Mixing masas in group factors

Paul Jolissaint, Université de Neuchâtel

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Introduction

J. Dixmier (1954) introduced 3 classes of maximal abelian $*$-subalgebras (masas) A in a type II_1 factor M in terms of the *normaliser* of A in M

$$\mathcal{N}(A) := \{u \in U(M) : uAu^* = A\} :$$

(i) A is *regular* if $\mathcal{N}(A)$ generates M (nowadays, such a masa is called a *Cartan* masa);
(ii) A is *semi-regular* if $\mathcal{N}(A)''$ is a proper subfactor of M;
(iii) A is *singular* if $\mathcal{N}(A) = U(A)$.

He gave examples of all three classes coming from pairs of groups $H < G$ where H is an infinite abelian group: if for each $g \in G \setminus H$, the set $\{hgh^{-1} : h \in H\}$ is infinite, then $L(H)$ is a masa in $L(G)$, and extra conditions on the pair $H < G$ ensure that one of the 3 properties above is satisfied.

Focus on the class of singular masas.

What about existence of singular masas in general II_1 factors?

Theorem 1 (S. Popa, 1983) *Let N be a separable type II_1 factor (i.e. N has separable predual) and let M be a subfactor of N such that $M' \cap N = \mathbb{C}1$. Then there is a singular masa A in N contained in M.*

In general, it is rather hard to check that $\mathcal{N}(A) = U(A)$.

G. Robertson, A. Sinclair and R. Smith, 2003 found an apparently stronger condition: *If $A \subset M$ is an abelian von Neumann algebra such that*

$$(*) \quad \|\mathbb{E}_A - \mathbb{E}_{uAu^*}\|_{\infty,2} \geq \|u - \mathbb{E}_A(u)\|_2 \quad \forall u \in U(M)$$

then A is a singular masa in M.
(Notation: $\|\Phi\|_{\infty,2} = \sup_{\|x\|\leq 1} \|\Phi(x)\|_2$.)

Geometric examples. (RSS, 2003) Let G be a cocompact lattice in a semisimple Lie group.

Then G is the fundamental group $\pi(\mathcal{M})$ of a suitable compact manifold $\mathcal{M}$.
If $T^r \subset \mathcal{M}$ is a flat r-torus, then its fundamental group $H = \mathbb{Z}^r$ embeds into G and:

Theorem 2 *The pair of von Neumann algebras $L(H) \subset L(G)$ satisfies $(*)$, hence $L(H)$ is a singular masa in $L(G)$.*

A (not necessarily abelian) von Neumann subalgebra $A \subset M$ that satisfies $(*)$ is called *strongly singular*.

Theorem 3 (A. Sinclair, R. Smith, S. White and A. Wiggins, 2007) *Let A be a masa in a separable type II_1 factor. Then TFAE:*

(1) *A is singular in M;*

(2) *A is strongly singular in M;*

(3) *A has the WAHP: for every finite set $F \subset M$, and every $\varepsilon > 0$, there is a unitary operator $u \in A$ such that*

$$\|\mathbb{E}_A(xuy) - \mathbb{E}_A(x)u\mathbb{E}_A(y)\|_2 < \varepsilon \quad \forall x, y \in F.$$

N.B. This is false for non-abelian subalgebras!

Weak mixing

Let $\alpha : \Gamma \to \mathrm{Aut}(M, \tau)$ be an action; it is called *weakly mixing* if, for every finite set $F \subset M$ and every $\varepsilon > 0$, one can find $g \in \Gamma$ such that

$$|\tau(a\alpha_g(b)) - \tau(a)\tau(b)| < \varepsilon \quad \forall a, b \in F.$$

The action is called *strongly mixing* if, for all $a, b \in M$, one has

$$\lim_{g \to \infty} |\tau(a\alpha_g(b) - \tau(a)\tau(b)| = 0.$$

Relative versions:

(1) (S. Popa, 2005) If $1 \in B \subset M$ and if Γ acts on M so that $\alpha_g(B) = B$ for every g, then α is called *weakly mixing relative to* B if, for every finite subset $F \subset M \ominus B (= \{x : \mathbb{E}_B(x) = 0\})$ and for every $\varepsilon > 0$, one can find $g \in \Gamma$ such that

$$\|\mathbb{E}_B(x\alpha_g(y))\|_2 < \varepsilon \quad \forall x, y \in F.$$

(2) Under the same general conditions, the action α is called *strongly mixing relative to* B if, for all $x, y \in M$, one has

$$\lim_{g \to \infty} \|\mathbb{E}_B(x\alpha_g(y))\|_2 = 0.$$

Consider then a triple $1 \in N_0 \subset N_1 \subset M$ and the action of $U(N_0)$ on M by conjugation ($\sigma_v(x) = vxv^*$).

