UFFE HAAGERUPS FOURTH TALK AT MASTERCLASS ON VON NEUMANN ALGEBRAS AND GROUP ACTIONS

1. Semi-direct product of groups

Definition 1. *Let H, K be two groups and $\sigma : K \to Aut(H)$ an action, then we define $H \rtimes_\sigma K$ as the set $H \times K$ equipped with the multiplication $(h_1, k_1)(h_2, k_2) = (h_1\sigma_{k_1}(h_2), k_1k_2)$ where we embed $H \hookrightarrow H \rtimes_\sigma K$ by the map $h \to (h, e_k)$ and define the quotine map $H \rtimes_\sigma K \to K$ by $(h, k) \to k$, this gives the sequence:*

$$0 \to H \to H \rtimes_\sigma K \to K \to 0 \tag{1.1}$$

Theorem 2 (Popa, PNAS 2004)**.** *Let σ be the standard action of $SL(2, \mathbb{Z})$ as automorphisms on $\mathbb{Z}^2$ and let $\Gamma \subseteq SL(2, \mathbb{Z})$ any subgroup of finite index, then $L(\mathbb{Z}^2 \rtimes_\sigma \Gamma)$ is a II_1-factor with trivial fundamental group. In particular $L(\mathbb{Z}^2 \rtimes_\sigma SL(2, \mathbb{Z}))$ has trivial fundamental group.*

Remark 3. *$\mathbb{F}_2 \hookrightarrow SL(2, \mathbb{Z})$ with index 12 by the embedding*

$$a \to \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \qquad b \to \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \tag{1.2}$$

Remark 4. *For $3 \leq n < \infty$ consider the group homomorphism $\phi : \mathbb{F}_2 \to \mathbb{Z}_{n-1} = \mathbb{Z}/(n-1)\mathbb{Z}$ given by $\phi(a) = [1]$, $\phi[b] = [0]$ then $\phi(\mathbb{F}_2) = \mathbb{Z}_{n-1}$ and $H = \ker(\phi) \subseteq \mathbb{F}_n$ as a subgroup of index $|\mathbb{Z}_{n-1}| = n-1$ which gives $H \cong \mathbb{F}_n$.*

Hence $\mathbb{F}_n \hookrightarrow \mathbb{F}_2 \hookrightarrow SL(2, \mathbb{Z})$ so $F_n \hookrightarrow SL(2, \mathbb{Z})$ with index $12(n-1)$.

Recall 5. *If $\Lambda \subseteq \Gamma$ of finite index then $C(\Lambda) - 1 = [\Gamma : \Lambda](C(\Gamma) - 1)$ and if $\Lambda = \mathbb{F}_n$ and $\Gamma = SL(2, \mathbb{Z})$ then $\mathbb{F}_n \hookrightarrow SL(2, \mathbb{Z})$ with finite index of $12(n-1)$ because $C(\mathbb{F}_n) - 1 = n - 1$ and $C(SL(2, \mathbb{Z})) - 1 = \frac{1}{12}$.*

2. Translation of theorem 2 to ergodic theory

If H is a locally compact abelian group and $\hat{H}$ is its dual group, and $\mathcal{F} : L^2(H) \to L^2(\hat{H})$ is its Fourier-Plancerel transform then $L(H) = \mathcal{F}^* L^\infty(\hat{H})\mathcal{F}$ and $L^\infty(\hat{H})$ is realized as multiplication operator on $l^2(\hat{H})$ and $L(H) \cong L^\infty(\hat{H})$.

If $H = \mathbb{Z}^2$ then $\hat{H} = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and $L(\mathbb{Z}^2) \cong L^\infty(\mathbb{T}^2)$ where (m, n) corresponds to the function on $\mathbb{T}^2$ given by $f(x, y) = \exp(i2\pi(mx + ny)$ for $(x, y) \in \mathbb{T}^2$.

If $\Gamma \subseteq SL(2, \mathbb{Z})$ as subgroup with finite index then $L(\mathbb{Z}^2 \rtimes_\sigma \Gamma) = L(\mathbb{Z}^2) \rtimes_{\tilde{\sigma}} \Gamma$ where the left hand side is semi direct product and the right hand side is crossed product and $\tilde{\sigma}$ is the natural extension of $\sigma : G \to \mathrm{Aut}(\mathbb{Z}^2)$ to $\tilde{\sigma} : G \to \mathrm{Aut}(L(\mathbb{Z}^2))$ where $\tilde{\sigma}_g(\exp(i2\pi(mx+ny)) = \exp(i2\pi(kx-ly))$ where $(k, l) = \sigma_g(m, n) = g\cdot(m, n)$ and $\tilde{\sigma}_g(f)(x, y) = f(g^T \cdot (x, y))$ for $f \in L^\infty(\mathbb{T}^2) = L^\infty(\mathbb{T}^2, \mu_{\mathbb{T}^2})$ where $\mu_{\mathbb{T}^2}$ is the Haarmeasure on $\mathbb{T}^2$.

