Property (T) for quantum groups from the dual point of view

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Rough outline

- Property (T) for groups
- Compact and discrete quantum groups
- Property (T) for quantum groups
- Different characterizations of property (T)

Property (T) for groups

We start with the following data:

- Γ discrete, countable group,
- $\pi\colon \Gamma \to U(H)$ a representation,
- $E \subseteq \Gamma$ finite, $\delta > 0$ and $\xi \in (H)_1$.

Then

- ξ is called (E, δ)-invariant if $\|\pi(\gamma)\xi - \xi\| < \delta$ for $\gamma \in E$.
- π is said to have almost invariant vectors if such ξ exists for all E and δ.
- Γ is said to have Kazhdan's property (T) if every π that has almost invariant vectors actually has a non-zero invariant vector.

Property (T) is of importance in many fields — in particular operator algebras.

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There are many ways to describe property (T). Here are two:

Theorem (Delorme-Guichardet, de la Harpe-Valette)

The following conditions are equivalent

- Γ *has property* (T)*.*
- *Any sequence* $\varphi_n\colon \Gamma \to \mathbb{C}$ *of normalized, positive definite functions converging pointwise to 1 has to converge uniformly.*
- *For any* $\pi\colon \Gamma \to U(H)$ *we have* $H^1(\Gamma, H) = 0$*.*

i.e. $\varphi_n(e) = 1$ and $\varphi_n(x^*x) \geq 0$ for $x \in \mathbb{C}\Gamma$.

First group cohomology of Γ=first Hochschild cohomology $H^1(\mathbb{C}\Gamma, {}_\pi H_\varepsilon)$.

The aim of the talk is to discuss a similar result for quantum groups. We first introduce these objects:

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Compact quantum groups

Definition (Woronowicz)

A compact quantum group $\mathbb{G}$ consists of a unital, separable C^*-algebra $C(\mathbb{G})$ together with a unital $*$-homomorphism $\Delta\colon C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ satisfying

- $(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta$,
- *a certain density condition.*

Example: $C(G)$ with G compact group and $\Delta(f)(s,t) = f(st)$.
Example: $C^*_{\mathrm{red}}(\Gamma)$ with Γ discrete and $\Delta\gamma = \gamma \otimes \gamma$.

Theorem (Woronowicz)

Any compact quantum group with $C(\mathbb{G})$ abelian is of the form $C(G)$ for an honest compact group G.

As groups have representations, quantum groups have corepresentations:

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As groups have representations, quantum groups have corepresentations:

A corepresentation of $\mathbb{G}$ on a finite dimensional Hilbert space H is a unitary $u \in C(\mathbb{G}) \otimes B(H)$ satisfying $(\Delta \otimes \mathrm{id})u = u_{(13)}u_{(23)}$. This leads to:

- Intertwiners (morphisms) of corepresentations.
- A notion of irreducibility.
- A notion of direct sums and tensor products of corepresentations.
- Complete decomposability into irreducibles.

We now choose a complete set $(u^\alpha)_{\alpha \in I}$ of representatives for the set of equivalence classes of irreducible corepresentations $\mathrm{Irred}(\mathbb{G})$:

$$u^\alpha \in C(\mathbb{G}) \otimes B(H_\alpha) \simeq \mathbb{M}_{n_\alpha}(C(\mathbb{G}))$$

Theorem (Woronowicz)

$\mathrm{Pol}(\mathbb{G}) := \mathrm{span}_{\mathbb{C}}\{u^\alpha_{ij} \mid \alpha \in I, 1 \leq i,j \leq n_\alpha\}$ is a Hopf $*$-algebra.

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A compact quantum group comes with a distinguished state $h\colon C(\mathbb{G}) \to \mathbb{C}$ called the Haar state. This yields

- A GNS space $L^2(\mathbb{G}) := L^2(C(\mathbb{G}), h)$.
- A GNS representation $\lambda\colon C(\mathbb{G}) \to B(L^2(\mathbb{G}))$ whose image $\lambda(C(\mathbb{G})) =: C(\mathbb{G}_{\mathrm{red}})$ is again a compact quantum group.
- A von Neumann algebra $L^\infty(\mathbb{G}) := \lambda(C(\mathbb{G}))''$ which becomes a von Neumann algebraic quantum group.