Definition Say that N_0 is *weakly mixing* in M relative to N_1 if, for every finite set $F \subset M \ominus N_1$ and every $\varepsilon > 0$, one can find $v \in U(N_0)$ such that

$$\|\mathbb{E}_{N_0}(xvy)\|_2 (= \|\mathbb{E}_{N_0}(x\sigma_v(y))\|_2) < \varepsilon \quad \forall x, y \in F.$$

If $N_0 = N_1$, we say that N_0 is *weakly mixing* in M.
If moreover $N_0 = A$ is a masa, then this is equivalent to singularity, by Theorem 3.

Inspired by strong singularity:

Proposition 4 *Let $N_0 \subset N_1 \subset M$ be a triple as above. If N_0 is weakly mixing in M relative to N_1, then one has for every $u \in U(M)$:*

$$\|\mathbb{E}_{N_0} - \mathbb{E}_{uN_0u^*}\|_{\infty,2} \geq \|u - \mathbb{E}_{N_1}(u)\|_2.$$

In particular, $\mathcal{N}_M(N_0)'' \subset N_1$.

In the case of group algebras, one has a complete characterisation of weak mixing in terms of the groups:

Theorem 5 *Let $G_0 < G_1 < G$ be groups and let $N_0 = L(G_0) \subset N_1 = L(G_1) \subset M$ be the corresponding von Neumann algebras. Then N_0 is weakly mixing in M relative to N_1 iff for every finite subset $C \subset G \setminus G_1$, there exists $\gamma \in G_0$ such that $g\gamma h \notin G_0$ for all $g, h \in C$.*

The latter condition on the triple $G_0 < G_1 < G$ is called *condition (SS)* (it generalizes the case where $G_0 = G_1$ is abelian studied by J. and Y. Stalder, 2008).

Strong mixing

Starting point: Let F be Thompson's group presented by

$$F = \langle x_0, x_1, \ldots | x_i^{-1} x_n x_i = x_{n+1}, n \geq 1, 0 \leq i < n \rangle$$

and set $G_0 = \langle x_0 \rangle$. Then

Proposition 6 (J, 2005) *The pair $G_0 < F$ satisfies the following condition:*
(ST) For every finite set $C \subset F \setminus G_0$, there exists a finite set $E \subset G_0$ such that

$$g\gamma h \notin G_0 \quad \forall g, h \in C, \forall \gamma \in G_0 \setminus E.$$

In particular, $L(G_0)$ is a singular MASA in $L(F)$.

Consider again a triple $G_0 < G_1 < G$ and let us say that it satisfies *condition (ST)* if, for every finite subset $C \subset G \setminus G_1$, there exists a finite set $E \subset G_0$ such that $g\gamma h \notin G_0$ for all $g, h \in C$ and every $\gamma \in G_0 \setminus E$.

Definition We say that N_0 is *strongly mixing in M relative to* N_1 if

$$\lim_{n\to\infty} \|\mathbb{E}_{N_0}(xu_n y)\|_2 = 0$$

for all $x, y \in M \ominus N_1$ and all sequences $(u_n) \subset U(N_0)$ which converges to 0 for the weak operator topology. If $N_0 = N_1$, we say that N_0 is *strongly mixing in M*.

Proposition 4 can be improved in the strong mixing case:

Proposition 7 *Let $N_0 \subset N_1 \subset M$ be a triple as above. If N_0 is strongly mixing in M relative to N_1, then for every diffuse unital von Neumann subalgebra B of M, one has for every $u \in U(M)$:*

$$\|\mathbb{E}_B - \mathbb{E}_{uBu^*}\|_{\infty,2} \geq \|u - \mathbb{E}_{N_1}(u)\|_2.$$

In particular, $\mathcal{N}_M(B)'' \subset N_1$.

As for weak mixing, we have a complete characterization in the case of group algebras:

Theorem 8 *Let* $G_0 < G_1 < G$ *be groups and let* $N_0 = L(G_0) \subset N_1 = L(G_1) \subset M = L(G)$ *be the corresponding triple. TFAE:*

(1) *the action of* G_0 *by conjugation on* M *is strongly mixing relative to* N_1*;*

(2) *the triple* $G_0 < G_1 < G$ *satisfies condition (ST);*

(3) N_0 *is strongly mixing relative to* N_1 *in* M*;*

(4) *for every* $v \in G \setminus G_1$*, the diagram*

$$\begin{array}{ccc} vN_0v^* & \subset & M \\ \cup & & \cup \\ vN_0v^* \cap N_0 & \subset & N_0 \end{array}$$

is a commuting square and $\dim(vN_0v^* \cap N_0) < \infty$.