Hence theorem 2 is equivalent to:

Theorem 6. *Let Γ be a finite index subgroup of $SL(2, \mathbb{Z})$ and let α be the natural action of $SL(2, \mathbb{Z})$ on $(\mathbb{T}^2, \mu_{\mathbb{T}^2})$. Then*

Date: 27/01/2010.

- (Γ, α) *acts freely and ergodicly on* $(\mathbb{T}^2, \mu_{\mathbb{T}^2})$
- *The corresponding crossed product* $\mathcal{M} = L^\infty(\mathbb{T}^2, \mu_{\mathbb{T}^2}) \rtimes_\alpha \Gamma$ *is a* II_1*-factor with trivial fundamental group.*

Remark 7. $\mathcal{A} = L^\infty(\mathbb{T}^2) \subseteq \mathcal{M}$ *is a Cartan MASA and* $\mathcal{F}(\mathcal{A} \subseteq \mathcal{M}) = \{1\}$ *because* $C(\Gamma) = 1 + \frac{1}{12}[SL(2,\mathbb{Z}) : \Gamma] > 1$.

Definition 8. *Let* G *be a discrete countable group and denote by* $P_1(G)$ *the set of positive definite functions* ϕ *on* G *with* $\phi(e) = 1$.

Recall 9. G *has property* H *iff there exists a sequence* $(\phi_n) \subseteq P_1(G) \cap C_0(G)$ *such that* $\phi_n(g) \to 1$ *for all* $g \in G$.

Recall 10. G *has property* T *iff whenever* $(\phi_n) \subseteq P_1(G)$ *with* $\phi_n(g) \to 1$ *for all* $g \in G$ *then* $\|\phi_n - 1\|_\infty \to 0$.

Definition 11. *A subgroup* $H \subseteq G$ *has relative property* T *if whenever* $(\phi_n) \subseteq P_1(G)$ *such that* $\phi_n(g) \to 1$ *for all* $g \in G$ *then* $\sup_{h \in H} \|\phi_n(h) - 1\| \to 0$.

Example 12 (Kazhdan)**.** $\mathbb{Z}^2 \subseteq \mathbb{Z}^2 \rtimes_\sigma SL(2,\mathbb{Z})$ *has relative property* T *and more generally* $\mathbb{Z}^2 \subseteq \mathbb{Z}^2 \rtimes_\sigma \Gamma$ *has relative property* T *for every finite subgroup* $\Gamma \subseteq SL(2,\mathbb{Z})$.

Definition 13. G *is rigid iff* G *has property* T *and* G *is weakly rigid if 4G4 contains an infinite normal subgroup* H *such that* $H \subseteq G$ *has relative property* T.

Example 14.
- $SL(n,\mathbb{Z}), \quad n \geq 3$ *is rigid*
- $\mathbb{Z}^2 \rtimes_\sigma SL(2,\mathbb{Z})$ *is weakly rigid, but not rigid*
- $SL(2,\mathbb{Z})$ *has property* H *and hence it is not weakly rigid*

Theorem 15 (Popa, PNAS 2004)**.** *Assume* $G = H \rtimes_\sigma K$ *is a semidirecte product such that*

- H *is abelian*
- *The corresponding action* $\hat{\sigma} : K \to Aut(\hat{H})$ *is free and ergodic on* $(\hat{H}, \mu_{\hat{H}})$
- *The inclusion* $H \subseteq H \rtimes_\sigma K$ *has relative property* T
- K *has property* H
- $C(K) > 1$ *(or one of the* l^2*-Betti-numbers* $\beta_k^{(2)}(K)$ *is nonzero)*

Then $\mathcal{F}(L(G)) = \{1\}$.

