Rieffel Deformation of Homogeneous Spaces

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Introduction

Rieffel Deformation:

- B - C^*-algebras
- $\mathbb{G} = (A, \Delta)$ - quantum groups
- $G \curvearrowright X$ - actions on spaces
- $G \curvearrowright B$ - actions on C^* - algebras
  - $\alpha : G \to \mathrm{Aut}(B) : \; G \ni g \mapsto \alpha_g(b) \in B$ - continuous $\forall b \in B$

Reiffel Deformation of Homogeneous Space X

Homogeneous spaces - classical language

Homogeneous spaces:

- X - locally compact space
- $G \curvearrowright X$ - continuous, transitive action
  - $\forall x_1, x_2 \in X \ \exists g \in G : \ g x_1 = x_2$
- Example: $X = G / G_0$
- $x_0 \in X$, $G_{x_0} \subset G$ - stabilizer of x_0
- $X \cong G / G_0$ - the space of left cosets

Homogeneous spaces - the language of algebras

C^* - **algebraic description:**

- $G \rightsquigarrow \mathbb{G} = (C_0(G), \Delta)$
  - $\Delta : C_0(G) \to M(C_0(G) \otimes C_0(G))$
  - $\Delta(f)(g_1, g_2) = f(g_1 g_2), \ \forall f \in C_0(G), \ \forall (g_1, g_2) \in G \times G$
- $G \curvearrowright X \rightsquigarrow \mathbb{X} = (C_0(X), \Delta_X)$
  - $\Delta_X : C_0(X) \to M(C_0(G) \otimes C_0(X))$
  - $\Delta_X(f)(g, x) = f(gx), \ \forall f \in C_0(X), \ \forall (g, x) \in G \times X$
- $(\Delta \otimes \iota)\Delta_X = (\iota \otimes \Delta_X)\Delta_X$
- Continuity $\rightsquigarrow [\Delta_X(C_0(X))(C_0(G) \otimes 1)] = C_0(G) \otimes C_0(X)$
- Homogeneity of X $\rightsquigarrow$ ergodicity, minimality of Δ_X

Quotient space G / G_0:

- $\mathcal{L}(G) = \{L_g \,|g \in G\}''$ - group von Neumann algebra
- $G_0 \subset G$ - closed subgroup
- $\mathcal{L}(G_0) \subset \mathcal{L}(G)$
- $\mathcal{R}(G) = \{R_g \,|g \in G\}'' = \mathcal{L}(G)'$
- $L^\infty(G / G_0) = \{x \in L^\infty(G) : xy = yx \;\forall\; y \in \mathcal{R}(G_0)\}$
- $C_0(G / G_0) \subset L^\infty(G / G_0)$ - strongly dense C^*-subalgebra
- $\Delta_{G / G_0} : C_0(G / G_0) \to M(C_0(G) \otimes C_0(G / G_0))$

Vaes's quotient $\mathbb{G}/\mathbb{G}_0$:

- $\mathbb{G} = (\mathrm{C}_0(\mathbb{G}), \Delta)$ - locally compact, regular quantum group
- $\mathcal{L}(\mathbb{G})$ - von Neumann quantum group algebra
- $\mathbb{G}_0$ - closed quantum subgroup: $\mathcal{L}(\mathbb{G}_0) \subset \mathcal{L}(\mathbb{G})$
- $\mathcal{R}(\mathbb{G}) = \mathcal{L}(\mathbb{G})', \ \mathcal{R}(\mathbb{G}_0) \subset \mathcal{R}(\mathbb{G})$
- $\mathrm{L}^\infty(\mathbb{G}/\mathbb{G}_0) = \{x \in \mathrm{L}^\infty(\mathbb{G}) : xy = yx \ \forall y \in \mathcal{R}(\mathbb{G}_0)\}$
- $\Delta_{\mathbb{G}/\mathbb{G}_0} : \mathrm{L}^\infty(\mathbb{G}/\mathbb{G}_0) \to \mathrm{L}^\infty(\mathbb{G}) \otimes \mathrm{L}^\infty(\mathbb{G}/\mathbb{G}_0)$

Vaes's quotient $\mathbb{G}/\mathbb{G}_0$

Theorem

There exists a unique strongly dense C^**-subalgebra* $C_0(\mathbb{G}/\mathbb{G}_0) \subset L^\infty(\mathbb{G}/\mathbb{G}_0)$ *such that*

- $\Delta|_{C_0(\mathbb{G}/\mathbb{G}_0)}$ *- continuous coaction of* $\mathbb{G}$ *on* $C_0(\mathbb{G}/\mathbb{G}_0)$
- $\Delta(L^\infty(\mathbb{G}/\mathbb{G}_0)) \subset M(\mathcal{K} \otimes C_0(\mathbb{G}/\mathbb{G}_0))$
- $\Delta : L^\infty(\mathbb{G}/\mathbb{G}_0) \to M(\mathcal{K} \otimes C_0(\mathbb{G}/\mathbb{G}_0))$ *- strict*