Examples

Example 1 Let $1 \in N_0 \subset N_1$ and Q be arbitrary finite von Neumann algebras with separable preduals. Then N_0 is strongly mixing in the free product algebra $M = N_1 * Q$.

Example 2 Let H be a *malnormal* subgroup of G, i.e. for every $g \in G \setminus H$, $gHg^{-1} \cap H = \{e\}$. Then $H < G$ satisfy condition (ST). Furthermore, if H is torsion free, then $H < G$ satisfies (ST) iff H is malnormal in G.

Example 3 Let $G = HNN(\Gamma, H, K, \phi)$ be an HNN-extention where H, K are subgroups of Γ and $\phi : H \to K$ is an isomorphism. Denote by t the stable letter such that $t^{-1}ht = \phi(h)$ for all $h \in H$ and by G_0 the subgroup generated by t. (G is generated by Γ and by t and it just has to satisfy relations of Γ and $t^{-1}ht = \phi(h)$.)

For every positive integer j, the domain of ϕ^j, denoted by $\mathrm{Dom}(\phi^j)$, is defined by $\mathrm{Dom}(\phi) = H$ for $j = 1$ and, by induction, $\mathrm{Dom}(\phi^j) = \phi^{-1}(\mathrm{Dom}(\phi^{j-1}) \cap K) \subset H$ for $j \geq 2$.

Theorem 9 (J, Y. Stalder, 2008) *Suppose that for all $j \in \mathbb{N}^*$, the homomorphism ϕ^j has no non trivial fixed point, that is, for all $h \in H$, $\phi^j(h) = h$ implies $h = 1$. Then the following hold:*

1. *the group G is ICC;*
2. *the pair $G_0 < G$ satisfies condition (ST);*
3. *the algebra $L(G)$ is a type II_1 factor, in which $L(G_0)$ is a strongly mixing masa.*

Example 4 Condition (SS) is strictly weaker than condition (ST): Fix a subgroup Γ of the multiplicative group $\mathbb{Q}^\times$ and set:

$$G=\left\{\begin{pmatrix} 1 & x & y \\ 0 & f_1 & 0 \\ 0 & 0 & f_2 \end{pmatrix} : f_j \in \Gamma,\ x,y\in\mathbb{Q}\right\},$$ and let

$$G_0=G_1=\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & f_1 & 0 \\ 0 & 0 & f_2 \end{pmatrix} : f_j \in \Gamma\right\}.$$ Then the pair $G_0 < G$ satisfies condition (SS) but not (ST).

An application to maximal injectivity

It turns out that some strongly mixing masas are also maximal injective, as will be seen below.
Let $\mathbb{F}_n$ be the free group on $n \geq 2$ generators $a_1, \ldots, a_n$ and let H be the cyclic group generated by a_1. Then it is easy to see that the pair $H < \mathbb{F}_n$ satisfies condition (ST). Moreover:

Theorem 10 (S. Popa, 1983) *The masa $A = L(H)$ is maximal injective in $L(\mathbb{F}_n)$.*

More recently: let $\mathcal{A}$ be the *radial* masa in $L(\mathbb{F}_n)$, *i.e.* the abelian von Neumann algebra generated by $\sum_{i=1}^{n}(a_i + a_i^{-1})$. Then Radulescu proved in 1991 that $\mathcal{A}$ is a singular masa in $L(\mathbb{F}_n)$, and, in 2008, J. Cameron, J. Fang, M. Ravichandran and S. White proved the following result (along the same general lines as Popa's proof).

Theorem 11 (CFRW, 2008) *The radial masa $\mathcal{A}$ is maximal injective in $L(\mathbb{F}_n)$.*

In fact, they also proved that if A is a singular masa in M that satisfies the so-called *asymptotic orthogonality property* (cf below), then A is maximal injective in M.

Popa's proof of Theorem 10 is divided into two parts (following A. Sinclair's and R. Smith's book, 2008):

Let $A \subset N \subset M := L(\mathbb{F}_n)$ be an intermediate von Neumann subalgebra. Then:

(1) *There is a partition of the unity* $(e_k)_{k\geq 0}$ *in the center of* N *such that* $Ne_0 = Ae_0$*, and, for every* $k \geq 1$*,* Ne_k *is a type* II_1 *factor such that* $(N' \cap A^\omega)e_k$ *has a non-zero atomic part.*

(2) *For every* $k \geq 1$ *such that* $e_k \neq 0$*,* Ne_k *is a full factor.*

Part (1) holds in more general situations:

Theorem 12 (J, 2010) *Let M be a separable type II_1 factor and let $1 \in A \subset M$ be a strongly mixing abelian von Neumann subalgebra of M. If N is a von Neumann subalgebra of M which contains A, then there exists a partition of the unity $(e_k)_{k \geq 0}$ in the center of N such that $Ne_0 = Ae_0$ and, for every $k \geq 1$, Ne_k is a type II_1 factor such that the von Neumann algebra $(N' \cap A^\omega)e_k$ has a non-zero atomic part.*

In particular, this result holds true for the hyperfinite factor since there are amenable ICC groups G that contain abelian groups H so that the pair $H < G$ satisfies condition (ST).