Lemma 16. *This theorem implies theorem 2.*

of lemma. Put $H = \mathbb{Z}^2$, $K = \Gamma \subseteq SL(2,\mathbb{Z})$ with finite index and σ as in theorem 2. Then the conditions of theorem 15 are fulfilled:

- $H = \mathbb{Z}^2$ is abelian
- The freeness and ergodicity of the action can easily be shown
- $\mathbb{Z}^2 \subseteq \mathbb{Z}^2 \rtimes_\sigma \Gamma$ has relative property T
- $\Gamma \subseteq SL(2,\mathbb{Z})$ has property h as this is preserved by taking subgroups
- $C(\Gamma) > 1$

Hence $\mathcal{F}(L(\mathbb{Z}^2 \rtimes_\sigma \Gamma)) = \{1\}$. □

Proposition 17 (Popa, 2006)**.** *If* $\mathcal{N}$ *is a* II_1*-factor and* $B \subseteq \mathcal{N}$ *is a sub-von Neumann algebra, then* $B \subseteq \mathcal{N}$ *has relative property* T *iff for all* $\epsilon > 0$ *there exists a finite subset* $F \subseteq \mathcal{N}$ *and a* $\delta > 0$ *such that for all* $\phi : \mathcal{N} \to \mathcal{N}$ *completely positive with* $\phi(1) \leq 1$ *and* $\tau \circ \phi \leq \tau$ *(e.g.* $\|\phi(x)\|_2 \leq \|x\|_2$*) one has that for all* $\|\phi(x) - x\|_2 < \delta$ *implies* $\sup_{b \in B, \|b\|_2 \leq 1} \|\phi(b) - b\|_2 \leq \epsilon$ *for all* $x \in F$.

Proposition 18 (Equivalent formulation)**.** *$B \subset \mathcal{N}$ has relative property T iff whenever $(\phi_n) : \mathcal{N} \to \mathcal{N}$ completely positive maps such that $\phi_n(1) \leq 1$, $\tau \circ \phi_n \leq \tau$ and $\|\phi(x) - x\|_2 0$ for all $x \in \mathcal{N}$ then $\sup_{b\in B, \|b\|_2 \leq 1} \|\phi_n(b) - b\| \to 0$.*

Remark 19. *ϕ_n extends uniquely to $\tilde{\phi}_n \in B(L^2(\mathcal{N}))$ to the last statement can be rewritten to $\|\tilde{\phi}_n - Id\|_{L^2(B)} \to 0$.*

Proposition 20 (Popa)**.** *If $H \subseteq G$ is a subgroup of an ICC-group G then $L(H) \subseteq L(G)$ has relative property T iff $H \subseteq G$ has relative property T.*

Proposition 21 (Popa)**.** *If $B \subseteq \mathcal{N}$ has relative property T then $pBp \subseteq p\mathcal{N}p$ has relative property T for all $p \in P(B)\backslash\{0\}$.*

3. Relative property H for von Neumann algebras

Let $B \subseteq \mathcal{N}$ be a sub-von Neumann algebra of a II_1-factor $\mathcal{N}$.

3.1. **Jones' basic construction.** $L^2(B, \tau|B) \subseteq L^2(\mathcal{N}, \tau)$ where $\tau = \tau_{\mathcal{N}}$.

Let e_B be the orthogonal projection on $L^2(B)$.

Then $B \subseteq \mathcal{N} \subseteq <\mathcal{N}, e_B>$ where $<\mathcal{N}, e_B> = (N \cup \{e_B\})'' \subseteq B(L^2(\mathcal{N}))$.

Define $J : L^2(\mathcal{N}) \to L^2(\mathcal{N})$ as the extension of $x \to x^*$ to $L^2(\mathcal{N})$. Then $<\mathcal{N}, e_B>' = JBJ = \{R_b | b \in B\}$ where $R_b : x \to xb, \quad x \in L^2(\mathcal{N})$.

Now $<\mathcal{N}, e_B> = <\mathcal{N}, e_B>'' = (JBJ)'$ and $B \subseteq \mathcal{N}$ is a finite von Neumann algebra, hnece JBJ is finite and hence $<\mathcal{N}, e_B>$ is a semifinite von Neumann algebra and e_B is a finite projection in $<\mathcal{N}, e_B>$.

If $\phi : \mathcal{N} \to \mathcal{N}$ is completely positive and $\phi(1) \leq 1$ and $\tau \circ \phi \leq \tau$ then ϕ extends to $\tilde{\phi} : L^2(\mathcal{N}) \to L^2(\mathcal{N})$ moreover if ϕ is a B-bimodule map i.e. $\phi(b_1 x b_2) = b_1 \phi(x) b_2$ for all $x \in \mathcal{N}, b_1, b_2 \in B$ then $\tilde{\phi} \in (JBJ)' = <\mathcal{N}, e_B>$.

