28.1.20
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Aud 10
Master Cla

G group $\rightsquigarrow$ dual of G?
$\hat{G} = \{\omega : G \to \mathbb{T} \; : \; \text{cts gp-homo}\}$

Thm (Pontryjagin ...)
If G is l-cpct, abln, then $\hat{\hat{G}} \cong G$

G simple $\rightsquigarrow$ $\hat{G} = \{*\}$ all info lost

Def (Woronowicz)
A compact quantum group (q-gp) is a pair (A, Δ) where A is a unital, sep C^*-alg and
$\Delta : A \to A \otimes_{\min} A$ unital $*$-homo
(i) $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$
(ii) $\overline{\text{span}}\{\Delta(a)(b \otimes 1) : a, b \in A\} = A \otimes A$
and analog. $[\Delta(A)(1 \otimes A)] = A \otimes A$

Ex: G cpct gp $\rightsquigarrow$ $A = C(G)$, $A \otimes A \cong C(G \times G)$
$\Delta : A \to A \otimes A$, $\Delta f(s,t) = f(st)$

Thm (Woronowicz)
If $\mathbb{G}$ is a cpct q-gp (CQG) with A abelian, then there ex a unique cpct gp (G, m) s.t.
$A = C(G)$, $\Delta = C(m)$

Ex: Γ discrete (ctbl) gp $\rightsquigarrow$ $A = C^*_r(\Gamma)$
$\Delta \gamma = \gamma \otimes \gamma$
$C^*(\Gamma)$ also works

Thm (Woronowicz)

For a CQG $\mathbb{G}$ $\exists!$ state $h : A \to \mathbb{C}$ s.t.

$(h \otimes id)\Delta a = h(a) 1_A = (id \otimes h)\Delta a$

Ex: $A = C(G) \rightsquigarrow h(f) = \int f \, d\mu$ ← Haar prob. msr

Ex: $A = C_r^* \Gamma \rightsquigarrow h(x) = \langle x \delta_e | \delta_e \rangle$

$\uparrow$
$C^* \Gamma$

To h is associated a GNS-repr

$L^2 \mathbb{G} = L^2(A, h)$, $\lambda : A \to B(L^2 \mathbb{G})$

$\lambda(A) =: A_{red} = C(\mathbb{G}_r)$

$M = \lambda(A)'' =: L^\infty \mathbb{G}$

Ex: $A = C(G)$, $L^2 \mathbb{G} = L^2(G, \mu)$

λ = left regular repr, $L^\infty \mathbb{G} = L^\infty(G, \mu)$

Ex: $A = C_r^* \Gamma \rightsquigarrow L^2(C_r^* \Gamma, h) \cong \ell^2 \Gamma$

Thm (Woronowicz, Baaj, Skandalis)

$\Lambda : A \to L(A, h)$ $a \mapsto \Lambda a$ from GNS-constr.

The relation $W^* \Lambda(a) \otimes \Lambda(b) = (\Lambda \otimes \Lambda) \Delta b (a \otimes 1)$

defines a unitary $W \in B(L^2 \mathbb{G} \otimes L^2 \mathbb{G})$ s.t.

$(\lambda \otimes \lambda)\Delta a = W^* (1 \otimes \lambda(a)) W$ $(\dagger)$

And

$A_r = C^* \{ (id \otimes \omega) W : \omega \in B(H)_* \}$

Rmk: The relation $(\dagger)$ extends Δ to a map $\Delta : L^\infty\mathbb{G} \to L^\infty\mathbb{G} \otimes L^\infty\mathbb{G}$

Def (Kustermans - Vaes)
A locally cpct q-gp is a pair (M, Δ) with M v.N. alg, $\Delta : M \to M \bar{\otimes} M$ an unital, coassoc, normal $*$-homo
+ two* weights $\varphi, \psi : M_+ \to [0, \infty]$ s.t. * normal semifinite,

$$(id \otimes \varphi)\Delta a = \varphi(a) \cdot 1$$
$$(\psi \otimes id)\Delta a = \psi(a) \cdot 1 \qquad \forall a \geq 0$$

In cpct case, existence of Haar state is theorem, for locally cpct case it is axiom (unfortunately, try to find "better" axioms)

Ex: $\mathbb{G}$ CQG $M = L^\infty\mathbb{G}$ with extension of Δ and $\varphi = \psi = h$

Ex: G loc. cpct gp, $M = L^\infty(G, \mu_e)$
μ_e - left Haar msr, $\Delta f(s,t) = f(st)$
$\varphi = \int - d\mu_e$, $\psi = \int - d\mu_r$

$H = L^2(M, \varphi)$, $M \hookrightarrow B(H)$, $\Lambda : M \hookrightarrow H$

Thm (Kustermans, Vaes)
The relation $W^* \Lambda(a) \otimes \Lambda(b) = (\Lambda \otimes \Lambda)\Delta b (a \otimes 1)$ gives a unitary W s.t.