Summing up we have:

$$\textit{General}: \mathbb{C} \xleftarrow{\ \varepsilon\ } \mathrm{Pol}(\mathbb{G}) \overset{S}{\circlearrowleft} \hookrightarrow C(\mathbb{G}_{\mathrm{red}}) \hookrightarrow L^\infty(\mathbb{G})$$

$$\textit{Example}: \mathbb{C} \xleftarrow{\ \gamma \mapsto 1\ } \mathbb{C}\Gamma \overset{\gamma \mapsto \gamma^{-1}}{\circlearrowleft} \hookrightarrow C^*_{\mathrm{red}}(\Gamma) \hookrightarrow \mathscr{L}(\Gamma)$$

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A compact quantum group $\mathbb{G}$ has a discrete dual quantum group $\hat{\mathbb{G}}$. It comes with 3 algebras

$$\begin{array}{ccccc} \mathrm{pol}(\hat{\mathbb{G}}) & \hookrightarrow & c_0(\hat{\mathbb{G}}) & \hookrightarrow & \ell^\infty(\hat{\mathbb{G}}) \\ \Big\| \text{def} & & \Big\| \text{def} & & \Big\| \text{def} \\ \bigoplus_{\alpha\in I}^{\mathrm{alg}} B(H_\alpha) & & \bigoplus_{\alpha\in I}^{c_0} B(H_\alpha) & & \bigoplus_{\alpha\in I}^{vNa} B(H_\alpha) \end{array}$$

and a comultiplication $\hat{\Delta}\colon \ell^\infty(\hat{\mathbb{G}}) \to \ell^\infty(\hat{\mathbb{G}})\bar{\otimes}\ell^\infty(\hat{\mathbb{G}})$.

Example: For $C^*_{\mathrm{red}}(\Gamma)$ we get the following

$$c_f(\Gamma) \subseteq c_0(\Gamma) \subseteq \ell^\infty(\Gamma).$$

Definition

A corepresentation of $\hat{\mathbb{G}}$ on a Hilbert space H is a unitary $V \in \ell^\infty(\hat{\mathbb{G}})\bar{\otimes}B(H)$ such that $(\hat{\Delta}\otimes \mathrm{id})V = V_{(13)}V_{(23)}$

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- $E \subseteq \mathrm{Irred}(\mathbb{G})$ and $\delta > 0$.

Definition (Fima)

- *$\xi \in H$ is called invariant if $V_\alpha(\eta \otimes \xi) = \eta \otimes \xi$ for all $\alpha \in I$ and $\eta \in H_\alpha$.*
- *$\xi \in (H)_1$ is called (E, δ)-invariant if $\|V_\alpha(\eta \otimes \xi) - \eta \otimes \xi\| < \delta$ for each $\alpha \in E$ and each $\eta \in (H_\alpha)_1$.*
- *V has almost invariant vectors if such ξ exists for each (E, δ).*
- *$\hat{\mathbb{G}}$ has property (T) if each corepresentation that has almost invariant vectors, has a non-zero invariant vector.*

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From discrete to compact

Notice that

$$\text{Corepresentations of } \hat{\mathbb{G}} \xleftrightarrow{1:1} *\text{-representations of } \mathrm{Pol}(\mathbb{G}).$$

We can now mimic the definition from the dual point of view:

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$\hat{\mathbb{G}}$ has property (T) iff every $*$-representation $\pi\colon \mathrm{Pol}(\mathbb{G}) \to B(H)$ which has almost invariant vectors has a non-zero invariant vector.

Recall: Γ has property (T) iff every sequence of normalized, positive definite functions $\varphi_n\colon \Gamma \to \mathbb{C}$ converging pointwise to 1, converges uniformly. A positive definite, normalized function on Γ corresponds to a state on $C^*_{\max}(\Gamma)$. The relation

$$\|a\|_{\max} = \sup\{\|\pi(a)\| \mid \pi\colon \mathrm{Pol}(\mathbb{G}) \to B(H) \text{ a } *\text{-rep}\}$$

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Theorem (Fima, K.)

The quantum group $\hat{\mathbb{G}}$ has property (T) iff every sequence of states $\varphi_n\colon C(\mathbb{G}_{\max}) \to \mathbb{C}$ converging pointwise to the counit ε, converges in the uniform norm.

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Recall: Γ has property (T) iff any 1-cocycle is inner. The description for quantum groups is a bit more involved:

Definition

Let $\pi\colon \mathrm{Pol}(\mathbb{G}) \to B(H)$ be a $*$-representation and consider a Hochschild 1-cocycle $c\colon \mathrm{Pol}(\mathbb{G}) \to {}_{\pi}H_{\varepsilon}$. Then c is called closable if the following holds: if $x_n \in \ker(\varepsilon) \subseteq \mathrm{Pol}(\mathbb{G})$ is a sequence such that $\|x_n\|_{\gamma} \to 0$ and $c(x_n) \to \eta$ then $\eta = 0$, where

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Spectral characterization

Let $\mathbb{G}$ be a matrix quantum group and let $E \subset \mathrm{Irred}(\mathbb{G})$ be a generating subset for $\mathrm{Irred}(\mathbb{G})$. Then we define

$$x = \sum_{\alpha \in E} \sum_{i,j=1}^{n_\alpha} (u_{ij}^\alpha - \varepsilon(u_{ij}^\alpha)1)^*(u_{ij}^\alpha - \varepsilon(u_{ij}^\alpha)1) \in \mathrm{Pol}(\mathbb{G}).$$

This element detects property (T):

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