Definition 22. *$J(<\mathcal{N}, e_B>)$ is a normclosed two-sided ideal generated by the finite projections in $<\mathcal{N}, e_B>$.*

$J_0(<\mathcal{N}, e_B>)$ is a normclosed two-sided ideal in $<\mathcal{N}, e_B>$ generated by e_B.

Theorem 23 (Popa, 2006)**.** *Let $B \subseteq \mathcal{N}$, TFAE*

- *There exists a sequence $(\phi_n) : \mathcal{N} \to \mathcal{N}$ completely bounded B-bimodule maps with $\phi_n(1) \leq 1$, $\tau \circ \phi_n \leq \tau$ and $\tilde{\phi}_n \in J(<\mathcal{N}, e_B>)$ such that $\|\phi_n(x) - x\|_2 \to 0$ for all $x \in \mathcal{N}$.*
- *The same with J replaced with J_0.*

Definition 24. *If the two equivalent conditions in theorem 23 are fullfilled then $B \subseteq \mathcal{N}$ has relative property H. ($\mathcal{N}$ has property H relative to B.)*

Proposition 25 (Popa, 2006)**.** *If $G = H \rtimes K$ is an ICC-group then $L(H) \subseteq L(H \rtimes K)$ has relative property H iff K has property H.*

Proposition 26 (Popa, 2006)**.** *If $B \subseteq \mathcal{N}$ has relative property H then $pBp \subseteq p\mathcal{N}p$ has relative property H for all $p \in P(B)\backslash\{0\}$.*

Definition 27. *If $\mathcal{N}$ is a II_1-factor and $\mathcal{A} \subseteq \mathcal{N}$ is a Cartan MASA then $\mathcal{A}$ has propert HTs (s for strong) iff $\mathcal{A} \subseteq \mathcal{N}$ has relative property T.*

Theorem 28 (Popa, 2006)**.** *Let $\mathcal{N}$ be a II_1-factor and A, B two Cartan MASA. If both A and B have property HTs then there exists a unitary $u \in \mathcal{N}$ such that $uAu^* = B$.*

of theorem 15. $\mathcal{N} = L(H \rtimes_\sigma K) = L(H) \rtimes K = L^\infty(\hat{H}) \rtimes K$ and K acts freely and ergodicly on $L^\infty(\hat{H}, \mu_{\hat{H}})$, so $\mathcal{N}$ is a II_1-factor and $\mathcal{A} = L^\infty(\hat{H})$ is a Cartan MASA in $\mathcal{N}$.

By assumption K has property H hence $L(H) \subseteq L(H \rtimes K)$ has relative property H and hence $\mathcal{A} \subseteq \mathcal{N}$ has relative property H.

By assumption $H \subseteq h \rtimes K$ has relative property T hence $L(H) \subseteq L(H \rtimes K)$ has relative property T.

Altogether $\mathcal{A}$ has property HTs and hence is a Cartan MASA.

We know that $\mathcal{F}(A \subseteq \mathcal{N}) = \{1\}$ by assumpition and Gaboriaus theorem. Now assume $\mathcal{F}(\mathcal{N}) \subsetneq \{1\}$ as $\mathcal{F}$ is a group there exists a $t \in]0,1[\cap \mathcal{F}(\mathcal{N})$.

For this t choose $p \in P(A)$ with $\tau(p) = t$. Then $p\mathcal{N}p \cong \mathcal{N}$ by definition of $\mathcal{F}(\mathcal{N})$, so choose $\theta : p\mathcal{N}p \to \mathcal{N}$ as von Neumann algebra isomorphism.

Because $\mathcal{A} \subseteq \mathcal{N}$ is a Cartan MASA with property HTs we have that $p\mathcal{A} = p\mathcal{A}p \subseteq p\mathcal{N}p$ is a Cartan MASA with property HTs.

Put $B = \theta(p\mathcal{A}p)$ a unital subalgebra of $\mathcal{N}$, then $(p\mathcal{A}p \subseteq p\mathcal{N}p) \cong_\theta (B \subseteq \mathcal{N})$, hence B is a Cartan MASA in $\mathcal{N}$.

But then $B = u\mathcal{A}u^*$ for some unitary $u \in \mathcal{N}$, hence $(\mathcal{A} \subseteq \mathcal{N}) \cong (B \subseteq \mathcal{N}) \cong (p\mathcal{A}p \subseteq p\mathcal{N}p)$ and hence $t = \tau(p) \in \mathcal{F}(\mathcal{A} \subseteq \mathcal{N}) = \{1\}$ which is a contradiction, and hence $\mathcal{F}(\mathcal{A} \subseteq \mathcal{N}) = \{1\}$. ∎