Strictness:

- $L^\infty(\mathbb{G}/\mathbb{G}_0) \ni x_i \underset{i\to\infty}{\longrightarrow} x \in L^\infty(\mathbb{G}/\mathbb{G}_0)$ - weakly convergent, uniformly bounded net
- $\Delta(x_i)y \underset{i\to\infty}{\longrightarrow} \Delta(x)y$ in norm, for any $y \in \mathcal{K} \otimes C_0(\mathbb{G}/\mathbb{G}_0)$

Rieffel Deformation of Homogeneous Spaces

Deformation data:

- X - G - homogeneous space
- $\Gamma \subset G$ - abelian, closed subgroup
- $\hat{\Gamma}$ - Pontryagin dual
- Ψ - 2 - cocycle on $\hat{\Gamma}$:
  $\Psi(\hat{\gamma}_1 + \hat{\gamma}_2, \hat{\gamma}_3)\Psi(\hat{\gamma}_1, \hat{\gamma}_2) = \Psi(\hat{\gamma}_1, \hat{\gamma}_2 + \hat{\gamma}_3)\Psi(\hat{\gamma}_2, \hat{\gamma}_3)$
- $\rho : \Gamma \to \mathrm{Aut}(C_0(X)) : \ \rho_\gamma(f)(x) = f(\gamma^{-1}x)$

Rieffel Deformation of Homogeneous Spaces

Deformation procedure of X:

- $B = \Gamma_\rho \ltimes C_0(X) = [C^*(\Gamma)\, C_0(X)]$
- $\hat{\rho} : \hat{\Gamma} \to \mathrm{Aut}(B)$ - the dual action
- $f \in C_0(X) : \ \hat{\rho}_{\hat{\gamma}}(f) = f$
- $\lambda_\gamma \in M(C^*(\Gamma)) : \ \hat{\rho}_{\hat{\gamma}}(\lambda_\gamma) = \langle \hat{\gamma}, \gamma \rangle \lambda_\gamma$
- $C_0(X) \subset M(B)^{\hat{\rho}}$ - Landstad invariants
- $(\Psi, \hat{\rho}) \rightsquigarrow \hat{\rho}^\Psi : \hat{\Gamma} \to \mathrm{Aut}(B)$ - the twisted dual action
- $f \in C_0(X) : \ \hat{\rho}^\Psi_{\hat{\gamma}}(f) \neq f$
- $\lambda_\gamma \in M(C^*(\Gamma)) : \ \hat{\rho}^\Psi_{\hat{\gamma}}(\lambda_\gamma) = \langle \hat{\gamma}, \gamma \rangle \lambda_\gamma$
- $C_0(X)^\Psi \subset M(B)^{\hat{\rho}^\Psi}$ - Landstad invariants of $\hat{\rho}^\Psi$

Rieffel Deformation of G

Deformation procedure of G**:**

- $\rho' : \Gamma^2 \to \mathrm{Aut}(\mathrm{C}_0(\mathrm{G}))$: $\rho'_{\gamma_1,\gamma_2}(f)(g) = f(\gamma_1^{-1} g \gamma_2)$
- $\mathrm{B}' = \Gamma^2_{\rho'} \ltimes \mathrm{C}_0(\mathrm{G}) = [\mathrm{C}^*(\Gamma^2)\,\mathrm{C}_0(\mathrm{G})]$
- $\hat{\rho}'^{\Psi} : \hat{\Gamma}^2 \to \mathrm{Aut}(\mathrm{B}')$ - twisted dual action
- $\mathrm{C}_0(\mathrm{G})^{\Psi} \subset \mathrm{M}(\mathrm{B}')^{\hat{\rho}'^{\Psi}}$ - Landstad invariants

