ into $L^1([0,1])$ is isometric. The Khintchine inequalities for Rademachers do not give the isometric result, as the best constant for the lower bound estimate is $1/\sqrt{2}$ (Szarek, 1974).

Problem: Does OH admit a completely isometric embedding into the predual of a von Neumann algebra?

Some results:

(Pisier 2004): OH does not embed (cb-isomorphically) into the predual of any semifinite von Neumann algebra.

(Junge 2005): OH admits a cb-embedding into the predual of a type III von Neumann algebra. A year later, Junge showed that OH cb-embeds into the predual of the hyperfinite type III_1-factor, with cb-isomorphism constant ≈ 200.

(Haagerup-M. 2007): OH admits a cb-embedding into the predual of the hyperfinite type III_1-factor, with cb-isomorphism constant $\leq \sqrt{2}$.

Our proof is based on an improvement of a result of Junge concerning Khintchine-type inequalities for subspaces of $R \oplus C$, for which we obtain a sharp lower bound of $1/\sqrt{2}$.

The fascinating question whether the analogue of the classical result that $L^2[0,1]$ embeds isometrically into $L^1[0,1]$ holds in the noncommutative setting remains open.

Idea of the proof of the OH **embedding**:

Let H be a closed subspace of $R \oplus C$, and associate to it $A \in \mathcal{B}(H)$, $0 \leq A \leq I_H$ so that the operator space structure on H is given by

$$\left\| \sum_{i=1}^{r} x_i \otimes \xi_i \right\|_{M_n(H)} =$$

$$= \max \left\{ \left\| \sum_{i,j=1}^{r} \langle (I_H - A)\xi_i, \xi_j \rangle_H x_i x_j^* \right\|^{\frac{1}{2}}, \left\| \sum_{i,j=1}^{r} \langle A\xi_i, \xi_j \rangle_H x_i^* x_j \right\|^{\frac{1}{2}} \right\}$$

for all $n, r \in \mathbb{N}$, $x_i \in M_n(\mathbb{C})$, $\xi_i \in H$.

Let $\mathcal{A}$ be the CAR-algebra built on H. Recall that $\mathcal{A}$ is a unital C^*-algebra (unique up to $*$-isomorphism) with the property that there exists a linear map

$$H \ni f \mapsto a(f) \in \mathcal{A}$$

whose range generates $\mathcal{A}$, satisfying for all $f, g \in H$ the anticommutation relations

$$\begin{aligned} a(f)a(g)^* + a(g)^*a(f) &= \langle f, g \rangle_H I \\ a(f)a(g) + a(g)a(f) &= 0 \,. \end{aligned}$$

Let ω_A be the gauge-invariant quasi-free state on $\mathcal{A}$ corresponding to A, that is, for all $n, m \in \mathbb{N}$,

$$\omega_A(a(f_n)^* \ldots a(f_1)^* a(g_1) \ldots a(g_m)) = \delta_{nm} \mathrm{det}(\langle Ag_i, f_j \rangle_H, i, j),$$

for all $f_1, \ldots, f_n, g_1, \ldots, g_m \in H$.

Use Riesz representation theorem to define a map $E_A : \mathcal{A} \to H$ by

$$\langle f, E_A(b) \rangle_H = \omega_A(a(f)b^* + b^* a(f)) \,, \quad f \in H \,,$$

for all $b \in \mathcal{A}$.

Let π_A be the unital $*$-representation from the GNS construction for $(\mathcal{A}\,, \omega_A)$. Then E_A extends to a bounded linear operator on

$$\mathcal{M} := \overline{\pi_A(\mathcal{A})}^{\text{s.o.t}} \,.$$

By a result of Powers-Størmer (1970), $\mathcal{M}$ is a hyperfinite factor.

Theorem (Haagerup-M. 2007):

The map $E_A : \mathcal{A} \to H$ yields a complete isomorphism

$$H \cong \mathcal{A}/\mathrm{Ker}(E_A)$$

with cb-isomorphism constant $\leq \sqrt{2}$. Furthermore, the dual map E_A^* is a complete isomorphism of H^* onto a subspace of $\mathcal{M}_*$.

Now, let $\mathcal{R}_\infty$ denote the hyperfinite type III_1-factor. Then

$$\mathcal{M} \bar{\otimes} \mathcal{R}_\infty \cong \mathcal{R}_\infty \,,$$

and hence $\mathcal{M}_*$ cb-embeds into $(\mathcal{R}_\infty)_*$. Therefore by the above Theorem, H^* cb-embeds into $(\mathcal{R}_\infty)_*$, with cb-isomorphism constant $\leq \sqrt{2}$.

Therefore any quotient (and further, any subspace of a quotient) of $(R \oplus C)^*$ cb-embeds into $(\mathcal{R}_\infty)_*$, with cb-isomorphism constant $\leq \sqrt{2}$. Pisier (2004) showed that OH is a subspace of a quotient of $R \oplus C$. Since OH is self-dual, OH is also a sub-quotient of $(R \oplus C)^*$. Hence OH cb-embeds into $(\mathcal{R}_\infty)_*$, with cb-isomorphism constant $\leq \sqrt{2}$.