Concerning part (2), if there is some $k > 0$ such that Ne_k has property Γ, then

- the relative commutant $(Ne_k)' \cap (Ne_k)^\omega$ is diffuse;
- therefore, by Theorem 12, it contains $(N' \cap A^\omega)e_k$ properly; thus choose $0 \neq x \in (Ne_k)' \cap (Ne_k)^\omega$ such that $\mathbb{E}_{A^\omega}(x) = 0$.
- **Key lemma** (Consequence of asymptotic orthogonality property):
  For any unitary $w \in Ne_k$, $\mathbb{E}_A(w) = 0$, one has $wx \perp xw$ in $L^2((Ne_k)^\omega)$.
- Hence

$$2\|x\|_2^2 = \|wx\|_2^2 + \|xw\|_2^2 = \|wx - xw\|_2^2 = 0$$

leads to a contradiction.

The asymptotic orthogonality property is a typical consequence of freeness, and it holds under more general hypotheses:
Let G be an ICC countable group and let H be an abelian subgroup of G. Put $A = L(H) \subset M = L(G)$.

Consider the following condition on the pair $H < G$: There exists a sequence $(W_m)_{m\geq 1}$ of subsets of $G \setminus H$ such that

(H1) $W_m \subset W_{m+1}$ for every $m \geq 1$ and $\bigcup_m W_m = G \setminus H$;

(H2) there exists an integer $m_0 > 0$ such that, for every $m > m_0$, one can find elements $h_{1,m}, \ldots, h_{n_m,m} \in H$ such that $n_m \to \infty$ when $m \to \infty$ and

$$h_{i,m} W_m h_{i,m}^{-1} \cap h_{j,m} W_m h_{j,m}^{-1} = \emptyset \quad \forall i \neq j \;;$$

(H3) if V_m denotes the complementary set of $W_m \cup W_m^{-1}$ in $G \setminus H$, then for all $g_1, g_2 \in G \setminus H$, there exists a positive integer $m_1 = m_1(g_1, g_2)$ such that $g_1 V_m \cap V_m g_2 = \emptyset$ for every $m > m_1$.

Lemma (Asymptotic orthogonality condition) *If the pair $H < G$ satisfies conditions (H1) to (H3), then for any $x_1, x_2 \in A' \cap M^\omega$ and $y_1, y_2 \in M$ such that $\mathbb{E}_{A^\omega}(x_j) = \mathbb{E}_A(y_j) = 0$ for $j = 1, 2$, one has*

$$y_1 x_1 \perp x_2 y_2$$

in $L^2(M^\omega)$.

Due to the fact that the pairs of groups below satisfy condition (ST) by [J, Y. Stalder, 2008], the following pairs of groups $H < G$ provide pairs such that $L(H)$ is a maximal injective masa in $L(G)$ and part (2) above is also true:

- $G = H_1 *_Z H_2$ is an amalgamated product and $H = H_1$, where H_1 is a finitely generated, infinite abelian group, $Z \neq H_2$ is finite, H_2 is finitely generated and G is ICC.
- $G = K * L$ is a free product such that $|K| \geq 2$ and L contains an element β of order at least 3. Let α be some non-trivial element of K and set $H = \langle \alpha\beta \rangle$.

Open questions

- Let G be a non-elementary, torsion free, hyperbolic group and let $H < G$ be a maximal abelian subgroup of G. It is known that H is malnormal in G, *i.e.* $gHg^{-1} \cap H = \{e\}$ for every $g \in G \setminus H$, hence the pair $H < G$ satisfies condition (ST). In particular, $L(H)$ is a strongly mixing masa in $L(G)$. It is also known that G is an ICC group.

  Does $L(H)$ satisfy the asymptotic orthogonality property in $L(G)$?

- Assume that $A \subset M$ is strongly mixing and satisfies the asymptotic orthogonality property, and let Q be a diffuse finite von Neumann algebra. Then (J., Y.S. 2008) A is strongly mixing in the free product $M * Q$.

  Does A satisfy also the asymptotic orthogonality property in $M * Q$?