Comultiplication on $\mathrm{C}_0(\mathrm{G})^{\Psi}$**:**

- Covariant properties: $\Delta(\rho'_{\gamma_1,\gamma_2}(f)) = (\rho'_{\gamma_1,e} \otimes \rho'_{e,\gamma_2})\Delta(f)$
- Universal properties of B': $\Delta \rightsquigarrow \Delta_{\mathrm{B}'} : \mathrm{B}' \to \mathrm{M}(\mathrm{B}' \otimes \mathrm{B}')$
  - $f \in \mathrm{C}_0(\mathrm{G})$: $\Delta_{\mathrm{B}'}(f) = \Delta(f)$
  - $\lambda_{\gamma_1,\gamma_2} \in \mathrm{M}(\mathrm{C}^*(\Gamma^2))$: $\Delta_{B'}(\lambda_{\gamma_1,\gamma_2}) = \lambda_{\gamma_1,e} \otimes \lambda_{e,\gamma_2}$
- $\Delta^{\Psi} = \mathrm{Ad}_{\Psi}\, \Delta_{\mathrm{B}'}$
- $\Delta^{\Psi}|_{\mathrm{C}_0(\mathrm{G})^{\Psi}} : \mathrm{C}_0(\mathrm{G})^{\Psi} \to \mathrm{M}(\mathrm{C}_0(\mathrm{G})^{\Psi} \otimes \mathrm{C}_0(\mathrm{G})^{\Psi})$
- $\mathbb{G}^{\Psi} = (\mathrm{C}_0(\mathrm{G})^{\Psi}, \Delta^{\Psi})$

Coaction of $\mathbb{G}^{\Psi}$ on $\mathrm{C}_0(\mathrm{X})^{\Psi}$

Deformation of Δ_{X}

- Covariance: $\Delta_{\mathrm{X}}(\rho_\gamma(f)) = (\rho'_{\gamma,e} \otimes \iota)\Delta_{\mathrm{X}}(f)$
- Universality of $\mathrm{B} = \Gamma_\rho \ltimes \mathrm{C}_0(X)$: $\Delta_{\mathrm{B}} : \mathrm{B} \to \mathrm{M}(\mathrm{B}' \otimes \mathrm{B})$
  - $f \in \mathrm{C}_0(\mathrm{X}) : \quad \Delta_{\mathrm{B}}(f) = \Delta_{\mathrm{X}}(f)$
  - $\lambda_\gamma \in \mathrm{M}(\mathrm{C}^*(\Gamma)) : \ \Delta_B(\lambda_\gamma) = \lambda_{\gamma,e} \otimes 1$
- $\Delta_{\mathrm{X}}^{\Psi} = \mathrm{Ad}_{\Psi}\, \Delta_{\mathrm{B}}$
- $\Delta_{\mathrm{X}}^{\Psi} : \mathrm{C}_0(\mathrm{X})^{\Psi} \to \mathrm{M}(\mathrm{C}_0(\mathrm{G})^{\Psi} \otimes \mathrm{C}_0(\mathrm{X})^{\Psi})$ - coaction of $\mathbb{G}^{\Psi}$
- $\mathbb{X}^{\Psi} = (\mathrm{C}_0(\mathrm{X})^{\Psi}, \Delta_{\mathrm{X}}^{\Psi})$ - Rieffel deformation of $\mathbb{X}$

Rieffel deformation and Vaes quotient

Assumption $\Gamma \subset G_0 \subset G$

- $\mathbb{G}_0 = (C_0(G_0), \Delta_0)$ $\mathbb{G} = (C_0(G), \Delta)$
- Rieffel deformation: $\mathbb{G}_0^\Psi \subset \mathbb{G}^\Psi$ - quantum subgroup
- Vaes quotient: $\mathbb{G}^\Psi/\mathbb{G}_0^\Psi = (C_0(\mathbb{G}^\Psi/\mathbb{G}_0^\Psi), \Delta_{\mathbb{G}^\Psi/\mathbb{G}_0^\Psi})$
- $X = G\,/\,G_0$, $\mathbb{X} = (C_0(X), \Delta_X)$
- $\mathbb{X}^\Psi$ - Rieffel deformation of $\mathbb{X}$

Theorem

$\mathbb{X}^\Psi \cong \mathbb{G}^\Psi/\mathbb{G}_0^\Psi$

Rieffel deformation of a minimal action

Case $\Gamma \not\subseteq G_0 \subset G$

- $G \curvearrowright Y$ - minimal $\Leftrightarrow G\,y \subset Y$ - dense $\forall y \in Y$
- $\mathbb{Y} = (C_0(Y), \Delta_Y)$ - continuous coaction

Definition

- $\mathbb{Y}$ - continuous coaction of $\mathbb{G}$
- $\mathbb{Y}$ is minimal if
  - $\forall$ continuous coaction $\mathbb{Z}$ of $\mathbb{G}$
  - $\forall\ \pi \in \mathrm{Mor}(C_0(\mathbb{Y}), C_0(\mathbb{Z}))$: $\ (\iota \otimes \pi) \circ \Delta_{\mathbb{Y}} = \Delta_{\mathbb{Z}} \circ \pi$
  - We have $\ker \pi = \{0\}$

Theorem

$\mathbb{Y}$ - *minimal* $\Rightarrow \mathbb{Y}^{\Psi}$ - *minimal*

Corollary

X - homogeneous space $\Rightarrow \mathbb{X}^{\Psi}$ - minimal coaction of $\mathbb{G}^{\Psi}$