$\Delta a = W^*(1\otimes a)W$ and

$M = \text{v.N.a}\,\{(\mathrm{id}\otimes\omega)W : \omega\in B(H)_*\}$

moreover $\hat M = \text{v.Na}\,\{(\omega\otimes\mathrm{id})W : \omega\in B(H)_*\}$

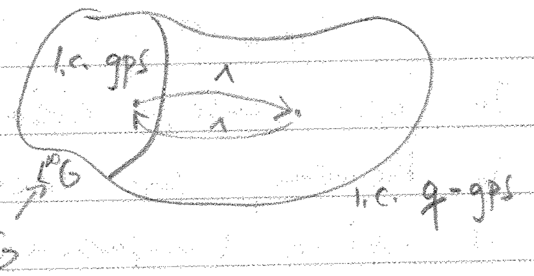
becomes a l.c. q.gp with comult

$\hat\Delta a = \hat W^*(1\otimes a)\hat W$, $\hat W = \Sigma W\Sigma$,

$\Sigma : H\otimes H$ flip

and $(\hat{\hat M}, \hat{\hat\Delta}) \cong (M,\Delta)$

$(\hat M,\hat\Delta)$ – the dual l.cpct q-gp



Eg: $A = C^*_r\Gamma \subset B(\ell^2\Gamma)$ $\quad M = L\Gamma$

$\widehat{L\Gamma} \subset B(\ell^2\Gamma)$

$\widehat{L\Gamma} = \text{vNa}\,\{(\mathrm{id}\otimes\omega)W : \omega\in B(H)_*\}$

$W^*(\Lambda\gamma\otimes\Lambda\mu) = (\Lambda\otimes\Lambda)(\mu\otimes\mu)\gamma\otimes 1 = \Lambda(\mu\gamma\otimes\gamma)$

$W(\Lambda\gamma\otimes\Lambda\mu) = \Lambda\otimes\Lambda\ \mu^{-1}\gamma\otimes\gamma \qquad \delta_\gamma = \Lambda\gamma$

$\omega_{\gamma,\mu}(T) = \langle T\delta_\gamma \mid \delta_\mu\rangle$

$(\omega_{\gamma,\mu}\otimes\mathrm{id})W|_{\delta_x} = (\omega_{\gamma,\mu}\circ\Lambda^{-1})[W[\delta_e\otimes\delta_x]]$

$= (\omega_{\gamma,\mu}\circ\Lambda^{-1})[\delta_{x^{-1}}\otimes\delta_x] = \omega_{\gamma,\mu}(\lambda_{x^{-1}})|_{\delta_x}$

$= \langle\delta_{x^{-1}\gamma}\mid\delta_\mu\rangle\,\delta_x$

$= \delta_{\gamma\mu^{-1}}(x)\,\delta_x$ multiplication op

so $(\omega_{\gamma,\mu}\otimes\mathrm{id})W = m(\delta_{\gamma\mu^{-1}}) \curvearrowright \ell^2\Gamma$

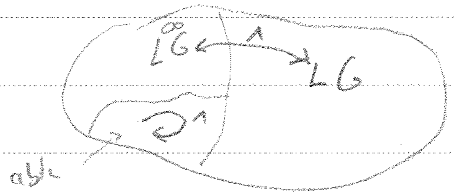
$\leadsto \; (\omega_{\gamma,e} \otimes id) W = m(\delta_\gamma)$

$\leadsto \; m(\ell^\infty \Gamma) \subset L\hat{\Gamma}$

$\omega \in B(H)_*$ has form $\omega(T) = \sum_k \langle T\xi_k, \eta_k \rangle$

where $\sum_k \|\xi_k\|^2 + \|\eta_k\|^2 < \infty$

$\Rightarrow \; L\hat{\Gamma} = m(\ell^\infty \Gamma)$



$$\begin{tikzcd} L^\infty G \arrow[r, bend left, "\wedge"] & LG \arrow[l, bend left, "\wedge"] \end{tikzcd}$$

Thm: (KV)

If G l.cpct + abelian, then $\widehat{L^\infty G} = L^\infty \hat{G}$

Ex: $\widehat{L\mathbb{Z}} = \ell^\infty \mathbb{Z} \leadsto \widehat{\ell^\infty \mathbb{Z}} = L\mathbb{Z} \underset{Ad_F}{\cong} L^\infty(S^1, \mu) = L^\infty(\hat{\mathbb{Z}})$

can recover group (Nakayama, Tatsuuma ?)

LG

$\leadsto \{ T \in LG : T \neq 0, \; \hat{\Delta}(T) = T \otimes T \}$

„group like elements"

$T = L_g$, SOT = top on G